

Supplementary Information for Knowledge-Concealing Evidencing of Knowledge about a Quantum State

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PROTOCOLS WITH AN ABORT OPTION

Before proving our main results, we want to extend our definitions to allow for the possibility that Alice may abort the protocol. In a relativistic KCEKQS protocol with abort option, each party may have several trusted agents occupying separate secure laboratories, with secure communications between them, lying within pre-agreed regions. One agent of Bob's initially possesses Q_B . The protocol requires Alice's and Bob's agents to carry out unitary operations and/or measurements on quantum systems in their possession and to send classical and/or quantum communications to given other agents of the same party and/or the other party, within their agreed location regions and within agreed time intervals. The protocol may specify these actions are probabilistically determined, according to given probability distributions.[12]

After each round of receiving data, each of Alice's agents generates one of two possible outcomes, 0 and 1, from the classical and quantum information in her possession. If she gets outcome 0, she announces to Bob's neighbouring agents, within a pre-agreed time interval, that the protocol is aborted. Bob's agent communicates this to Bob's other agents. Alice's agent also announces the abort to all of Alice's other agents. Any agent who receives an abort message stops participating in the protocol from that point.

If there is no abort, the protocol terminates after a fixed finite number of communications between Alice's and Bob's agents. The allowed timings of abort announcements and of Bob's final announcement are fixed so that, in the event of any valid abort announcement, no agent of Bob's will announce the protocol outcome. If there is no abort, then the final prescribed action is for one of Bob's agents to generate one of two possible outcomes, 0 and 1, from the classical and quantum information in his possession. These correspond to Bob rejecting or accepting that Alice has provided evidence of knowledge of η . The possible outcomes are thus disjoint events 0, 1, abort. We write $p(0), p(1), p(\text{abort})$ for the outcome probabilities.

A non-relativistic protocol with abort option proceeds similarly, with Alice having the option to announce an abort after each Bob-to-Alice communication before Bob's final announcement.

Non-triviality for protocols with an abort option:

We define ϵ_C, ϵ_K and ϵ_S as before. We say a protocol with abort is *non-trivial* if $1 - \epsilon_C > \epsilon_S$, as before.

Comments: 1. Our definition of non-triviality is intended to characterise what might reasonably be considered to be a useful knowledge-evidencing protocol. Consider two possible hypotheses. The first is that Alice has no classical or quantum information correlated with η . The second is that Alice knows the classical value of η precisely and follows the protocol honestly. If the protocol allows Alice to attain a higher or equal value of $p(1)$ when the first hypothesis holds than the value attained when the second holds, then outcome 1 gives Bob no evidence to prefer the second hypothesis over the first. Our non-triviality condition excludes this possibility, meaning that outcome 1 gives Bob at least some evidence to prefer the second hypothesis over the first. (Of course, there are other possible hypotheses. For example, Alice could have some definite classical information about η , or some beliefs about η that she expresses in a probability distribution, or some quantum information correlated with η , or combinations of these. Non-triviality does not necessarily imply that the outcome 1 gives Bob evidence in favour of the hypothesis that Alice knows η precisely and follows the protocol honestly compared to any of these hypotheses.)

Outcome 1 is the only outcome that can sensibly have this property. Bob can never rely on an abort as giving evidence that Alice knows η , since a dishonest Alice can always abort. And it makes no sense for an outcome 0 to give positive evidence that Alice knows η compared to outcome 1: if this were the case then the protocol outcomes should be relabelled.

2. In fact, a weaker definition of non-triviality suffices for our no-go theorem below. Let $p(x|\psi; \eta)$ be the probability distribution for the three outcomes $x \in \{0, 1, \text{abort}\}$ when the state of Q_B is ψ , Bob performs the protocol correctly, and Alice performs the version of the protocol which would be correct if she knew that the state of Q_B were η . Then it is sufficient that $p(x|\psi; \eta)$ depends non-trivially on ψ for fixed η .

NO-GO THEOREMS

Zero-knowledge

Theorem 1. *There exists no non-trivial KCEKQS protocol which is zero-knowledge.*

Proof. Consider a non-trivial KCEKQS protocol \mathcal{P} applied to a system Q_B with Hilbert space \mathcal{H}_B . In any such protocol Bob may replace the unknown state η by any state of his choice. Suppose that he inputs a randomly chosen pure state ϕ , but that the protocol is otherwise honestly performed by both parties, with Alice acting in the belief that the unknown state is η . Non-triviality implies that $P(\text{outcome } 1) < P(\text{outcome } 1 | \eta = \phi)$, where the probability on the left hand side is obtained by averaging over all possible values for η . Using Bayesian inversion, we infer that $p(\eta = \phi | \text{outcome } 1) > p(\eta = \phi)$. It then follows from the continuity of the conditional probability as a function on Hilbert space that there is a neighbourhood N of ϕ with nonzero measure δ (with respect to the uniform measure), such that

$$P(\eta \in N | \text{outcome } 1) > P(\eta \in N) = \delta.$$

A non-trivial protocol necessarily also satisfies $\epsilon_C < 1$, since $1 - \epsilon_C > \epsilon_S \geq 0$, and hence $P(\text{outcome } 1) > 0$. Thus with nonzero probability, Bob gains some nontrivial information about the distribution of η from this operation. Since he retains η , he also can carry out any measurement he wishes on η . Combining information from the two processes gives him on average strictly more information than is available from the measurement alone. Hence $\epsilon_K > \epsilon_M$ and the protocol is not zero-knowledge. \square

Completeness vs soundness

Theorem 2. *In a KCEKQS protocol for a pure quantum state in a Hilbert space of dimension d , the completeness and soundness parameters obey*

$$\frac{\epsilon_S}{1 - \epsilon_C} \geq \frac{1}{d}.$$

Proof. A protocol P for KCEKQS may require either party to carry out measurements, to make random choices from a classical probability distribution, to introduce quantum states, and/or to send classical data. For any such protocol, we can define a related fully quantum protocol QP in which all data are introduced as quantum ancillae at the start and kept at the quantum level until the outcomes are obtained. Thus, to define QP from P , measurements are replaced by unitary measurement interactions (without extracting measurement data), classical random choices are replaced by interactions with entangled “quantum dice” (without extracting data about the dice outcome) and classical communications are replaced by quantum communications of states in a pre-agreed orthonormal basis (for instance the computational basis). We take the protocol QP to proceed thus until the last steps. The first of these is for all the information in Alice’s possession to be sent to one of Alice’s agents, who carries out a measurement giving her the outcome abort or not abort. She communicates this outcome to all of Alice’s and all of Bob’s agents, within an agreed time window, so that Bob’s agents all know the outcome after a prespecified coordinate time. If no abort is communicated to any of Bob’s agents within the prescribed time, they send all the information in their possession to one of Bob’s agents, who carries out a single measurement giving him the outcome 1 or 0.

We take the Hilbert spaces under Alice’s and Bob’s control initially to be H_A and H_B respectively, and the Hilbert spaces under Alice’s and Bob’s control at the end of the protocol to be H_{A_f} and H_{B_f} . The initial and final Hilbert spaces for each party are not necessarily identical, since the protocol may require states to be sent from one party to another. By introducing ancillae as necessary, we may take the final measurements by Alice and Bob to be projective measurements $(P_{A_f}, I_{A_f} - P_{A_f})$ and $(P_{B_f}, I_{B_f} - P_{B_f})$.

If both parties are honest, the probabilities of all outcomes are the same in QP as in P . If Bob is honest, then it cannot be disadvantageous to Alice to replace P by QP : all strategies available to her in the former can be replicated in the latter, and the latter also generally offers her further strategies. In particular, the strategy we define below for Alice, when she does not know η , has the same success probabilities in QP and P . We may therefore, for simplicity, without loss of generality assume a fully quantum protocol. We will write η_B for Bob’s unknown state when we wish to emphasize that it is initially under Bob’s control. Let S_{AB} be the state of all the ancillae introduced by Alice and Bob in QP , and let U^η be the unitary operation defined by the protocol up to the final outcome measurements, when Alice honestly follows the protocol and believes the state is η . Here U^η

includes any state transfers between the parties, as well as local unitaries applied by each party. Thus U^η maps $H_A \otimes H_B$ to $H_{A_f} \otimes H_{B_f}$. [13] We have

$$\int \text{Tr}(((I_{A_f} - P_{A_f}) \otimes P_{B_f})U^\eta(\eta_B \otimes S_{AB})(U^\eta)^\dagger)d\mu(\eta) = 1 - \epsilon_C$$

where $d\mu(\cdot)$ denotes the uniform measure over quantum states, and the integral is performed over the entire Hilbert space of Q_B .

Now suppose Alice does not in fact know η . Then she may always adopt the strategy of choosing a random state ϕ from the Hilbert space of Q_B and proceeding with the protocol as if she knows that Q_B is in the state ϕ . Since aborting cannot increase $p(1)$, an optimum strategy to maximise $p(1)$ is never to abort, i.e. to take $P_{A_f} = 0$. Assuming that Bob always performs his part of the protocol honestly, the expected value of $p(1)$ is then

$$\int \int \text{Tr}((I_{A_f} \otimes P_{B_f})U^\phi(\eta_B \otimes S_{AB})(U^\phi)^\dagger)d\mu(\eta)d\mu(\phi) \leq \epsilon_S.$$

Moving the integral inside the trace, and noting that $\int \psi d\mu(\psi) = \frac{1}{d}I_B$, where I_B is the d -dimensional identity matrix on Q_B , we obtain

$$\frac{1}{d} \int \text{Tr}((I_{A_f} \otimes P_{B_f})U^\phi(I_B \otimes S_{AB})(U^\phi)^\dagger)d\mu(\phi) \leq \epsilon_S.$$

The left hand side is bounded below by

$$\frac{1}{d} \int \text{Tr}((I_{A_f} \otimes P_{B_f})U^\phi(\phi_B \otimes S_{AB})(U^\phi)^\dagger)d\mu(\phi),$$

which is bounded below by

$$\frac{1}{d} \int \text{Tr}(((I_{A_f} - P_{A_f}) \otimes P_{B_f})U^\phi(\phi_B \otimes S_{AB})(U^\phi)^\dagger)d\mu(\phi) = \frac{1}{d}(1 - \epsilon_C).$$

Hence $\frac{\epsilon_S}{1 - \epsilon_C} \geq \frac{1}{d}$ as required. □

CLASSICAL PROTOCOLS

As noted in the main text, Horodecki et al. [7] argue, inter alia, that non-trivial zero-knowledge classical A-to-B protocols with $\epsilon_C = 0$ are impossible for a qubit. For such protocols, Alice must predict some measurement outcome, and any measurement prediction that holds with certainty for a pure qubit η and is not certain for a random qubit allows Bob to identify η exactly, and so has $\epsilon_K = 1$. Horodecki et al. did not analyse protocols involving qudits and also did not consider protocols with $\epsilon_C > 0$. One way of realising such protocols is for Alice to choose a projective measurement including a randomly chosen projector P from those with $\langle \eta | P | \eta \rangle \geq 1 - \epsilon_C$ and predict the likeliest outcome to Bob.

Clearly, however, any qudit protocol in which Alice simply sends Bob a classical prediction of a measurement outcome has similar issues to the qubit protocols which Horodecki et al. analysed [7]. Whenever the prediction is highly likely if Q_B is in state η and not so likely if Q_B is in a random state, Bob can obtain a significant amount of information about η simply by examining the prediction and calculating the set of states for which it is highly likely. Moreover, if Bob assumes that Alice is honest, he may choose to refrain from performing the measurement necessary to confirm her prediction and instead carry out some other measurement, thus gaining additional information.

Horodecki et al. also did not consider relativistic protocols. Moving to the setting of relativistic quantum cryptography provides ways to strengthen prediction protocols somewhat, since Alice can use secure relativistic bit commitments [1, 2, 8–10] to commit her predictions, conferring two potential advantages. First, Alice can commit to more than one outcome and subsequently reveal to Bob only the outcome corresponding to the result that he has obtained, thus decreasing the amount of information that Bob obtains simply from the fact that she has made a certain prediction. Second, Alice need not reveal her prediction to Bob unless he first tells her the predicted outcome. The intuition is that this essentially forces Bob to carry out (something close to) the specified measurement if he wishes to have a significant chance of getting information from Alice. This prevents him from carrying out a different measurement that gives him (much) additional information.

We now consider classical protocols which use relativistic bit commitment protocols for one or both of these purposes.

Classical relativistic KCEKQS protocol 1:

There is some agreed frame in which the agents are agreed to remain at approximately the same position coordinates throughout the protocol, respecting the configurations given below. We take space and time coordinates with respect to this frame. We work in units with $c = 1$ throughout this paper.

1. Alice and Bob each have two agents, A_1, A_2 and B_1, B_2 , configured so that $d = d(A_1, B_1) \approx d(A_2, B_2) \ll D = d(A_1, B_2) \approx d(A_2, B_1)$, as in the relativistic bit commitment protocol of Ref. [9].
2. Alice chooses a projective measurement $\{P_i\}$ such that $\exists x : \text{Tr}(P_x \eta) \geq 1 - \epsilon_C$. The value of x is secretly shared by both her agents.
3. At $t = 0$ Alice's agent A_1 tells Bob's agent B_1 the measurement $\{P_i\}$. Also at $t = 0$, Alice's agent A_2 and Bob's agent B_2 initiate a relativistic bit string commitment protocol committing A_2 to the binary encoding of the index x .
4. At $t = \delta \ll D$, B_1 performs the measurement $\{P_i\}$ on Q_B and reports his result to A_1 .
5. At $t = \delta'$, where $\delta' > \delta$ and $\delta' \ll D$, if B_1 reported the result P_x , A_1 unveils the commitment made by A_2 . If not, the commitment is not unveiled.
6. If A_1 unveils a commitment that matches P_x , Bob accepts the proof, after verifying the unveiled commitment by collecting data from his two agents. Otherwise, he rejects.

Note: The time coordinates given for this and later relativistic protocols are merely examples of possible timings. The key requirement is that the timings should ensure that at each step, relativistic signalling constraints ensure that the relevant agent of Alice can have no information about data supplied to the other agent by Bob's corresponding agent at the previous commitment round (if any).

Security discussion: First, note that there is no exchange of quantum information in this protocol. For the purposes of security analysis, it can be treated as a relativistic bit string commitment protocol, in which Alice commits to the bit string defining x via parallel repetition of the protocol of Ref. [9]. For Alice to cheat, she must be able to unveil the possible bit strings y with success probabilities p_y for which $\sum_y p_y$ is significantly greater than 1. Conversely, to prove it secure against Alice requires showing that the relativistic bit string commitment protocol ensures that $\sum_y p_y \leq 1 + \epsilon(N, d)$, where N is the protocol security parameter and $\epsilon(N, d) \rightarrow 0$ as $N \rightarrow \infty$ for each finite d .

A full proof of security against Alice for this protocol thus requires analysis of the behaviour of the bit commitment protocol of ref [9] under parallel repetition. The security proof of [9] for of single round classical relativistic bit commitment holds for both classical and quantum attacks. The extension of this argument to bit strings is an interesting topic in its own right, and we leave discussion of this for future work.

In the present discussion we simply assume without proof that, in the limit as the security parameters for the bit commitments become large, Alice can effectively commit to only one outcome of the measurement (or to some convex combination of outcomes if she makes a probabilistic commitment). Under this assumption, if Alice has no information about the state η , her optimal strategy is simply to commit to a randomly chosen outcome. Hence the value of ϵ_S is $\frac{1}{d}$, and so $\epsilon_S \rightarrow 0$ as $d \rightarrow \infty$.

Comment: On the other hand, the protocol does not provide a good trade-off between completeness and knowledge concealment. Assuming Alice does indeed know the state η , the protocol succeeds with probability at least $(1 - \epsilon_C)$. In this case Bob learns the value of a projection P_x with $\text{Tr}(P_x \eta) \geq 1 - \epsilon_C$.

Hence

$$\epsilon_K \geq (1 - \epsilon_C)^2. \quad (1)$$

This is a very poor trade-off; in particular $\epsilon_C \approx 0$ implies $\epsilon_K \approx 1$.

For comparison, if Bob does not take part in the protocol but instead carries out operations on Q_B , the maximum expected fidelity of his estimate for η is $\epsilon_M = \frac{2}{d+1}$. For $\epsilon_C \approx 0$ we thus have $\epsilon_K - \epsilon_M \gtrsim \frac{d-1}{d+1} \geq \frac{1}{3}$, and for d large we have $\epsilon_K - \epsilon_M \approx 1$.

One way to improve on the bound (1), at the price of increasing ϵ_S , is to allow Alice to commit to more than one outcome of the projective measurement. We now consider a protocol of this type.

Classical relativistic KCEKQS protocol 2:

As with the previous protocol, there is some agreed frame in which the agents are agreed to remain at approximately the same position coordinates throughout the protocol, respecting the configurations given below. We take space and time coordinates with respect to this frame.

1. Alice and Bob each have two agents, A_1, A_2 and B_1, B_2 , configured so that $d = d(A_1, B_1) \approx d(A_2, B_2) \ll D = d(A_1, B_2) \approx d(A_2, B_1)$, as in the relativistic bit commitment protocol of Ref. [9].
2. Alice chooses a projective measurement $\{P_i\}$ for which there exists a set S consisting of q measurement elements such that $\text{Tr}((\sum_{i \in S} P_i)|\eta\rangle\langle\eta|) = 1 - \epsilon_C$ for some agreed value of ϵ_C . The indices $x \in S$ are secretly shared by both her agents.
3. At $t = 0$, Alice's agent A_1 tells Bob's agent B_1 the measurement $\{P_i\}$. Also at $t = 0$, Alice's agent A_2 and Bob's agent B_2 initiate a relativistic bit string commitment protocols committing A_2 to the binary encoding of each index $x \in S$.
4. At $t = \delta \ll D$, B_1 performs the measurement $\{P_i\}$ on Q_B and reports his result to A_1 .
5. At $t = \delta'$, where $\delta' > \delta$ and $\delta' \ll D$, if B_1 reported a result P_x with $x \in S$, A_1 unveils the commitment to x made by A_2 . The remaining commitments are not unveiled.
6. If A_1 unveils a commitment that matches P_x , Bob accepts the proof, after verifying the unveiled commitment by collecting data from his two agents. Otherwise, he rejects.

We now show that, conditional on our previously stated assumptions about the security of relativistic bit commitment protocols:

$$\epsilon_K \geq q \left(\frac{1 - \epsilon_C}{q} \right)^2 \quad \text{and} \quad \epsilon_S = \frac{q}{d}.$$

Proof. As in the previous proof, we assume that the bit commitment protocol remains secure under parallel repetition. Hence we limit our discussion to strategies where Alice commits honestly to a single set of q out of d possible measurement outcomes in the bit commitment protocol.

If Alice has no information about the state η , then she has no better strategy than choosing q random orthogonal one-dimensional projections. Hence $\epsilon_S = \frac{q}{d}$, i.e. $q = \epsilon_S d$.

The relativistic bit commitment protocol [9] is perfectly secure against Bob both for a single bit commitment and when composed to define a bit string commitment: in either case, he obtains no information about Alice's commitments unless and until she unveils them.

Suppose then that Alice knows η and randomly chooses a projective measurement $\{P_i\}$ for which there exists a set S consisting of q measurement elements such that $\text{Tr}((\sum_{i \in S} P_i)\eta) = 1 - \epsilon_C$. If Bob follows the protocol and carries out this measurement, he obtains outcome P_i with probability $p_i = \text{Tr}(P_i \eta)$. If Alice's unveiled commitment confirms that $i \in S$, Bob's maximum expected squared fidelity guess is $\eta = P_i$, which has squared fidelity p_i . Hence

$$\epsilon_K \geq \sum_{i \in S} (p_i)^2 \geq q \left(\frac{1 - \epsilon_C}{q} \right)^2 = (1 - \epsilon_C)^2 q^{-1}.$$

□

For comparison, if Bob does not take part in the protocol but instead carries out operations on Q_B , the maximum expected fidelity of his estimate for η is $\epsilon_M = \frac{2}{d+1}$. For $\epsilon_C \approx 0$ and $q \ll \frac{d}{2}$ we thus have $\epsilon_K \gg \epsilon_M$.

Comment A variation on this protocol is for Alice to randomly choose a projective measurement $\{P_i\}$ from among those for which there exists a set S consisting of q measurement elements such that $\text{Tr}((\sum_{i \in S} P_i)\eta) \geq 1 - \epsilon$. She could, for example, use the uniform measure on the complete projective decompositions satisfying this criterion. This variation has $\epsilon_C < \epsilon$, since Alice's average success probability, when she knows η , is greater than $1 - \epsilon$. Similarly, it has $\epsilon_K > (1 - \epsilon)^2 q^{-1} > (1 - \epsilon_C)^2 q^{-1}$.

QUANTUM ALICE-TO-BOB PROTOCOLS

Throughout this section, we use the notation $w(a, b) = \binom{a+b-1}{a}$.

We first show that the value of ϵ_S for the quantum Alice-to-Bob protocol set out in the main article is $\frac{1}{N+1} + \frac{N}{d(N+1)}$.

If Alice does not in fact know η , the probability that Bob's measurement obtains a positive result is maximized if she chooses a single state ϕ and prepares all N systems $\{S_i\}$ in the state ϕ .

We use the fact that $\text{Tr}(\Pi_S(|\phi\rangle\langle\phi|^{\otimes N} \otimes \mathbb{I}^{\otimes M})\Pi_S) = \frac{w(M+N, d)}{w(N, d)}$ [11]. Thus when Alice employs the optimum strategy, the probability that Bob's measurement obtains a positive outcome is given by:

$$\begin{aligned}
\epsilon_S &= \int \text{Tr}(\Pi(|\phi\rangle\langle\phi|^{\otimes N} \otimes \psi)\Pi) d\mu(\psi) = \text{Tr}(\Pi(|\phi\rangle\langle\phi|^{\otimes N} \otimes \frac{\mathbb{I}}{d})\Pi) = \frac{w(N+1, d)}{w(N, d)d} \\
&= \frac{1}{N+1} + \frac{N}{d(N+1)}
\end{aligned} \tag{2}$$

In particular, for $N = 1$, we have

$$\epsilon_S = \frac{1}{2} + \frac{1}{2d}$$

and $\epsilon_S > \frac{1}{2}$ for any d .

We can also bound ϵ_K . If Alice follows the protocol honestly, Bob has available $N + 1$ copies of the state η . His average squared fidelity between the true state η and the best possible guess from measuring $N + 1$ states is $\frac{N+2}{N+1+d}$ [4]. Thus $\epsilon_K = \frac{N+2}{N+1+d}$. Similarly, setting $N = 0$, we have that Bob's optimal average squared fidelity guess from the original copy of η is $\epsilon_M = \frac{2}{d+1}$.

From equation (2), we obtain $\epsilon_K > \frac{1}{d\epsilon_S}$ and $\epsilon_K - \epsilon_M > \frac{d-1}{d+1} \frac{N}{N+2} \frac{1}{d\epsilon_S}$. In particular, near CS-optimality ($\epsilon_S \approx \frac{1}{d}$) requires N large, which implies near-zero concealment ($\epsilon_K \approx 1$) and significant knowledge gain ($\epsilon_K - \epsilon_M \approx \frac{d-1}{d+1}$). For large d , this also implies near-complete knowledge gain ($\epsilon_K - \epsilon_M \approx 1$).

Note that Bob can both follow this protocol honestly in order to gain evidence about Alice's knowledge of the quantum state, and then also afterwards attempt to estimate η from the $N + 1$ states in his possession since the states of these copies of η are not changed in the course of a protocol in which both parties are honest. Thus, if Alice honestly follows the protocol, Bob can obtain evidence of her knowledge of η and still attain the value of $\epsilon_K = \frac{N+2}{N+1+d}$ above.

QUANTUM BOB-TO-ALICE PROTOCOL

We begin with a complete description of the quantum B-to-A protocol summarised in the main text, including the configuration of agents required for the relativistic bit commitment sub-protocols. As with the earlier protocols, there is some agreed frame in which the agents are agreed to remain at approximately the same position coordinates throughout the protocol, respecting the configurations given below. We take space and time coordinates with respect to this frame.

1. Alice and Bob each have two agents, A_1, A_2 and B_1, B_2 , configured so that $d = d(A_1, B_1) \approx d(A_2, B_2) \ll D = d(A_1, B_2) \approx d(A_2, B_1)$, as in the relativistic bit commitment protocol of Ref. [9].
2. Alice and Bob agree in advance on positive integer security parameters N and q .
3. Bob gives Q_B to B_1 , who prepares N additional quantum systems $\{S_i\}$ in states chosen uniformly at random.
4. B_1 randomly permutes the systems $\{S_i\}$ and the system Q_B , assigns them all indices from 1 to $N + 1$, and then gives all $N + 1$ systems, labelled by their indices, to A_1 .
5. A_1 carries out the projective measurement $\{\eta, \mathbb{I} - \eta\}$ on each of the $N + 1$ systems that B_1 gave her. Write C' for the list of indices for which she obtains outcome η ; let $|C'| = q'$. If $q' \leq q$, she forms a list $C = C' \cup D$, where D is a list of $(q - q')$ copies of the dummy index 0.[14]. If $q' > q$, she picks a random size q sublist C of C' .
6. A_1 randomly permutes C . At $t = 0$, B_1 and A_1 initiate q relativistic bit string commitments [9] committing A_1 to each of the indices in the permuted list. Each bit string commitment is set up so that Alice can commit to any index in $\{1, 2, \dots, N + 1\}$. This commitment is sustained by B_2 and A_2 at time $t = \delta$, where $0 < \delta \ll D$. These commitments involve a further security parameter N' .
7. At $t = \delta'$, where $0 < \delta' \ll D$ (and for definiteness we may take $\delta' > \delta$) B_1 tells A_1 the index $x \in \{1, \dots, N + 1\}$ that he assigned to Q_B .
8. If $x \in C$, A_1 unveils her commitment to that index (which she initiated and A_2 sustained). Otherwise she announces failure (or aborts, if the protocol includes an abort option) and Bob rejects.[15]
9. If Alice's unveiled commitment is indeed x , Bob accepts, once he is able to verify this by comparing data from his agents. Otherwise he rejects.

Security against Alice

Alice's most general possible strategy starts by pre-sharing some quantum state between A_1 and A_2 . A_1 receives additional quantum systems from B_1 , namely $(N + 1)$ qudits. She can then carry out any quantum operation on the quantum systems in her control, depending on the classical data sent by B_1 , to generate responses to B_1 that purport to initiate the bit commitment protocols. A_2 can similarly carry out any quantum operation on the quantum systems in her control, depending on the classical data sent by B_2 , to generate responses to B_2 that purport to sustain the protocols. A_1 can then carry out any further quantum operation on the quantum systems under her control, depending on the index x sent by B_1 , to generate data that purport to unveil x as one of her committed strings.

A full security analysis against Alice thus requires a discussion of security for a protocol composed of two rounds of a series of relativistic bit commitment sub-protocols together with the remaining steps of the protocol above. We leave this discussion for a future more general analysis of protocols within which relativistic bit commitments are suitably composable. For the present discussion, we will make the restrictive assumption that Alice honestly follows each relativistic bit commitment protocol [9] for the first two rounds and uses no quantum information in these relativistic bit commitment protocols. Thus we assume that A_1 and A_2 have pre-shared classical random numbers which they use, following the protocol [9], to commit Alice to some definite classical bit value in the first round of each bit commitment, and to sustain each of these commitments in the second round.

Given that A_1 and A_2 follow this honest strategy to commit and sustain the commitment to a definite classical bit value b for the first two rounds, let $p_S(\bar{b})$ be the probability (for any unveiling strategy S) that A_1 can successfully unveil the opposite bit value \bar{b} at the unveiling (third) stage. Each individual relativistic bit commitment used in the protocol has the property [9] that $p_S(\bar{b}) \leq \epsilon(N')$, where $\epsilon(N') \rightarrow 0$ as the security parameter $N' \rightarrow \infty$. We will assume N' is very large compared to N and neglect terms of order $\epsilon(N')$ in the following discussion. (Note that the equalities and inequalities proven are thus correct only up to order $\epsilon(N')$). In this limit, and modulo our assumptions, the individual relativistic bit commitments are perfectly secure against Alice. Given her restricted strategy, her only options are to honestly commit to strings of bits $\{b_i\}$ and then either to unveil any given bit commitment, revealing the actual committed bit value b_i , or to decline to unveil it.

Soundness

We now argue that, given these assumptions, if the protocol involves an unknown d -dimensional state, and we take $(N + 1) = Md$ for integer M , and $q = \frac{N+1}{d} = M$, then $\epsilon_S = \frac{1}{d}$.

Proof. Alice can achieve a success probability of $\frac{q}{N+1}$ by simply committing to q random distinct indices, and hence $\epsilon_S \geq \frac{q}{N+1}$. We now show that also $\epsilon_S \leq \frac{q}{N+1}$, and hence $\epsilon_S = \frac{q}{N+1}$.

Suppose that A_1 and A_2 begin the KCEKQS protocol with no information whatsoever about η , and that Bob honestly follows the protocol. From Alice's perspective, she simply receives $(N + 1)$ random pure qudits, since η is a random qudit and the remaining qudits are independently randomly chosen by Bob. Thus until B_1 gives A_1 classical information about the index assigned to Q_B , A_1 and A_2 's state of knowledge is exactly symmetrical with respect to all of the $N + 1$ qudits sent by B_1 . A_1 is required to initiate commitments, and A_2 to sustain them, before they receive the index assigned to Q_B , and therefore their commitment strategy on these rounds cannot depend on that index.

Thus we may without loss of generality calculate Alice's success probability by considering some fixed commitment strategy and averaging over all $N + 1$ possible values for the index of Q_B . For simplicity, we will analyse a related protocol T in which Alice always makes q commitments and subsequently unveils all q commitments. We will say that Alice is successful in T if and only if at least she unveils at least one valid commitment to the correct index for Q_B (whether or not other unveiled commitments turn out to be valid commitments to any index). Alice's success probability ϵ_S in the KCEKQS protocol is no greater than her success probability in T , since in either case Alice succeeds if and only if she can unveil at least one commitment to the correct index.

Now, for the purpose of obtaining a contradiction, suppose that Alice can succeed in protocol T with probability greater than $\frac{q}{N+1}$. If the probability distribution over the values of Alice's unveiled commitments in task T depends only on her initial commitment strategy and not on the classical information she subsequently obtains from Bob, then the values of her q unveiled commitments must be uncorrelated with the index of Q_B . Therefore the probability that one of these q unveiled commitments is to the correct index for Q_B can be no greater than $\frac{q}{N+1}$, with this bound saturated whenever Alice uses a strategy which always produces q valid commitments to different bit values in $\{1, 2, \dots, N + 1\}$. Thus if the success probability is greater than $\frac{q}{N+1}$, the probability distribution over the values of her unveiled commitments must be correlated with the classical information that she receives from Bob.

If so, under our assumptions about Alice's restricted strategy, Alice must have some freedom to choose whether to unveil 0 or 1 for at least one bit i out of the $\approx q \log(N + 1)$ bits to which she commits. More precisely, for at least two different indices q, r

in $\{1, 2, \dots, N+1\}$, if $p_i(0|q)$ is the probability that Alice unveils a 0 for bit value i when Bob tells her the index of Q_B is q , and $p_i(1|r)$ is similarly defined, then $p_i(0|q) + p_i(1|r) > 1$. But this contradicts the security of the relativistic bit commitment protocol [9], under our assumptions about Alice's restricted strategy. (Recall again that we neglect terms of order $\epsilon(N')$ in this discussion.) Hence, given those assumptions, Alice cannot succeed in protocol T with probability greater than $\frac{q}{N+1}$. Hence, again given those assumptions, in our KCEKQS protocol, $\epsilon_S \leq \frac{q}{N+1}$. \square

Completeness

We now show that $\epsilon_C \rightarrow 0$ as $N \rightarrow \infty$.

Proof. If Alice does know the state η , she will get a positive outcome on Q_B . The probability that her commitment fails to be accepted is thus given by:

$$\epsilon_C = \sum_{x=q}^N P(X_N = x) \frac{x+1-q}{x+1}, \quad (3)$$

where $p(X_N = x)$ is the probability that x out of Alice's N measurements on the N systems S_i obtain the result η .

If Bob is honest and chooses the states of the systems S_i at random, the result η is obtained on each run with probability $\frac{1}{d}$. Hence X_N is binomially distributed:

$$P(X_N = x) = \binom{N}{x} \frac{1}{d^x} \left(1 - \frac{1}{d}\right)^{N-x}. \quad (4)$$

For $q = \frac{N+1}{d}$, the distribution X_N has mean $q - \frac{1}{d} = \frac{N}{d}$. Hoeffding's inequality implies that

$$P(X_N \geq \frac{N}{d} + \epsilon N) \leq \exp(-2\epsilon^2 N). \quad (5)$$

It follows from equation (3) that

$$\epsilon_C \leq \epsilon d p_{\text{mode}} + \exp(-2\epsilon^2 N) \quad (6)$$

for any $\epsilon > 0$, where p_{mode} is the maximum over x of the binomial distribution $P(X_N = x)$. We can obtain an adequate bound simply by using $p_{\text{mode}} \leq 1$. [16]

Taking, for example, $\epsilon = \frac{1}{2} N^{-\frac{1}{2}} (\log N)^{\frac{1}{2}}$, we see that $\epsilon_C \rightarrow 0$ as $N \rightarrow \infty$. \square

Knowledge concealment

Bob begins with one copy of η . If dishonest, he may combine this with any ancillae he wishes, carry out any quantum operations he wishes, and produce a (perhaps highly entangled) state including $(N+1)$ qudits that he sends to Alice, together with a system that he retains. He may then carry out any measurement he wishes to produce an index x . Alice responds, effectively, with either a 1 (if she unveils a commitment to x) or a 0 (if she fails to unveil a commitment to x). Bob may then carry out any measurement he wishes on his retained system to produce his maximum possible expected squared fidelity estimate of η . This measurement choice may depend on Alice's response.

Effectively, Bob's task is to optimize his state estimation in this scenario. His overall strategy S is fixed up to Alice's response, but then may involve different state estimation strategies depending on the one bit of information supplied by Alice. For any given overall strategy S , the overall expected squared fidelity of his estimate is

$$p_S(0)f_S(0) + p_S(1)f_S(1), \quad (7)$$

where $p_S(b)$ is the probability of outcome b given strategy S and $f_S(b)$ is the expected squared fidelity obtained from S conditioned on outcome b .

Suppose that Alice does not respond at all. Bob may simply follow the fixed strategy S_b given by following S and assuming outcome b . This produces an expected squared fidelity of at least $p_S(b)f_S(b)$. Since S_b is a possible strategy for the standard task of state estimation given one copy of an unknown qudit (and no further information), we have that

$$p_S(b)f_S(b) \leq f_{\text{max}} = \frac{2}{d+1} \quad (8)$$

where the right hand side is the maximum expected square fidelity obtainable from any state estimation strategy on an unknown qudit [4]. Hence Bob's overall expected squared fidelity

$$p_S(0)f_S(0) + p_S(1)f_S(1) \leq \frac{4}{d+1}. \quad (9)$$

That is, $\epsilon_K \leq \frac{4}{d+1}$. For comparison, we have that Bob's optimal expected squared fidelity if he does not participate in the protocol but simply carries out measurements on η is $\epsilon_M = \frac{2}{d+1}$. Thus, for large d , the protocol is highly knowledge concealing and close to zero knowledge.

Comment

Our discussion to date has been based on the assumption that the state η is pure. This excludes the possibility that Alice has access to one or more systems which are entangled with Q_B . However, another natural scenario (call it Sc_1) in which a KCEKQS protocol might be useful is that Q_B is maximally entangled with a system Q_A in Alice's possession, and that Alice knows the joint state.

One option for Alice is then to perform a projective measurement on Q_A before the start of the protocol, so that the system Q_B is subsequently in a pure state η known to her. This recreates our original scenario (call it Sc_0). Thus, if Alice can succeed in any given KCEKQS protocol with probability $(1 - \epsilon_C)$ if she knows η , she can also succeed with probability at least $(1 - \epsilon_C)$ in the scenario of the previous paragraph. Conversely, while there are clearly protocols that give Bob evidence favouring Sc_1 over Sc_0 , no KCEKQS protocol can give Bob evidence favouring Sc_0 over Sc_1 .

Allowing an abort option

Our relativistic quantum Bob-to-Alice protocol could be altered to achieve $\epsilon_C = 0$ by allowing Alice to abort the protocol whenever she obtains a positive outcome more than q times at step five [17]. An honest Alice with perfect knowledge will abort whenever the binomial random variable $X_N > q$. For $q = \frac{N+1}{d}$ and large N , this implies an abort probability of roughly $\frac{1}{2}$. Taking $q = \frac{N}{d} + \epsilon N$, with $\epsilon > 0$, we see from equation (5) that the abort probability can be bounded by $\exp(-2\epsilon^2 N)$. This gives a protocol with $\epsilon_C = 0$, $\epsilon_S = \frac{1}{d} + \epsilon$ and with abort probability that tends to zero for large N .

These parameters may represent a reasonable tradeoff in some circumstances. However, introducing an abort option does not eliminate the possibility of unjustified mistrust. Without an abort option, it is possible that Alice and Bob may both honestly follow the protocol, and that Alice may know η precisely and thus correctly identify Q_B as a candidate, but that Alice may be unable to persuade Bob of this because she had more than q candidates and her random choice of a size q subset did not include the index of Q_B . Introducing an abort option removes the possibility of honest Alice being falsely suspected of cheating by honest Bob *for this specific reason*. However, an honest Bob may now unfairly suspect an honest Alice of cheating if she honestly aborts. A dishonest Alice might abort because she had no information about η , carried out no measurements, made random or invalid commitments, and thus effectively used the protocol to steal Bob's copy of η . Bob cannot tell whether or not the protocol has been honestly aborted.

No version of this protocol can guarantee that honest Bob accepts the honesty of honest Alice. This is because KCEKQS is a one-shot procedure, and this protocol has nonzero probability of either failure or abort (if there is an abort option). Bob has only one copy of η , and so if he gives that copy to Alice and the protocol fails or is aborted, he has no further opportunity to learn anything about η or about Alice's knowledge of it. Of course, the parties might be able to repeat the protocol using a new state, but even if this next protocol succeeds, Alice will have proved only that she knows the new state, which gives Bob no direct evidence about whether she knew the previous state.

In summary, in most scenarios, it seems to us that the abort option version of the protocol gives no clear advantage.

FURTHER SECURITY ISSUES

Here we discuss several potential security issues and possible weaknesses for KCEKQS protocols which are not covered by our earlier security definitions.

What precisely does Alice give evidence of?

We have defined security for KCEKQS in terms of two extremes, the value of ϵ_S giving the probability that Alice's proof is accepted if she knows nothing at all about the state, and the value of $1 - \epsilon_C$ giving the probability that Alice's proof is accepted if she knows the state exactly. As noted above, there are intermediate possibilities. For example, Alice could have a significant amount of classical knowledge about the quantum state without knowing it exactly; she could have quantum information correlated with the state (such as a number of copies), without any additional classical information; she could

have beliefs about the state encoded in a probability distribution. Such beliefs need not necessarily be well founded or correct: for instance she might believe the classical description of the state is $\eta' \neq \eta$.

In each case, if Alice's information or beliefs allow her to produce a guess η' at the classical description of η such that $\text{Tr}(\eta\eta') \approx 1$, she can produce what we have called "evidence of knowledge". In each of the KCEKQS protocols described above, Bob will accept this "evidence of knowledge" with probability close to $1 - \epsilon_C$ – even though Alice may not only not have complete knowledge of η but may even have false beliefs about its classical value. As this illustrates, the notion of knowledge of a quantum state needs careful analysis. One might be inclined to frame a definition so that if Alice believes the state is the pure state η' , and it is actually $\eta \neq \eta'$, then she has no knowledge about the state, even if $\text{Tr}(\eta\eta') \approx 1$. But if one does, one has to accept that someone who has no knowledge about a state can nonetheless appear to give strong evidence of knowledge. Alternatively, one might frame a definition so that, if Alice has probabilistic beliefs about the state η encapsulated in the density matrix ρ , then $\text{Tr}(\rho\eta)$ is a measure of her knowledge. If so, one has to accept that false (not merely uncertain) beliefs are consistent with a high degree of knowledge.

Without giving a precise general definition of knowledge of a quantum state, our security definitions still establish that the protocols give evidence of knowledge of η in an interesting, if restricted, sense. Namely, they strengthen the evidence for the hypothesis that Alice knows the precise classical value of η compared to the hypothesis that she has no classical or quantum information correlated with η .

One might frame additional security definitions that allow more to be established. For example, one could require *strong non-triviality*: if Alice does not have and is not able to obtain a precise classical description of η , then the probability that her proof is accepted is strictly less than $(1 - \epsilon_C)$. This requirement holds for our classical relativistic KCEKQS protocol 1 and for our quantum B-to-A protocol but does not hold for our classical relativistic KCEKQS protocol 2, in which Alice can ensure acceptance probability $(1 - \epsilon_C)$ even if she only knows a subspace of Hilbert space of dimension $q = \epsilon_S d$ in which the state lies. Nor does it hold for our quantum A-to-B protocols, in which Alice can ensure acceptance probability $(1 - \epsilon_C)$ even if she only has some way of producing precisely N copies of η (and no more), and no other classical or quantum information correlated with η .

Composability and Accumulation of Information

Zero-knowledge proofs are often used as sub-protocols in the construction of more complex protocols, such as electronic voting schemes and digital signature schemes. One might hope that our knowledge-concealing protocols could be used as building blocks for quantum protocols of this type. One security concern here is composability: our security arguments hold for single-shot instances of the protocol, and collective attacks on combinations of several protocols would need to be addressed separately. Among other things, since the amount of information obtained by Bob in a KCEKQS protocol will never be *exactly* zero for any finite-dimensional quantum state, one would have to consider carefully whether Bob's information gains could accumulate in such a way as to undermine the overall security. We leave further investigation for future work.

Alternative Measures of Bob's information gain

Density matrices characterise an agent's knowledge about the outcomes of measurements that may be made on a system; they do not completely characterise an agent's knowledge of how that system was prepared. Our knowledge-concealing criterion thus does not necessarily characterise all the information Bob might learn in the course of any conceivable protocol. As a hypothetical illustration for the case of a qubit, if Bob were somehow to learn in the course of a protocol that the original state was definitely either the state $|+\rangle$ or the state $|-\rangle$, there is a sense in which he has gained significant information. However, the density matrix ρ describing his new state of knowledge and the density matrix describing his knowledge before the protocol are both the maximally mixed state. Our fidelity-based knowledge-concealment criterion would thus suggest that he has gained no information.

We believe that ϵ_K is nonetheless a good measure of Bob's information gain for a single shot protocol, because it quantifies Bob's ability to predict the outcome of measurements on a system in the state η and/or to prepare a state that successfully simulates the state η . These operationally defined measures of information about a state are relevant in most cryptographic contexts and are the only relevant measures in some.

That said, there is no unique measure of information about a quantum state that characterises every property that is relevant in every possible scenario. Even assuming that Bob's aim is to produce a guess η' at η that optimizes some cost function, and assuming also that the cost function depends only and monotonically on $F(\eta, \eta')$, there are still infinitely many cost functions that may be considered. As a simple example, Bob might be given a fixed reward if and only if $F(\eta, \eta') > 1 - \epsilon$ for some small $\epsilon > 0$. In this case, a sensible parameter might be ϵ'_K , defined as the probability of this condition holding after the protocol if

Bob follows an optimal strategy. A sensible comparison would then be to ϵ'_M , the probability of it holding if Bob instead carries out his optimal strategy that involves operations only on the unknown state.

What does Alice learn?

We have considered in detail the amount of information that Bob gains about the state η in the course of our protocols, but we have not thus far considered how much *Alice* could potentially learn about η if she does not in fact know this state. In some applications it might not matter if Alice gains information about η , but if we are using a KCEKQS protocol precisely because knowledge of the state η has value in some context (such as quantum money[3] or quantum voting[6]), then it may be important to limit the amount of information that might potentially be gained by a dishonest Alice.

Both parties may also be concerned about the information available to an eavesdropper Eve who intercepts protocol communications. In principle, eavesdropping can be completely prevented by using secure classical and quantum channels. Secure classical channels can be ensured by using one time pads. In practice, though, secure quantum channels require distributing and storing perfect entangled states, so that quantum eavesdropping presently remains a practical concern. We will assume secure classical channels but potentially insecure quantum channels below.

The quantum Alice-to-Bob protocols give Alice no additional information. Bob need not tell Alice whether or not his measurement obtained a positive result, so Alice learns nothing whatsoever from the protocol. However, these protocols are vulnerable to quantum eavesdropping, since Eve could intercept the state copies sent by an honest Alice, and obtain information by state estimation.

In our classical protocols, assuming that Bob performs the protocol correctly and honestly, Alice can gain some information about the state since Bob must tell Alice the result of a measurement that he has performed on the system Q_B . But clearly since Alice obtains information about the state only via Bob's classical communication, she learns no more than Bob does in the course of such protocols, if she begins with no knowledge about the state.[18]

Our quantum Bob-to-Alice protocol is significantly vulnerable to a dishonest Alice who wishes to obtain rather than prove knowledge. If Bob performs this protocol honestly he learns nothing whatsoever about η from the protocol. However, a dishonest Alice can store all the quantum states sent to her and make commitments to random indices, or fail to make valid commitments at all. She can then perform a measurement on Q_B after Bob has told her the index of Q_B . Effectively, she can thus steal η from Bob and obtain as much information about it as he could have obtained unaided. For large d , this strategy has a low probability of successfully persuading Bob that Alice has honestly followed the protocol and knew the state. Nonetheless, Alice dishonestly gains some information about η , whereas an honest Bob sacrifices his copy of η and gains no information at all. If Eve intercepts the quantum communications but not the classical communications, she can steal all the states sent by Bob but will not learn which one is η . If N is large, she can thus obtain very little information about η .

For large d , the information a dishonest Alice can gain is small, assuming she begins with no information; for large N , the information Eve can obtain is even smaller. Still, in contexts where even limited information about η is highly valuable, these are undesirable weaknesses.

What if Bob has additional information?

We have calculated bounds on the information that Bob gains during the course of each protocol assuming he starts with strictly zero knowledge about the state of Q_B and only ever has a single copy of the state of Q_B .

In other possible scenarios, Bob might also have some limited classical information about η , or have additional correlated quantum information (for example further copies of η), or both. Our bounds do not necessarily apply in such scenarios. For example, suppose that Bob has Mq copies of η , for some large M , and that he and Alice perform classical relativistic KCEKQS protocol 2. Bob can apply the projective measurement specified by Alice on every copy in his possession. Suppose that Alice is honest, and suppose also that $\epsilon_C \ll (Mq)^{-1}$. Bob is then likely to obtain positive outcomes for r elements of the measurement basis, where $r \leq q$. This allows him to identify a subspace V of dimension r such that $\text{Tr}(P_V \eta) \approx 1$.

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- [11] V. Scarani, S. Iblisdir, N. Gisin, and A. Acín. Quantum cloning. *Reviews of Modern Physics*, 77:1225–1256, October 2005.
- [12] One could also allow protocols that specify nondeterministic actions but do not specify probability distributions, as considered for bit commitment by D’Ariano et al.[5]. We do not consider such protocols here.
- [13] We assume the protocol does not require any states to be discarded, since neither party can trust that the other will in fact discard states as required. Thus $H_A \otimes H_B$ and $H_{A_f} \otimes H_{B_f}$ are isomorphic, although the factors may be different.
- [14] This dummy index prevents cheating strategies in which Bob uses the number of Alice’s commitments, made at the next step, to extract additional information about the state.
- [15] In the ideal error-free case, if Alice knows η precisely and both parties honestly perform the protocol, failure is possible if and only if $q' > q$.
- [16] A tighter bound for $N \gg d$ follows from the normal approximation to the binomial distribution, which gives $p_{\text{mode}} \approx \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{N \frac{1}{d} (1 - \frac{1}{d})}}$.
- [17] One could also include an abort option for the case where Alice does not obtain a positive outcome for any one of these measurements. However, there is a significant distinction. In that case, in the error-free model, Alice knows for certain that Bob is trying to cheat. Since announcing an abort leaves open the possibility that Bob was honest and a statistically unlikely outcome was obtained, she may prefer to directly state that Bob is cheating, or simply stop the protocol.
- [18] Alice can however learn more than Bob does if she begins with some information about the state.