Discrete and free subgroups of SL_2

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration, except as specified in the text. It is not substantially the same as any work that I have submitted, or is being concurrently submitted, for a degree, diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no substantial part of my dissertation has already been submitted, or is being concurrently submitted, for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution.

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Abstract

In this thesis, we study finitely generated subgroups of the matrix group SL_2 (over various locally compact fields) which are both discrete and free.

We first examine the existing literature on two- and three-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$. Some such subgroups are known to be free, and this can be proved by applying a 'combination' theorem (such as Klein's Combination Theorem, or the Ping Pong Lemma) to the action of these groups by Möbius transformations on the Riemann sphere $\hat{\mathbb{C}}$. It remains, however, an open problem to determine freeness of such subgroups in general. On the other hand, applying the Ping Pong Lemma to the action of $\mathrm{SL}_2(\mathbb{R})$ by Möbius transformations on the hyperbolic plane $\mathbb{H}^2$ is known to give necessary and sufficient conditions for a two-generated subgroup of $\mathrm{SL}_2(\mathbb{R})$ to be both discrete (with respect to the topology inherited from $\mathbb{R}^4$) and free of rank two. This forms the basis of an existing practical algorithm which, given a two-generated subgroup $G \leq \mathrm{SL}_2(\mathbb{R})$, determines after finitely many steps whether or not G is both discrete and free of rank two.

We then look at two-generated subgroups of $\mathrm{SL}_2(K)$, where K is a non-archimedean local field (such as the p-adic numbers $\mathbb{Q}_p$). Such groups act by isometries and without inversions on a locally finite regular simplicial tree, called the Bruhat-Tits tree. We demonstrate that applying the Ping Pong Lemma to this action gives a practical algorithm which, given a two-generated subgroup $G \leq \mathrm{SL}_2(K)$, determines after finitely many steps whether or not G is both discrete (with respect to the topology inherited from K^4) and free of rank two. The basis of this algorithm involves computing and comparing various translation lengths.

Finally, we show that similar techniques can be used to give another algorithm which, given a three-generated subgroup $G \leq \mathrm{SL}_2(K)$, determines after finitely many steps whether or not G is both discrete and free of rank three. We demonstrate that both algorithms can be applied more generally in the setting of two- or three-generated subgroups of the isometry group of any locally finite simplicial tree (when equipped with the topology of pointwise convergence, and a method of computing translation lengths) and have relevance to the constructive membership problem.

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Chapter 1

Introduction

1.1 Background

The problem of deciding whether or not two elements of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ generate a free group of rank two has been considered in the literature for many years. One approach to this problem is to study the action of these groups (as subgroups of $\mathrm{GL}_2(\mathbb{C})$) by homeomorphisms on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This action is via Möbius transformations: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$ acts on $z \in \hat{\mathbb{C}}$ by

$$z \mapsto \frac{az+b}{cz+d},$$

where $\infty \mapsto \infty$ if $c = 0$, and $\infty \mapsto \frac{a}{c}$ and $\frac{-d}{c} \mapsto \infty$ otherwise.

It is well-known that two matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ have a common fixed point in $\hat{\mathbb{C}}$ if and only if the trace of the commutator $[A, B] = A^{-1}B^{-1}AB$ is 2; see [8, Theorem 4.3.5(i)]. The proof uses the following trace identity, which is straightforward to verify:

$$\mathrm{tr}([A,B]) = \mathrm{tr}(A)^2 + \mathrm{tr}(B)^2 + \mathrm{tr}(AB)^2 - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB) - 2. \tag{1.1}$$

If $A, B \in \mathrm{SL}_2(\mathbb{C})$ do have a common fixed point in $\hat{\mathbb{C}}$, then (by conjugation) one can assume this point to be ∞, which is also fixed by every element in the subgroup

$G = \langle A, B \rangle \leq \mathrm{SL}_2(\mathbb{C})$ generated by these two matrices. Hence G is conjugate to a group of upper triangular matrices, and is therefore soluble. This shows the following lemma; see also [17, Lemma 3.4 (b)].

Lemma 1.1.1. *Let $A, B \in \mathrm{SL}_2(\mathbb{C})$. If $\mathrm{tr}([A, B]) = 2$, then $G = \langle A, B \rangle \leq \mathrm{SL}_2(\mathbb{C})$ is not free of rank two.*

Demonstrating that a pair of matrices in $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ do generate a free group is a little more complicated. One of the most widely studied examples is the subgroup $F_{\alpha,\beta} \leq \mathrm{SL}_2(\mathbb{C})$ which, given $\alpha, \beta \in \mathbb{C}$, is generated by the matrices

$$A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}.$$

In 1947, Sanov proved that $F_{2,2}$ is free of rank two and, in 1955, Brenner showed that the subgroup $F_{\alpha,\alpha} \leq \mathrm{SL}_2(\mathbb{R})$ is free of rank two whenever $\alpha \in \mathbb{R}$ and $\alpha \geq 2$; see [51] and [10] respectively. Chang, Jennings and Ree observed in 1958 that, if $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ satisfy $\gamma\delta = \alpha\beta \neq 0$, then conjugation gives an isomorphism between $F_{\alpha,\beta}$ and $F_{\gamma,\delta}$. In particular, if $\lambda = \frac{\alpha\beta}{2} \neq 0$, then

$$F_\lambda = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \right\rangle \cong F_{\alpha,\beta},$$

which is free of rank two whenever $\lambda \in \mathbb{C}$ satisfies $|\lambda| \geq 1, |\lambda - 1| \geq 1$ and $|\lambda + 1| \geq 1$; see [13, Theorem 2].

In the late 1960's, Lyndon and Ullman observed in [38, 39] that all these results follow from a theorem of Macbeath (see [40, Theorem 1]). This 'combination' theorem gives set-theoretic conditions for a group to be a free product of certain subgroups, and first appeared in an 1883 paper of Klein (see [35]) in the context of groups of Möbius transformations; see also [18, Chapter II, Theorem 13]. We present below a more general version of this theorem:

Theorem 1.1.2 (Klein's Combination Theorem)**.** *Let G be a group of permutations of a set Ω, and let G_1 and G_2 be non-trivial subgroups of G. Suppose that Ω_1 and Ω_2 are non-empty disjoint subsets of Ω such that, for all $g_1 \in G_1\backslash\{1\}$ and $g_2 \in G_2\backslash\{1\}$,*

$$g_1\Omega_1 \subseteq \Omega_2 \quad \textit{and} \quad g_2\Omega_2 \subseteq \Omega_1.$$

Then the subgroup $H = \langle G_1, G_2\rangle \leq G$ generated by the subgroups G_1 and G_2 is either the free product $G_1 * G_2$, or $|G_1| = |G_2| = 2$ and H is dihedral.

Proof. If $G_1 = \{1, x\}$ and $G_2 = \{1, y\}$, then either there are no relations between x and y (in which case $H \cong C_2 * C_2$), or there are only relations of the form $(xy)^n = 1$ (in which case, if n is minimal, then H is the dihedral group of order $2n$).

Hence we may suppose that $|G_1| \geq 3$. Let $w = g_1 \dots g_k$ be a reduced word in H (that is, each g_i alternately lies in $G_1\backslash\{1\}$ or $G_2\backslash\{1\}$). After conjugating by an appropriate element of G_1, if necessary, we can assume that $g_1, g_k \in G_1\backslash\{1\}$. Then

$$w\Omega_1 \subseteq g_1 \dots g_{k-1}\Omega_2 \subseteq \cdots \subseteq g_1\Omega_1 \subseteq \Omega_2.$$

Since $\Omega_1 \cap \Omega_2 = \varnothing$, this implies that $w \neq 1$ in H. Thus $H \cong G_1 * G_2$; see [37, Proposition 12.2] for further details. $\square$

By considering the action of the matrices $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}$ (with $\alpha \in \mathbb{C}$) on the subsets $\{z \in \mathbb{C} : |z| < 1\}$ and $\{z \in \mathbb{C} : |z| > 1\}$ of $\hat{\mathbb{C}}$, one can use Klein's Combination Theorem to verify the work of Sanov and Brenner, and further show that $F_{\alpha,\alpha}$ is free whenever $|\alpha| \geq 2$. This method leads to an alternative proof of the theorem of Chang, Jennings and Ree, and also gives further values of $\lambda \in \mathbb{C}$ for which F_λ is free; for instance, see [24, 25, 39]. In fact, the values of λ for which F_λ is free are dense in $\mathbb{C}$; see [13, Theorem 3].

On the other hand, there are many values of $\lambda \in \mathbb{C}$ for which F_λ is not free. (Note that Lemma 1.1.1 is not applicable here, since the commutator of the generators has trace $2 + 4\lambda^2 \neq 2$.) Chang, Jennings and Ree constructed infinitely many such values of λ in Theorem 4 of [13], and Ree proved in Corollary 1 of [47] that the values of λ for which F_λ is not free are dense in the disk $\{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$. This is done by finding matrices in F_λ which can be diagonalised to have roots of unity as the diagonal entries. Alternatively, if some matrix $C \in F_{\alpha,\beta} \backslash \langle B \rangle$ is lower triangular (where A and B denote the generators of $F_{\alpha,\beta} \cong F_\lambda$ as above), then $C^{-1}BC$ and B commute, and hence $F_{\alpha,\beta}$ is not free. This method is used in [7] and [39] to find additional values of α and β such that $F_{\alpha,\beta}$ is not free: for instance, if $\alpha = \beta = \frac{p}{q}$, where p and q are integer solutions to Pell's equation $p^2 - Nq^2 = 1$ (with N being some non-square positive integer), then $A^{q^2}B^NA^{-1}$ is lower triangular. Further values of $\lambda \in \mathbb{C}$ for which F_λ is not free were given in [6, 11, 21, 25–27] and, more recently, in [33].

Despite this progress, there remain many values of $\alpha, \beta \in \mathbb{C}$ for which it is unknown whether or not $F_{\alpha,\beta}$ is free of rank two. For instance, it is still an open question to decide if $F_{\alpha,\alpha} \leq \mathrm{SL}_2(\mathbb{R})$ is not free for every rational number $\alpha \in (-2, 2)$; see [32, Problem 15.83]. Bearing in mind that this is just one particular class of subgroups, it seems a very difficult problem, in general, to determine freeness of two-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$.

This open problem also extends to subgroups of higher rank. Some three-generated subgroups of $\mathrm{SL}_2(\mathbb{C})$ are known to be free: in 1976, Bachmuth and Mochizuki used a variant of Klein's Combination Theorem to prove that, for $\alpha, \beta, \gamma \in \mathbb{C}$, the subgroup

$$F_{\alpha,\beta,\gamma} = \left\langle \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}, \begin{bmatrix} 1-\gamma & -\gamma \\ \gamma & 1+\gamma \end{bmatrix} \right\rangle$$

of $\mathrm{SL}_2(\mathbb{C})$ is free of rank three whenever $|\alpha|, |\beta|, |\gamma| \geq 4.45$, and is not contained in any known (at that time) free subgroup of rank two; see [5]. This was strengthened by Merzljakov in 1978, who showed that $F_{\alpha,\beta,\gamma}$ is also free when $|\alpha|, |\beta|, |\gamma| \geq 3$; see [41]. This agrees with the work of Scharlemann, who proved in 1979 (using similar methods) that $F_{\alpha,\beta,\gamma} \leq \mathrm{SL}_2(\mathbb{R})$ is free of rank three whenever $\alpha, \beta, \gamma \in \mathbb{R}$ are such that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \leq 1$; see [52, Theorem 2.2]. It is, however, an open question to decide whether or not there are rational numbers satisfying $|\alpha|, |\beta|, |\gamma| < 3$ for which $F_{\alpha,\beta,\gamma}$ is free of rank three; see [32, Problem 15.84]. Besides this class of examples, it does not appear that many other subgroups of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ are known to be free on three or more generators.

Restricting to the real case, a much simpler problem is to determine whether or not a given two-generated subgroup of $\mathrm{SL}_2(\mathbb{R})$ is both discrete (with respect to the topology inherited from $\mathbb{R}^4$) and free of rank two. As first observed by Newman in 1968, many examples of two-generated free subgroups of $\mathrm{SL}_2(\mathbb{R})$ are also discrete; see [43]. As noted by Lyndon and Ullman in [38], this can be seen by applying a particular variant of Klein's Combination Theorem (known as the Ping Pong Lemma) to the action of $\mathrm{SL}_2(\mathbb{R})$ by Möbius transformations on the hyperbolic plane $\mathbb{H}^2$.

In 1972, Purzitsky and Rosenberger each used this idea to give necessary and sufficient conditions, depending on matrix trace, for any two elements of $\mathrm{PSL}_2(\mathbb{R})$ to generate a discrete and free group; see [45, Section 4] and [48, Satz 1]. It is observed in both papers that these conditions can be checked systematically by using a sequence of 'trace minimising' Nielsen transformations on the generators of G. In 2014, Eick, Kirschmer and Leedham-Green formalised this by giving a practical algorithm that takes as input a two-generated subgroup G of $\mathrm{SL}_2(\mathbb{R})$ (or, equivalently, of $\mathrm{PSL}_2(\mathbb{R})$) and determines after finitely many steps whether or not G is both discrete and free of rank two; see [17, Algorithm 2]. This algorithm can be used to solve the constructive membership problem for discrete and free two-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{R})$: given such a subgroup G, and an element h in the corresponding overgroup, one can determine algorithmically whether or not h is an element of G and, if it is, give an explicit expression of h as a word in the generators of G.

Determining whether or not a given two-generated subgroup of $\mathrm{SL}_2(\mathbb{C})$ is both discrete and free of rank two is a much harder problem. It is known that the subgroup F_λ is both discrete and free for every λ in a subset of $\mathbb{C}$ known as the *Riley slice*; see [30]. However, to construct analogues of the results in [17, 45, 48] would likely involve studying the action of $\mathrm{SL}_2(\mathbb{C})$ by extended Möbius transformations on threedimensional hyperbolic space $\mathbb{H}^3$, and this is much more intricate than the real case; see [8, Section 4.1] for details. Some progress has been made in this area (see [9], for instance), but we will not discuss this here.

It also seems a very difficult problem to determine whether or not a given subgroup of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ is both discrete and free on at least three generators: there is very little existing literature (if any) on such subgroups. Similarly, the study of discrete and free subgroups of SL_2 over infinite fields other than $\mathbb{R}$ or $\mathbb{C}$ is not at all prominent in the literature. In this thesis, we will consider subgroups of $\mathrm{SL}_2(K)$, where K is a non-archimedean local field (for instance, the p-adic numbers $\mathbb{Q}_p$), and show that a method exists to determine whether or not a two- or three-generated subgroup of $\mathrm{SL}_2(K)$ is both discrete and free.

The group $\mathrm{SL}_2(K)$ acts continuously by isometries and without inversions on a locally finite regular simplicial tree (called the Bruhat-Tits tree), and applying the Ping Pong Lemma to this action yields an analogue of the discrete and free algorithm for two-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$. Namely, given a two-generated subgroup G of $\mathrm{SL}_2(K)$ (or, equivalently, of $\mathrm{PSL}_2(K)$), we show that 'translation length minimising' Nielsen transformations can be performed on the generators of G in order to determine after finitely many steps whether or not G is both discrete and free of rank two. This method, introduced by the author in [14], also gives rise to algorithms deciding whether or not three-generated subgroups of $\mathrm{SL}_2(K)$, or two- or three-generated subgroups of the isometry group of any locally finite simplicial tree (when equipped with an appropriate topology and a method of computing translation lengths) are both discrete and free. All of these algorithms have applications to the constructive membership problem.

1.2 Chapter summary

In Chapter 2, we present an original version of the Ping Pong Lemma and discuss how it can be applied to the action of $\mathrm{SL}_2(\mathbb{R})$ on the hyperbolic plane $\mathbb{H}^2$. We show that our version of the Ping Pong Lemma can be used to reconstruct necessary and sufficient conditions (given in [45] and [48], in the context of two-generated subgroups of $\mathrm{PSL}_2(\mathbb{R})$) for a two-generated subgroup of $\mathrm{SL}_2(\mathbb{R})$ to be both discrete and free of rank two, and we present the resulting algorithm from [17].

In Chapter 3, we give an overview of local fields and describe the action of the group $\mathrm{SL}_2(K)$ on the Bruhat-Tits tree. We discuss aspects of groups acting by isometries and without inversions on a simplicial tree. In particular, we use a theorem of Morgan and Shalen (see [42, Proposition II.3.15]) to classify elements of $\mathrm{SL}_2(K)$ based on their translation length. We also prove some important translation length formulae, one of which provides a correction to a theorem of Paulin (see [44, Proposition 1.6]), and we prove that a discrete and free subgroup of $\mathrm{SL}_2(K)$ cannot contain any elliptic isometries of the Bruhat-Tits tree. At the end of the chapter, we show how translation length can be used to determine if two hyperbolic elements of $\mathrm{SL}_2(K)$ satisfy the hypotheses of the Ping Pong Lemma. This gives rise to an algorithm which determines after finitely many steps whether or not any given two-generated subgroup of $\mathrm{SL}_2(K)$ is both discrete and free of rank two. We discuss the implementation of this algorithm and give some examples which compare and contrast it with the algorithm from [17].

In Chapter 4, we generalise the methods used in Chapter 3 and show how translation length can be used to determine if three hyperbolic elements of $\mathrm{SL}_2(K)$ satisfy the hypotheses of the Ping Pong Lemma. This leads to an algorithm which determines after finitely many steps whether or not a given three-generated subgroup of $\mathrm{SL}_2(K)$ is both discrete and free of rank three. The algorithm gives a constructive method of deciding between the two outcomes of a theorem of Weidmann (see [55, Theorem 7]).

Finally, in Chapter 5, we demonstrate that both these algorithms generalise to two- or three-generated subgroups of the isometry group of a locally finite simplicial tree, equipped with the topology of pointwise convergence (which, in this setting, is equivalent to the compact-open topology) and a method of computing translation lengths. Given a subgroup which is verified by any of these algorithms to be both discrete and free, we discuss how the constructive membership problem can be solved.

Chapter 2

Discrete and free two-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$

In this chapter, we summarise some existing theory of discrete and free two-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$. We show that any two-generated subgroup of $\mathrm{SL}_2(\mathbb{R})$ is discrete and free if and only if the corresponding subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is, and we present necessary and sufficient conditions (in the form of [45] and [48]) for such subgroups to be both discrete and free of rank two. Sufficiency of these conditions can be shown directly by applying the Ping Pong Lemma to the action of these groups on the hyperbolic plane $\mathbb{H}^2$. We give an original version of the Ping Pong Lemma that we use throughout this thesis, and summarise the practical algorithm from [17] which uses these conditions to determine whether or not a given two-generated subgroup of $\mathrm{SL}_2(\mathbb{R})$ is both discrete and free of rank two.

2.1 The Ping Pong Lemma

Recall that $\mathrm{SL}_2(\mathbb{R})$ acts by homeomorphisms on the Riemann sphere $\hat{\mathbb{C}}$. Restricting to the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ (or, equivalently, the hyperbolic plane $\mathbb{H}^2$) gives an action by isometries. It is well-known that elements of $\mathrm{SL}_2(\mathbb{R})$ can be classified by the number of fixed points of this action on the boundary $\partial\mathbb{H}^2 \cong \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and this corresponds to the trace of each matrix:

Definition 2.1.1. A matrix $\pm I_2 \neq A \in \mathrm{SL}_2(\mathbb{R})$ is said to be

- *elliptic* if $|\mathrm{tr}(A)| < 2$;
- *parabolic* if $|\mathrm{tr}(A)| = 2$;
- *hyperbolic* if $|\mathrm{tr}(A)| > 2$.

Elliptic matrices fix no point of the boundary $\partial\mathbb{H}^2 \cong \hat{\mathbb{R}}$, and are conjugate to rotation matrices of the form $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ for some angle θ. On the other hand, parabolic and hyperbolic elements respectively fix one and two points of the boundary $\partial\mathbb{H}^2 \cong \hat{\mathbb{R}}$; see [29, Section 2.1] for further details.

Recall that a *topological group* is a group equipped with a topology with respect to which the inversion and multiplication maps are continuous. The group $\mathrm{SL}_2(\mathbb{R})$, viewed as a subset of $\mathbb{R}^4$, is a topological group via the subspace topology. Similarly, $\mathrm{PSL}_2(\mathbb{R})$ is a topological group, with the quotient topology inherited from $\mathrm{SL}_2(\mathbb{R})$.

A topological group is said to be *discrete* if the corresponding topology is discrete. Given a topological group G, and any $y \in G$, the map $x \mapsto xy$ is a homeomorphism from G to itself. To determine discreteness of G, it therefore suffices to check that the singleton set $\{1\}$ is open. Hence any metrisable topological group G (in particular, this includes $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R})$) is discrete if and only if any sequence of elements in G converging to the identity is eventually constant. This observation leads to the following lemma:

Lemma 2.1.2. *If $G \leq \mathrm{SL}_2(\mathbb{R})$ is discrete and free, then it contains no elliptic elements.*

Proof. Suppose that $X \in G$ is elliptic. If X has finite order, then G is not free, so suppose that X has infinite order. Since X is conjugate to a rotation matrix, it follows that G is not discrete. See also [29, Theorem 2.2.3]. □

On the other hand, as mentioned in the introduction, a variant of Klein's Combination Theorem (known as the Ping Pong Lemma) can be used to show that certain subgroups of $\mathrm{SL}_2(\mathbb{R})$ are both discrete and free. As stated below, our original version of this lemma applies only to metrisable topological groups acting continuously on a topological space. This makes it more specialised than other variants of Klein's Combination Theorem (see [17, 38, 40], for instance), but it differs from these by additionally proving discreteness.

By a *continuous* action of a topological group G on a topological space X, we will mean that the map $G \times X \to X$ (given by $(g, x) \mapsto gx$) is continuous with respect to the product topology. For example, the action of the group $\mathrm{SL}_2(\mathbb{R})$ (and also that of $\mathrm{PSL}_2(\mathbb{R})$) on the hyperbolic plane $\mathbb{H}^2$ by Möbius transformations is given by polynomials and is therefore continuous.

Lemma 2.1.3 (The Ping Pong Lemma)**.** *Let G be a metrisable topological group acting continuously on a topological space X, and let $g_1, \ldots, g_n \in G \backslash \{1\}$. Suppose that $X_1^+, X_1^-, \ldots, X_n^+, X_n^-$ are non-empty, closed and pairwise disjoint subsets of X, which do not cover X and for all $1 \leq i \leq n$ satisfy*

$$g_i(X \backslash X_i^-) \subseteq X_i^+ \quad \textit{and} \quad g_i^{-1}(X \backslash X_i^+) \subseteq X_i^-.$$

Then the subgroup $H = \langle g_1, \ldots, g_n \rangle \leq G$ is both discrete and free of rank n.

Proof. We first fix some $x \in D = X \backslash (X_1^+ \cup X_1^- \cup \cdots \cup X_n^+ \cup X_n^-) \neq \varnothing$.

To show freeness, suppose that $w \in H$ is a non-trivial word in $g_1, \ldots, g_n$. Then $w(x) \in X \backslash D$. In particular, $w \neq 1$ in H and so H is free of rank n. Note that if $n = 2$, then freeness also follows from Theorem 1.1.2 by setting $G_1 = \langle g_1 \rangle, G_2 = \langle g_2 \rangle$, $\Omega_1 = X_2^- \cup X_2^+$ and $\Omega_2 = X_1^- \cup X_1^+$.

On the other hand, suppose that H is not discrete. Then one can find a sequence $(h_n)_{n \in \mathbb{N}}$ of non-identity elements of H which converges to $1 \in H$. Since $h_n(x) \in X \backslash D$ for each $n \in \mathbb{N}$, and G acts continuously on X, this gives a sequence $(h_n(x))_{n \in \mathbb{N}}$ of elements of $X \backslash D$ which converges to $x \in D$. But $X \backslash D$ is closed, so this is impossible. Thus H is discrete and free of rank n. (See Figure 2.1 for an example when $n = 2$.) $\square$

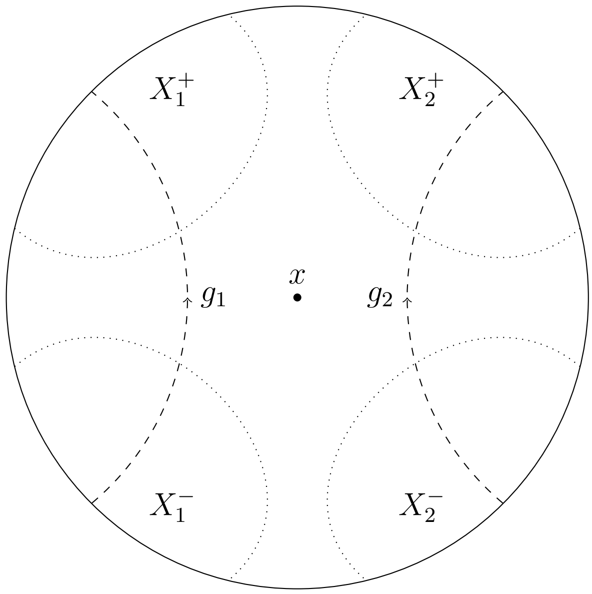

Figure 2.1: The Ping Pong Lemma ($n = 2$).

We conclude this section by noting that the problem of determining whether or not a finitely generated subgroup of $\mathrm{SL}_2(\mathbb{R})$ is both discrete and free is equivalent to the same problem for the corresponding subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

Proposition 2.1.4. *Let $G \leq \mathrm{SL}_2(\mathbb{R})$ be n-generated. Then G is both discrete and free of rank n if and only if the corresponding subgroup $\overline{G} \leq \mathrm{PSL}_2(\mathbb{R})$ (its image under the quotient map) is both discrete and free of rank n.*

Proof. By the remarks preceding Lemma 2.1.2, G is discrete if and only if $\overline{G}$ is. On the other hand, if either G or $\overline{G}$ is free of rank n, then the quotient map $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ restricts to an isomorphism $G \cong \overline{G}$; see [17, Lemma 4.1] for further details when $n = 2$. $\square$

2.2 Deciding whether a two-generated subgroup of $\mathrm{SL}_2(\mathbb{R})$ is discrete and free

We now discuss the work of Purzitsky and Rosenberger from [45] and [48], and explain how this leads to the practical algorithm from [17] which determines whether or not a two-generated subgroup $G \leq \mathrm{SL}_2(\mathbb{R})$ is both discrete and free of rank two. The key idea is to apply certain Nielsen transformations (defined below) to the generators of G. Nielsen transformations arise naturally in the study of free groups: given any two distinct generating sets of a free group of finite rank, there is a Nielsen transformation between them; see [37, Chapter I, Proposition 4.1].

Definition 2.2.1. Given n elements $(g_1, \ldots, g_n)$ of a group, a *Nielsen transformation* is some finite sequence of the following operations:

- Swap g_i and g_j (for $i \neq j$);
- Replace g_i by g_i^{-1};
- Replace g_i by $g_j^{-1} g_i$ (for $i \neq j$).

Note that Nielsen transformations preserve generation of the subgroup generated by $g_1, \ldots, g_n$. Since any pair of matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ satisfies Equation (1.1) and the well-known trace identity

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B), \tag{2.1}$$

it follows that applying Nielsen transformations to a pair of matrices $A, B \in \mathrm{SL}_2(\mathbb{R})$ also preserves the trace of the commutator $\mathrm{tr}([A, B])$.

It is observed in [45] and [48] that, given a two-generated subgroup $G \leq \mathrm{PSL}_2(\mathbb{R})$, Nielsen transformations can be performed on the generators of G in a ‘trace minimising’ manner in order to determine whether or not G is both discrete and free of rank two. This idea of systematically reducing trace via Nielsen transformations also appears in [20, 22, 34, 46, 50], in the context of algorithms for determining discreteness of two-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R})$.

The following is a slight reformulation of necessary and sufficient conditions, given in Section 4 of [45] and Satz 1 of [48], for a two-generated subgroup of $\mathrm{PSL}_2(\mathbb{R})$ to be both discrete and free of rank two. In both papers, it is shown that condition (i) can be tested by performing these 'trace minimising' Nielsen transformations.

Theorem 2.2.2. *Let $A, B \in \mathrm{SL}_2(\mathbb{R})$. Then $G = \langle A, B\rangle \leq \mathrm{SL}_2(\mathbb{R})$ is discrete and free of rank two if and only if one of the following holds:*

(i) $\mathrm{tr}([A,B]) > 2$*, and there exist* $X, Y \in \mathrm{SL}_2(\mathbb{R})$ *(whose images* $\overline{X}, \overline{Y} \in \mathrm{PSL}_2(\mathbb{R})$ *generate* $\overline{G} \leq \mathrm{PSL}_2(\mathbb{R})$*) which satisfy* $\mathrm{tr}(X), \mathrm{tr}(Y) \geq 2$ *and* $\mathrm{tr}(X^{-1}Y) \leq -2$*;*

(ii) $\mathrm{tr}([A,B]) \leq -2$.

Proof. First note that, by Proposition 2.1.4, G is discrete and free if and only if $\overline{G} = \langle \overline{A}, \overline{B}\rangle \leq \mathrm{PSL}_2(\mathbb{R})$ is. Moreover, it follows from Lemma 4.3 of [17] that, for any $X, Y \in \mathrm{SL}_2(\mathbb{R})$ whose images $\overline{X}, \overline{Y} \in \mathrm{PSL}_2(\mathbb{R})$ generate $\overline{G}$, we have $\mathrm{tr}([A,B]) = \mathrm{tr}([X,Y])$. Hence these conditions follow directly from those for discrete and free two-generated subgroups of $\mathrm{PSL}_2(\mathbb{R})$ in Section 4 of [45] and Satz 1 of [48].

For a proof of (i) in the setting of $\mathrm{PSL}_2(\mathbb{R})$, see [45, Theorems 2-7] or [48, Satz 1(i)]. For a proof of (ii) in the setting of $\mathrm{PSL}_2(\mathbb{R})$, see [48, Satz 1(ii)] or, after observing from Equation (1.1) that A and B must both be hyperbolic of positive trace, see [45, Theorem 8]. Note also that sufficiency of these conditions can be proved more directly by applying the Ping Pong Lemma to the subsets of $\mathbb{H}^2$ constructed in Sections 5.1 and 5.2 of [17]. □

We now present Algorithm 2 of [17], slightly adapting the steps so that they better align with the algorithms we introduce in future chapters. For completeness, we also include a proof that the algorithm is correct and terminates after finitely many steps; see also [17, Theorem 4.6], and the ideas used in [31, 49].

Algorithm 2.2.3. *Given $A, B \in \mathrm{SL}_2(\mathbb{R})$, we proceed as follows: If the subgroup $G = \langle A, B\rangle \leq \mathrm{SL}_2(\mathbb{R})$ is discrete and free of rank two, then the algorithm will return true and output representatives in $\mathrm{SL}_2(\mathbb{R})$ of a generating pair for $\overline{G} \leq \mathrm{PSL}_2(\mathbb{R})$ which satisfies the hypotheses of the Ping Pong Lemma, and otherwise it will return false.*

(1) *Set $X = A$ and $Y = B$. If $|\mathrm{tr}(X)| < 2$, $|\mathrm{tr}(Y)| < 2$ or $\mathrm{tr}([X, Y]) \in (-2, 2]$, then return false.*

(2) *If $\mathrm{tr}([X, Y]) \leq -2$, then return true and the pair (X, Y).*

(3) *If $\mathrm{tr}(X) < 0$, then replace X by $-X$. If $\mathrm{tr}(Y) < 0$, then replace Y by $-Y$.*

(4) *If $\mathrm{tr}(X) > \mathrm{tr}(Y)$, then swap X and Y.*

(5) *Compute $m = \min\{\mathrm{tr}(XY), \mathrm{tr}(X^{-1}Y)\}$. If $|m| < 2$, then return false.*

(6) *If $m \geq 2$, then replace Y by the element from $\{XY, X^{-1}Y\}$ which has trace m and go back to (4).*

(7) *If $m < \mathrm{tr}(X^{-1}Y)$, then replace X by X^{-1}.*

(8) *Return true and the pair (X, Y).*

Theorem 2.2.4. *Algorithm 2.2.3 terminates after finitely many steps and produces the correct output.*

Proof. If step (1) returns false, then G is not both discrete and free by either Lemma 1.1.1 or Lemma 2.1.2. If step (2) returns true, then G is both discrete and free of rank two by Theorem 2.2.2 (*ii*). Moreover, the images $\overline{A}, \overline{B} \in \mathrm{PSL}_2(\mathbb{R})$ generate $\overline{G}$ and they satisfy the hypotheses of the Ping Pong Lemma; see [17, Section 5.1] for the relevant subsets of $\mathbb{H}^2$. The replacements in step (3) preserve both generation of $\overline{G}$ and the equality $\mathrm{tr}([A, B]) = \mathrm{tr}([X, Y])$. Hence if the algorithm reaches step (5), then we must have $2 \leq \mathrm{tr}(X) \leq \mathrm{tr}(Y)$ and $\mathrm{tr}([X, Y]) = \mathrm{tr}([A, B]) > 2$.

If step (5) returns false, then G is not both discrete and free by Lemma 2.1.2. Otherwise we continue through steps (6) and (7), performing Nielsen transformations which preserve both the equality $\mathrm{tr}([A,B]) = \mathrm{tr}([X,Y])$ and generation of $\overline{G}$ by the images $\overline{X}, \overline{Y} \in \mathrm{PSL}_2(\mathbb{R})$. Finally, once step (8) is reached, we necessarily have $m = \mathrm{tr}(X^{-1}Y) \leq -2$. Thus G is discrete and free of rank two by Theorem 2.2.2 (i). Moreover, the elements $\overline{X}, \overline{Y} \in \mathrm{PSL}_2(\mathbb{R})$ satisfy the hypotheses of the Ping Pong Lemma; see [17, Section 5.2] for the relevant subsets of $\mathbb{H}^2$.

To prove that the algorithm terminates after finitely many steps, we consider the trace triples $(x,y,z) = (\mathrm{tr}(X), \mathrm{tr}(Y), \mathrm{tr}(XY))$, and show that the sequence of triples obtained by performing steps (4) – (6) cannot continue indefinitely. We start by assuming (swapping X and X^{-1} if necessary) that $2 \leq \mathrm{tr}(X^{-1}Y) \leq \mathrm{tr}(XY)$. Using Equation (2.1), this implies $z \geq \frac{xy}{2}$ and $2 \leq x \leq y \leq z$. If $y \leq xy - z$, then viewing Equation (1.1) as a quadratic in $z = \mathrm{tr}(XY)$ gives

$$y \leq xy - z = \frac{xy}{2} - \sqrt{\frac{x^2y^2}{4} - x^2 - y^2 + 2 + \mathrm{tr}([X,Y])}.$$

Rearranging and squaring gives $y^2(x-2) \leq x^2 - 2 - \mathrm{tr}([X,Y]) < x^2 - 4$, which implies that $x^2 \leq y^2 < x + 2$. This is a contradiction because $x \geq 2$. Hence we must have $y > xy - z$, that is, $\mathrm{tr}(X^{-1}Y) < \mathrm{tr}(Y)$. Since step (6) replaces the triple (x,y,z) with $(x, xy-z, y)$, after returning to and performing step (4), one obtains a component-wise decreasing sequence (x_n, y_n, z_n) of trace triples for which $2 \leq x_n \leq y_n \leq z_n$ for each $n \in \mathbb{N}$.

If this sequence were to continue indefinitely, then each component would converge to some real number - say to x_0, y_0 and z_0, respectively. It follows from observing the replacements in step (6), and taking limits, that $y_0 = z_0$ and $x_0 + y_0 = x_0y_0 - z_0 + x_0$, which implies that $x_0 = 2$. Since Equation (1.1) is also satisfied by each triple, we get

$$4 < 2 + \mathrm{tr}([X,Y]) = x_0^2 + y_0^2 + z_0^2 - x_0y_0z_0 = 4,$$

which is a contradiction. Thus the algorithm must eventually terminate. $\square$

As an example to illustrate Algorithm 2.2.3, we verify the theorem of Sanov (from [51]) mentioned in the introduction. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

so that $\mathrm{tr}(A) = \mathrm{tr}(B) = 2$, $\mathrm{tr}([A, B]) = 18$ and $-2 = \mathrm{tr}(A^{-1}B) < \mathrm{tr}(AB) = 6$. Then the algorithm will return true and the pair (A, B) at step (8). On the other hand,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

do not generate a discrete and free subgroup of $\mathrm{SL}_2(\mathbb{R})$. In this case, $\mathrm{tr}(A) = \mathrm{tr}(B) = 2$, $\mathrm{tr}([A, B]) = 3$ and $1 = \mathrm{tr}(A^{-1}B) < \mathrm{tr}(AB) = 3$, so these elements return false at step (5) of the algorithm.

We conclude this chapter by noting that Algorithm 2.2.3 has been implemented in the software package MAGMA for pairs of matrices in SL_2 over any subfield of $\mathbb{R}$ where, for each element x, it is computationally possible to test whether $x > 0$ (for instance, finite extensions of $\mathbb{Q}$); see [17, Section 6] for further details. The algorithm can also be applied to two-generated subgroups of $\mathrm{PSL}_2(\mathbb{R})$: by Proposition 2.1.4, one can run the algorithm for any representatives of the generators in $\mathrm{SL}_2(\mathbb{R})$ to determine whether or not the given subgroup is both discrete and free of rank two.

Chapter 3

Discrete and free two-generated subgroups of $\mathrm{SL}_2(K)$

In this chapter, we define the class of local fields and summarise some properties of non-archimedean local fields. Given a non-archimedean local field K, we describe the action of $\mathrm{SL}_2(K)$ by isometries and without inversions on the corresponding Bruhat-Tits tree. Isometries that act without inversions on a simplicial tree are very well understood: in particular, they can be classified as either elliptic or hyperbolic, based upon their translation length. We present a key theorem of Morgan and Shalen (see [42, Proposition II.3.15]) which relates the translation length of matrices in $\mathrm{SL}_2(K)$ to their trace, and we show that discrete and free subgroups of $\mathrm{SL}_2(K)$ cannot contain any elliptic elements.

We also prove some important formulae for the translation length of the product and commutator of two hyperbolic elements. The product formulae provide a correction to those given by Paulin in Proposition 1.6 of [44] and, combined with the Ping Pong Lemma, they give rise to a simple condition for two hyperbolic elements of $\mathrm{SL}_2(K)$ to generate a discrete and free group of rank two. This forms the basis of a practical algorithm which, given a two-generated subgroup $G \leq \mathrm{SL}_2(K)$, uses 'translation length minimising' Nielsen transformations on the generators of G in order to determine after finitely many steps whether or not G is both discrete and free of rank two; see

[14, Algorithm 4.1]. We discuss the implementation of this algorithm, and give some examples which compare and contrast it to the discrete and free algorithm in [17].

3.1 Local fields and the Bruhat-Tits tree

Recall that an *absolute value* on a field K is a function $|-|: K \to \mathbb{R}$ such that

(1) $|x| \geq 0$,

(2) $|x| = 0$ if and only if $x = 0$,

(3) $|xy| = |x||y|$, and

(4) $|x + y| \leq |x| + |y|$

for all $x, y \in K$. For example, the *trivial absolute value* is given by $|x| = 1$ for all $x \in K^\times$. Defining a distance function $d(x, y) = |x - y|$ for all $x, y \in K$ gives K the structure of a metric (and hence topological) space, so one can associate various topological properties to K.

Definition 3.1.1. A *local field* is a field K which is locally compact with respect to some non-trivial absolute value $|-|$. Such a field K is said to be *non-archimedean* if the corresponding absolute value $|-|$ is non-archimedean, meaning it satisfies the *ultrametric inequality*

$$|a + b| \leq \max\{|a|, |b|\}$$

for all $a, b \in K$. Otherwise, K is said to be *archimedean.*

The ultrametric inequality is a strengthened version of the triangle inequality (see condition (4) above), and it is known that equality holds whenever $|a| \neq |b|$; see [12, Chapter 2, Lemma 1.4]. Moreover, every archimedean local field is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$, with the same topology as the one induced by the standard absolute values; see [12, Chapter 3, Theorem 1.1]. Hence we restrict our interest to non-archimedean local fields, which have an equivalent characterisation in terms of discrete valuations.

Definition 3.1.2. A *valuation* on a field K is a group homomorphism $v\colon K^\times \to \mathbb{R}$ such that, when extended by defining $v(0) = \infty$, the *ultrametric inequality*

$$v(x+y) \geq \min\{v(x), v(y)\}$$

holds for all $x, y \in K$. Additionally, we say that v is *discrete* if $v(K^\times) \cong \mathbb{Z}$.

Given any valuation v on a field K, the *ring of integers* $\mathcal{O} = \{x \in K : v(x) \geq 0\}$ is a principal ideal domain with unique maximal ideal $\mathcal{P} = \{x \in K : v(x) > 0\}$. The quotient $k = \mathcal{O}/\mathcal{P}$ is called the *residue field* of K. Furthermore, setting $|x|_v = c^{-v(x)}$ for some $c \in (1, \infty)$ defines a non-archimedean absolute value on K. Any field K with discrete valuation v which is complete with respect to $|-|_v$ and has finite residue field k is a non-archimedean local field. The converse also holds, giving two equivalent definitions of a non-archimedean local field; see [12, Chapter 4] for further details.

For a non-archimedean local field K, the maximal ideal $\mathcal{P}$ is generated by a *uniformiser* $\pi \in \mathcal{O}$ (that is, any element of K with $v(\pi) = 1$), and hence the residue field k is of the form $\mathcal{O}/\pi\mathcal{O}$. For a fixed finite set S of coset representatives of $\pi\mathcal{O}$ in $\mathcal{O}$, every $a \in K^\times$ can be uniquely expressed as a sum

$$a = \sum_{i=N}^{\infty} a_i \pi^i \tag{3.1}$$

for some integer $N = v(a)$ such that $a_N \neq 0$, and with $a_i \in S$ for all $i \geq N$; see [12, Chapter 4, Lemma 1.4]. Note also that non-archimedean local fields satisfy the *Bolzano-Weierstrass property*: every bounded sequence (in terms of the corresponding absolute value) has a convergent subsequence.

Example 3.1.3. An important example of a non-archimedean local field is the field of p-adic numbers, defined using the *p-adic valuation* v_p on $\mathbb{Q}$: if p is a prime and $x \in \mathbb{Q}$ is of the form $p^r \frac{a}{b}$ with $p \nmid a, b$, then $v_p(x) = r$. The corresponding absolute value is usually defined to be $|x|_p = p^{-r}$, and the *p-adic numbers* $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ with respect to $|-|_p$. In fact, every non-archimedean local field is isomorphic to either a finite extension of $\mathbb{Q}_p$, or the field of formal Laurent series $\mathbb{F}_{p^r}((t))$, for some prime p and positive integer r; see [12, Exercise 25 of Chapter 4 and Lemma 1.1 of Chapter 8].

For the remainder of this thesis, we let K be a non-archimedean local field with discrete valuation v. We denote its ring of integers by $\mathcal{O} = \{x \in K : v(x) \geq 0\}$ and its residue field by $k = \mathcal{O}/\pi\mathcal{O}$, for some fixed uniformiser π. Given such a field K, there is a $(|k|+1)$-regular (and hence locally finite) simplicial tree T_v, known as the *Bruhat-Tits tree*, upon which the group $\mathrm{GL}_2(K)$ acts. The vertices of T_v are equivalence classes of free $\mathcal{O}$-modules of rank two (called *lattices*), where lattices L and L' are *equivalent* if $L = xL'$ for some $x \in K^\times$. Furthermore, given a lattice L, each equivalence class of lattices has a unique representative $L_0 \subseteq L$ for which L/L_0 is isomorphic (as an $\mathcal{O}$-module) to $\mathcal{O}/\pi^n\mathcal{O}$ for some $n \in \mathbb{Z}_{\geq 0}$. This gives rise to the edge structure of T_v: there is an edge between the vertices represented by L and L_0 if and only if $n = 1$. For further details, see [53, Chapter II, Section 1].

Note that $\mathrm{GL}_2(K)$ (and hence $\mathrm{SL}_2(K)$) inherits the structure of a metrisable topological group from K^4.

Proposition 3.1.4. *The group* $\mathrm{SL}_2(K)$ *acts continuously by isometries and without inversions on the Bruhat-Tits tree* T_v*.*

Proof. There is a natural action of $\mathrm{GL}_2(K)$ on the set of lattices, and this gives rise to an isometric action of $\mathrm{GL}_2(K)$ on the tree T_v (where we use the standard path metric on T_v). This action is given by polynomials and is therefore continuous (see also the discussion in Section 5.1), so it remains to show that $\mathrm{SL}_2(K)$ acts on T_v without inversions. Note that $\mathrm{GL}_2(K)$ acts with inversions on T_v (see [53, Chapter II, Section 1.3]), so this does not immediately follow. We outline the proof of Corollary II.3.14 of [42]; see also the corollary to Proposition 1 of [53, Chapter II, Section 1].

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(K)$, choose an entry α with minimal valuation. Since $ad - bc = 1$, the ultrametric inequality implies that $v(\alpha) \leq 0$. Perform the elementary row (respectively column) operation that adds an appropriate multiple of the row (respectively column) containing α to the other row (respectively column), in order to clear out the rest of the column (respectively row) containing α. By minimality of $v(\alpha)$, this process gives matrices $B, C \in \mathrm{SL}_2(\mathcal{O})$ for which $A = BMC$, where M is (without loss of generality) of the form $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$. Note that $d(p, Ap) = d(B^{-1}p, MCp) = d(p, Mp) = -2v(\alpha)$, where d denotes the standard path metric on T_v and p is the vertex of T_v representing the standard lattice $\mathcal{O}^2$, which is preserved under the action of $\mathrm{SL}_2(\mathcal{O})$. Thus

$$d(p, Ap) = -2\min\{v(a), v(b), v(c), v(d)\}. \tag{3.2}$$

Now, given an arbitrary vertex x of T_v, there exists $D \in \mathrm{GL}_2(K)$ for which $Dx = p$. It follows from Equation (3.2) that $d(x, Ax) = d(Dx, DAx) = d(p, DAD^{-1}p) \equiv 0 \mod 2$, and hence $\mathrm{SL}_2(K)$ acts on T_v without inversions. □

It is a consequence of Proposition 3.1.4 that the Ping Pong Lemma can be applied to the action of $\mathrm{SL}_2(K)$ on T_v. Moreover, it also enables us to classify matrices of $\mathrm{SL}_2(K)$ based upon their *translation length*: given an isometry g that acts without inversions on a simplicial tree T, this is the integer

$$l(g) = \min_{x \in V(T)} d(x, gx),$$

where $V(T)$ denotes the vertex set of T and d is the standard path metric on T. Note that $l(g) = l(g^{-1})$ and $l(hgh^{-1}) = l(g)$ for all such isometries g, h of T.

Definition 3.1.5. An isometry g of a simplicial tree T which acts without inversions is said to be:

- *elliptic* if $l(g) = 0$;
- *hyperbolic* if $l(g) > 0$.

Every elliptic isometry fixes some vertex of T. On the other hand, it is well-known that a hyperbolic isometry g acts by translations of length $l(g)$ on a straight path $\{p \in V(T) : d(p, gp) = l(g)\}$, called the *axis* of g.

Proposition 3.1.6. *Let g be a hyperbolic isometry of a simplicial tree T. If a vertex $p \in V(T)$ is at distance k from the axis of g, then $d(p, gp) = l(g) + 2k$. Moreover, an edge $p - q$ of T is contained in the axis of g if and only if $d(p, gp) = d(q, gq)$.*

Proof. See Proposition 24 (iv) of [53, Chapter I] and its corollary. □

We now give a key theorem of Morgan and Shalen, showing that the translation length of a matrix in $\mathrm{SL}_2(K)$ (with respect to the Bruhat-Tits tree) depends only on the valuation of its trace.

Proposition 3.1.7. *If $A \in \mathrm{SL}_2(K)$, then $l(A) = -2\min\{0, v(\mathrm{tr}(A))\}$.*

Proof. Recall from the proof of Proposition 3.1.4 that, for any vertex x of T_v, we have $d(x, Ax) = d(Dx, DAx) = d(p, DAD^{-1}p)$, where d is the standard path metric on T_v, the vertex p corresponds to the standard lattice $\mathcal{O}^2$, and $D \in \mathrm{GL}_2(K)$ is such that $Dx = p$. Since trace is preserved under conjugacy, it follows from Equation (3.2) and the ultrametric inequality that

$$d(x, Ax) = -2\min\{v(a), v(b), v(c), v(d)\} \geq -2v(a + d) = -2v(\mathrm{tr}(A)),$$

where $DAD^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Hence $l(A) \geq -2\min\{0, v(\mathrm{tr}(A))\}$.

On the other hand, any $\pm I_2 \neq A \in \mathrm{SL}_2(K)$ is conjugate (via some matrix $\tilde{D} \in \mathrm{GL}_2(K)$) to a matrix in rational canonical form $\begin{bmatrix} 0 & -1 \\ 1 & \mathrm{tr}(A) \end{bmatrix}$. It then follows from Equation (3.2) that $d(x, Ax) = d(p, \tilde{D}A\tilde{D}^{-1}p) = -2\min\{0, v(\mathrm{tr}(A))\}$, where $x = \tilde{D}^{-1}p$. If $A = \pm I_2$, then clearly the same equality holds for any vertex x of T_v, and this completes the proof. See [42, Proposition II.3.15] for further details. □

Using Proposition 3.1.7, elements of $\mathrm{SL}_2(K)$ can be classified as elliptic or hyperbolic. We conclude this section by giving an analogue of Lemma 2.1.2, which shows that discrete and free subgroups of $\mathrm{SL}_2(K)$ contain no elliptic elements. Recall that $\mathrm{SL}_2(K)$ inherits the structure of a metrisable topological group from K^4, so we can use the same criterion for discreteness as in the previous chapter.

Proposition 3.1.8. *Let $A \in \mathrm{SL}_2(K)$. Then the subgroup $\langle A \rangle \leq \mathrm{SL}_2(K)$ is discrete if and only if either A has finite order, or $v(\mathrm{tr}(A)) < 0$.*

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $t = \mathrm{tr}(A)$. If A has finite order, then it generates a discrete group, so suppose that $v(t) < 0$, that is, $|t|_v > 1$. Using the ultrametric inequality, without loss of generality, we may assume that $|a|_v > 1$.

For each $n \in \mathbb{N}$, let a_n denote the top left entry of the matrix A^n. By the Cayley-Hamilton Theorem, we have $A^n = tA^{n-1} - A^{n-2}$. If $|a_{n-1}t|_v > |a_{n-2}|_v$, then the ultrametric inequality implies that

$$|a_n t|_v > |a_n|_v = |a_{n-1}t - a_{n-2}|_v = |a_{n-1}t|_v > |a_{n-1}|_v.$$

Since $|a_1 t|_v > 1 = |a_0|_v$, this inductively proves that $|a_n t|_v > |a_{n-1}|_v$, and hence that $|a_{n+1}|_v = |a_n t|_v$, for all $n \in \mathbb{N}$. Therefore $|a_n|_v$ tends to ∞ as n does, so the subgroup $\langle A \rangle \leq \mathrm{SL}_2(K)$ is discrete.

On the other hand, suppose that A (with entries a, b, c and d, as above) has infinite order and $v(t) \geq 0$, that is, $|t|_v \leq 1$. For each $n \in \mathbb{N}$, let a_n, b_n, c_n and d_n denote the corresponding entries of the matrix A^n. Note that if both $|a_{n-1}|_v$ and $|a_{n-2}|_v$ are bounded above, then so is $|a_n|_v$ by the ultrametric inequality and the Cayley-Hamilton Theorem. It follows by induction that $|a_n|_v$ is bounded above for all $n \in \mathbb{N}$. Similarly, $|b_n|_v$, $|c_n|_v$ and $|d_n|_v$ are bounded above for all $n \in \mathbb{N}$. The Bolzano-Weierstrass property then implies that the subgroup $\langle A \rangle \leq \mathrm{SL}_2(K)$ is not discrete. $\square$

Corollary 3.1.9. *If $G \leq \mathrm{SL}_2(K)$ is both discrete and free, then it contains no elliptic elements.*

Proof. Suppose that $g \in G$ is elliptic. Then either g has finite order, in which case G is not free, or otherwise Proposition 3.1.7 implies that $v(\mathrm{tr}(A)) \geq 0$. But then G cannot be discrete, by Proposition 3.1.8. $\square$

3.2 Translation length formulae

In this section, we prove some formulae for computing the translation length of both the product and the commutator of two hyperbolic isometries of a simplicial tree. The product formulae will be particularly useful later, when deciding whether or not certain two- or three-generated subgroups are both discrete and free.

Similar formulae appear in independent papers of Alperin and Bass, and Culler and Morgan, in the context of isometries of Λ- and $\mathbb{R}$-trees (where distances take values in some totally ordered abelian group Λ, or $\mathbb{R}$, instead of $\mathbb{Z}$ as in the case of simplicial trees); see [2, Section 8] and [16, Section 1] respectively. These formulae were refined and made more transparent by Paulin in Proposition 1.6 of [44], but there is an extra case that was not considered - this is given by case (3)(*iii*) below.

We now present a full, corrected version of these product formulae, with additional details about how the axes of various products interact. For completeness, we also provide an independent proof, in the context of simplicial trees. An alternative version of this (established in joint work with the author of [44], and in the context of $\mathbb{R}$-trees) can be found in the appendix of [14].

Proposition 3.2.1. *Let A and B be hyperbolic isometries of a simplicial tree, such that AB and BA act without inversions. Then precisely one of the following holds:*

(1) *The axes of A and B do not intersect, are separated by a path P of minimum distance k, and*

$$l(AB) = l(BA) = l(A) + l(B) + 2k.$$

The axes of AB and BA intersect with opposite orientations exactly along P.

(2) *The axes of A and B intersect with the same orientation along a (possibly infinite) path P and*

$$l(AB) = l(BA) = l(A) + l(B).$$

The axes of AB and BA intersect with the same orientation exactly along P.

(3) *The axes of A and B intersect with opposite orientations along a (possibly infinite) path P of length $\Delta = \Delta(A, B) \geq 0$ and one of the following holds:*

(i) $\Delta < \min\{l(A), l(B)\}$ *and* $l(AB) = l(BA) = l(A) + l(B) - 2\Delta$. *The axes of AB and BA do not intersect and are distance Δ apart;*

(ii) $\Delta > \min\{l(A), l(B)\}$ *and* $l(AB) = l(BA) = |l(A) - l(B)|$. *If $l(A) \neq l(B)$, then the axes of AB and BA either do not intersect and are distance $2\min\{l(A), l(B)\} - \Delta$ apart (if $\Delta < 2\min\{l(A), l(B)\}$), or intersect with the same orientation only along a subpath of P (if $\Delta \geq 2\min\{l(A), l(B)\}$), which is of length $\Delta - 2\min\{l(A), l(B)\}$ if Δ is finite, and infinite otherwise;*

(iii) $\Delta = \min\{l(A), l(B)\}$, *either the axes of B and $A^{-1}BA$ (if $l(A) \leq l(B)$) or the axes of A and $B^{-1}AB$ (if $l(A) > l(B)$) intersect along a (possibly infinite) path of length $\Delta' \geq 0$, and*

$$l(AB) = l(BA) = \begin{cases} |l(A) - l(B)| - 2\Delta' & \text{if } \Delta' < \frac{|l(A)-l(B)|}{2} \\ 0 & \text{otherwise.} \end{cases}$$

If $\Delta' < \frac{|l(A)-l(B)|}{2}$, then the axes of AB and BA do not intersect and are distance $\Delta + 2\Delta'$ apart.

Proof. For each case, we follow the same general argument: we find edges $x - x'$ and $y - y'$ of the tree for which

$$d(x, ABx) = d(x', ABx') = m, \text{ and}$$
$$d(y, BAy) = d(y', BAy') = m$$

for some non-negative integer m. If $m = 0$, then clearly $l(AB) = l(BA) = 0$. If $m \neq 0$, then AB and BA are both hyperbolic and it follows from Proposition 3.1.6 that $l(AB) = l(BA) = m$. Moreover, the axis of AB contains the edges $x - x'$ and $ABx - ABx'$ (and the path between them), and the axis of BA contains the edges $y - y'$ and $BAy - BAy'$ (and the path between them).

For case (1), suppose that the axes of A and B do not intersect, and that P is the path of minimum distance k between them, with endpoints p (on the axis of A) and q (on the axis of B). Choose some edge $p' - p''$ contained in P, such that the vertex p' is closer to p than p'' is. Then

$$\begin{aligned} d(B^{-1}p', Ap') = d(B^{-1}p'', Ap'') = l(A) + l(B) + 2k, \text{ and} \\ d(A^{-1}p'', Bp'') = d(A^{-1}p', Bp') = l(A) + l(B) + 2k, \end{aligned}$$

so AB and BA are both hyperbolic with $l(AB) = l(BA) = l(A) + l(B) + 2k$. Moreover, the axis of AB contains the path from $B^{-1}p'$ to Ap'', and the axis of BA contains the path from $A^{-1}p''$ to Bp'. Hence the axes of AB and BA intersect with opposite orientation exactly along the path P; see Figure 3.1.

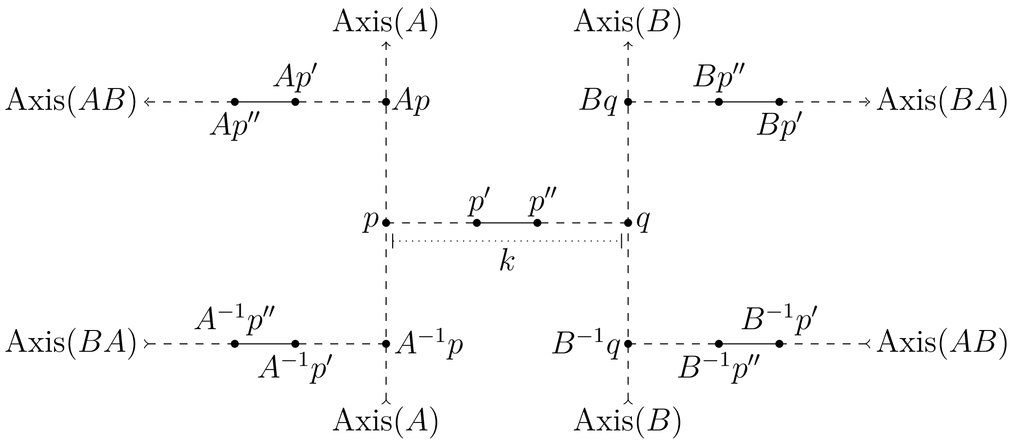

Figure 3.1: The axes of A and B do not intersect.

For case (2), suppose that the axes of A and B intersect with the same orientation along a path P and, without loss of generality, that $l(A) \leq l(B)$. If P is of infinite length, then either the axes of A, B, AB and BA are all given by P (with identical orientations), in which case the conclusions are clear, or otherwise there is one endpoint p of P. In this latter case, we may suppose (by swapping A and B with their inverses, if necessary) that A and B both translate p onto P. Denote by p' and p'' the vertices immediately preceding p on the axes of A and B respectively. Then

$$d(B^{-1}p', Ap') = d(B^{-1}p, Ap) = l(A) + l(B), \text{ and}$$
$$d(A^{-1}p'', Bp'') = d(A^{-1}p, Bp) = l(A) + l(B),$$

so AB and BA are both hyperbolic with $l(AB) = l(BA) = l(A) + l(B)$. Moreover, the axis of AB contains the path from $B^{-1}p'$ to Ap, and the axis of BA contains the path from $A^{-1}p''$ to Bp. It follows that the axes of AB and BA intersect with the same orientation exactly along the path P; see the left-hand diagram of Figure 3.2.

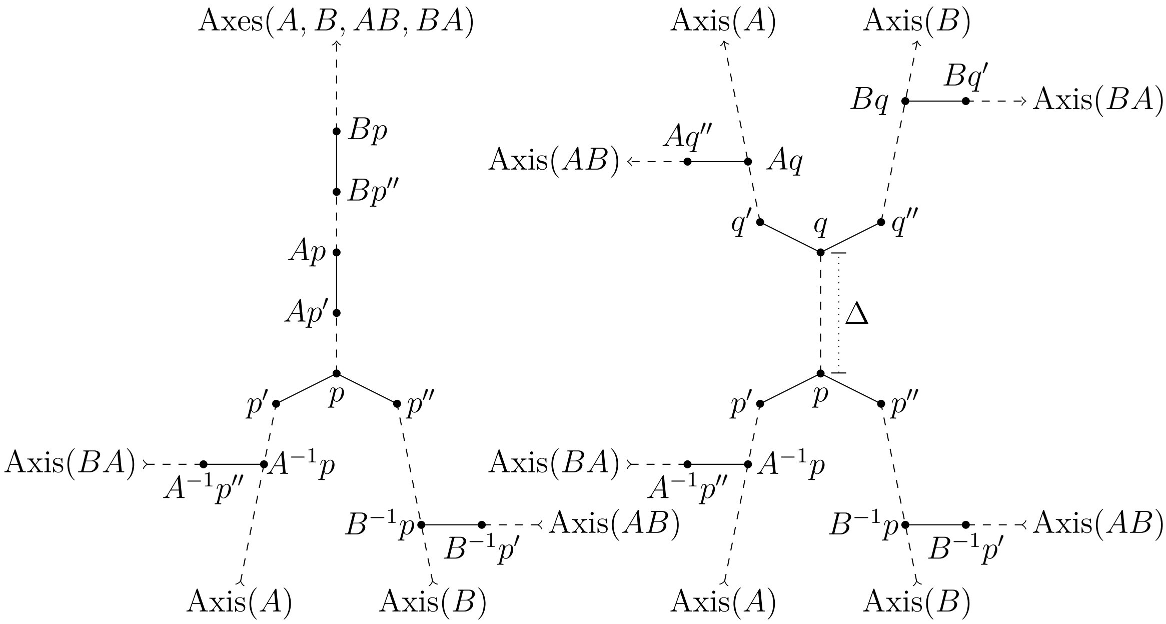

Figure 3.2: The axes of A and B intersect with the same orientation.

To finish case (2), we suppose that P is of finite length, and that p and q are its initial and terminal vertices respectively. Define p' and p'' as the vertices immediately preceding p on the axes of A and B respectively. Similarly, define q' and q'' as the vertices immediately following q on the axes of A and B respectively. By the same argument as before, AB and BA are both hyperbolic with $l(AB) = l(BA) = l(A)+l(B)$. Moreover, the edge $A^{-1}p'' - A^{-1}p$ lies on the axis of BA and the edge $B^{-1}p' - B^{-1}p$ lies on the axis of AB. By symmetry, the edge $Bq - Bq'$ lies on the axis of BA and the edge $Aq - Aq''$ lies on the axis of AB. This shows that the axes of AB and BA intersect with the same orientation exactly along the path P; see the right-hand diagram of Figure 3.2.

For case (3), suppose that the axes of A and B intersect with opposite orientations along a path P of length $\Delta = \Delta(A, B) \geq 0$. Let us again assume, without loss of generality, that $l(A) \leq l(B)$. If P is of infinite length, then it is straightforward to check that $l(AB) = l(BA) = l(B) - l(A)$ and, if $l(A) \neq l(B)$, then the axes of AB and BA coincide with the axis of B along an infinite subpath of P. This proves part of subcase (3)(ii).

We now suppose that P has finite length, with endpoints p and q. If $p = q$, then the result follows from case (2), since two axes which intersect at a single vertex have no relative orientations. We can therefore assume that P has finite and positive length and, without loss of generality, that A translates p towards q. Define p' (respectively p'') to be the vertex immediately preceding p on the axis of A (respectively, immediately following p on the axis of B). Similarly define q' (respectively q'') to be the vertex immediately following q on the axis of A (respectively, immediately preceding q on the axis of B). We consider three subcases, depending on the value of Δ.

For subcase (3)(i), we suppose that $\Delta < \min\{l(A), l(B)\} = l(A)$. Then

$$d(B^{-1}p', Ap') = d(B^{-1}p, Ap) = d(B^{-1}p'', Ap'') = l(A) + l(B) - 2\Delta, \text{ and}$$
$$d(A^{-1}q'', Bq'') = d(A^{-1}q, Bq) = d(A^{-1}q', Bq') = l(A) + l(B) - 2\Delta,$$

so AB and BA are both hyperbolic with $l(AB) = l(BA) = l(A) + l(B) - 2\Delta$. Moreover, the axis of AB contains the path from $B^{-1}p'$ to Ap'', and the axis of BA contains the path from $A^{-1}q''$ to Bq'. Therefore the axes of AB and BA do not intersect and are distance Δ apart; see Figure 3.3.

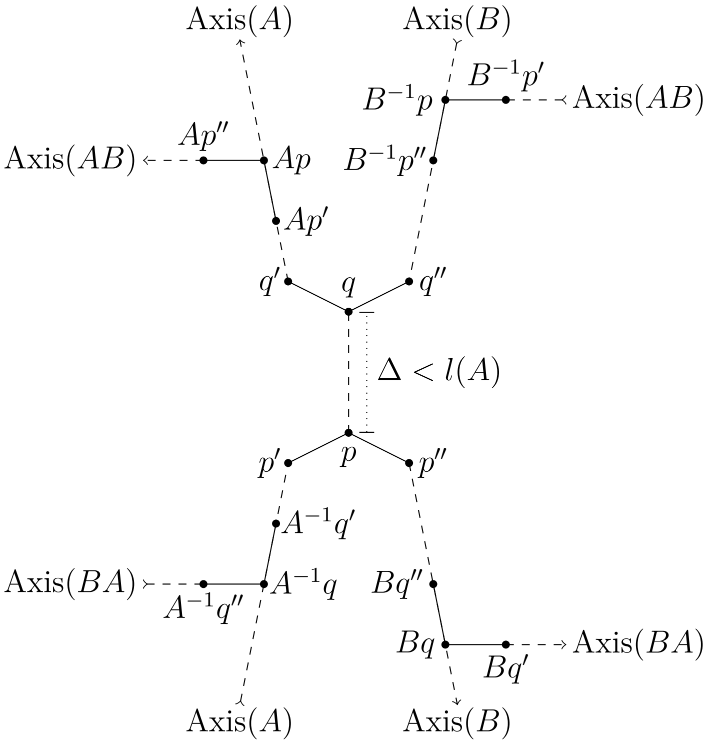

Figure 3.3: The axes of A and B intersect with opposite orientations along a path of length $\Delta < \min\{l(A), l(B)\} = l(A)$.

For subcase (3)(ii), we suppose that $\Delta > \min\{l(A), l(B)\} = l(A)$. As in subcase (3)($i$), the edge $Ap - Ap''$ lies on the axis of AB and the edge $A^{-1}q'' - A^{-1}q$ lies on the axis of BA. In this case, however, the vertices Ap and $A^{-1}q$ lie on the path P. Also

$$d(B^{-1}A^{-1}q'', q'') = d(B^{-1}A^{-1}q, q) = l(B) - l(A), \text{ and}$$
$$d(p, BAp) = d(p'', BAp'') = l(B) - l(A).$$

Hence, if $l(A) \neq l(B)$, then AB and BA are both hyperbolic with $l(AB) = l(BA) = l(B) - l(A)$. In this situation, the axis of AB contains the path between $B^{-1}A^{-1}q''$ and Ap'', and the axis of BA contains the path between $A^{-1}q''$ and BAp''. Therefore, depending on the relative positions of $A^{-1}q$ and Ap along P, the axes of AB and BA either do not intersect and are distance $2l(A) - \Delta$ apart (if $\Delta < 2l(A)$) or intersect with the same orientation along a subpath of P of length $\Delta - 2l(A)$ (if $\Delta \geq 2l(A)$); see the left- and right-hand diagrams of Figure 3.4 respectively.

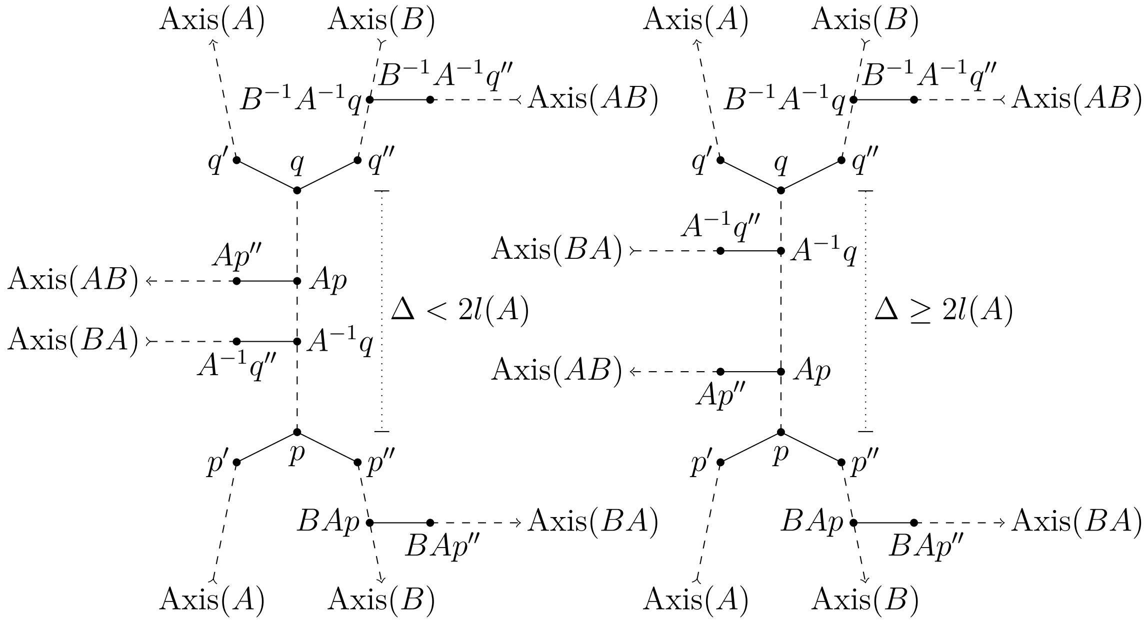

Figure 3.4: The axes of A and B intersect with opposite orientations along a path of length $\Delta > \min\{l(A), l(B)\} = l(A)$.

Finally, for subcase (3)(iii), we suppose that $\Delta = \min\{l(A), l(B)\} = l(A)$. The axes of B and $A^{-1}BA$ (that is, Axis(B) and $A^{-1} \cdot$ Axis(B)) intersect along a path of length Δ': this path is between the vertex p and some other vertex further along the axis of B, which we will denote by r. (Note that r could coincide with p if $\Delta' = 0$.)

If $\frac{l(B)-l(A)}{2} \leq \Delta' < l(B) - l(A)$, then it follows that BAr lies on the path between p and r. Moreover, $(BA)^2 r = r$ and BA inverts the path between r and BAr; see the left-hand diagram of Figure 3.5. Since BA acts without inversions, this path between r and BAr must have even length and hence BA fixes its midpoint, giving $l(BA) = l(AB) = 0$. Similarly, if $\Delta' \geq l(B) - l(A)$, then BAp lies on the path between p and r, and BA inverts the path between p and BAp; see the right-hand diagram of Figure 3.5. Since BA acts without inversions, it follows that the path between p and BAp has even length and BA fixes its midpoint, giving $l(BA) = l(AB) = 0$.

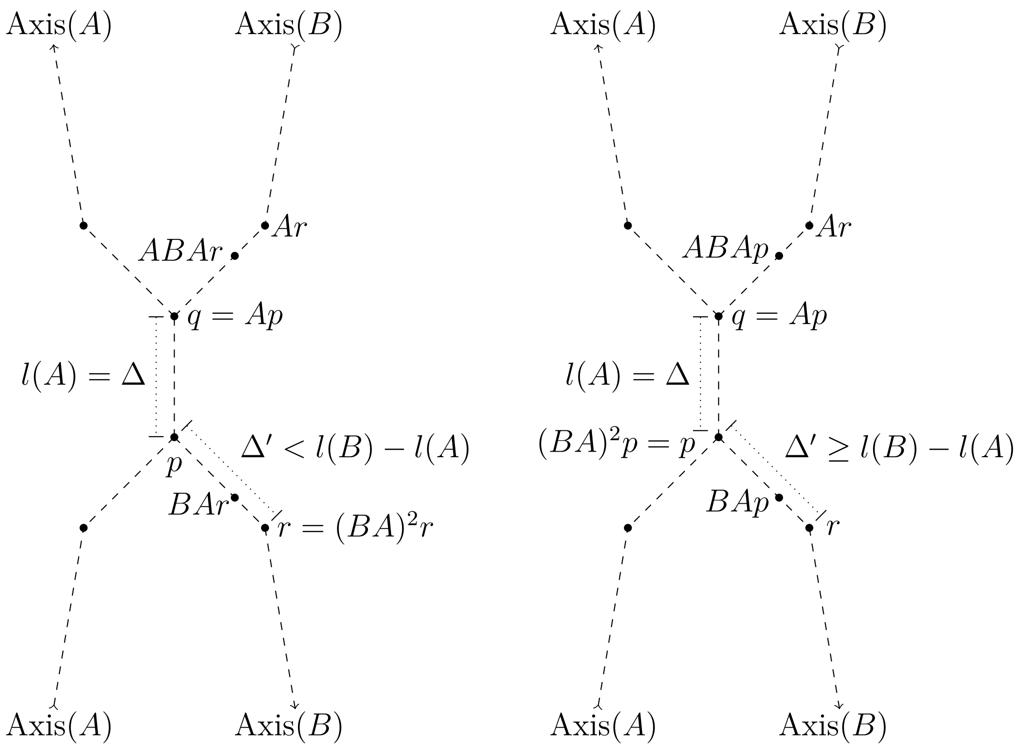

Figure 3.5: The axes of A and B intersect with opposite orientations along a path of length $\Delta = \min\{l(A), l(B)\} = l(A)$. The axes of B and $A^{-1}BA$ intersect along a path of length $\Delta' \geq \frac{l(B)-l(A)}{2}$.

On the other hand, if $\Delta' < \frac{l(B)-l(A)}{2}$, then let r' denote the vertex immediately following r on the axis of B, and let r'' be the vertex such that Ar'' immediately precedes Ar on the axis of B. Then

$$d(B^{-1}r'', Ar'') = d(B^{-1}r, Ar) = d(B^{-1}r', Ar') = l(B) - l(A) - 2\Delta', \text{ and}$$
$$d(r'', BAr'') = d(r, BAr) = d(r', BAr') = l(B) - l(A) - 2\Delta',$$

so AB and BA are both hyperbolic with $l(AB) = l(BA) = l(B) - l(A) - 2\Delta'$. Moreover, the axis of AB contains the path from $B^{-1}r''$ to Ar', and the axis of BA contains the path from r'' to BAr'. Therefore the axes of AB and BA do not intersect and are distance $\Delta + 2\Delta'$ apart; see Figure 3.6. This completes the proof. □

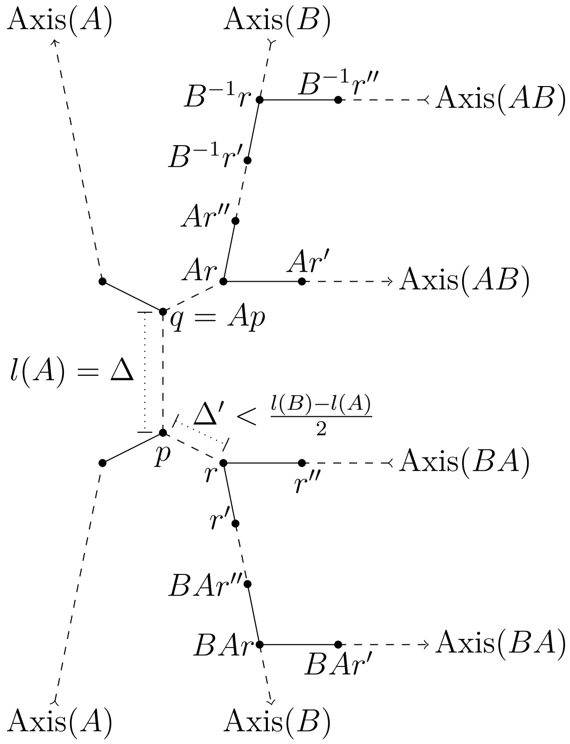

Figure 3.6: The axes of A and B intersect with opposite orientations along a path of length $\Delta = \min\{l(A), l(B)\} = l(A)$. The axes of B and $A^{-1}BA$ intersect along a path of length $\Delta' < \frac{l(B)-l(A)}{2}$.

We note that the missing case from Proposition 1.6 of [44] was discovered when considering various examples in $\mathrm{SL}_2(\mathbb{Q}_7)$. Specifically, given the matrices

$$X = \begin{bmatrix} 7^3 & 0 \\ 0 & \frac{1}{7^3} \end{bmatrix} \text{ and } Y = \begin{bmatrix} \frac{2}{7^7} & 7^3 \\ \frac{1}{7^3} & 7^7 \end{bmatrix},$$

setting $A = XY$ and $B = X^3Y^3$ yields hyperbolic elements with respective translation lengths on the Bruhat-Tits tree of 8 and 32. Moreover, the axes of A^{-1} and B overlap with opposite directions of translation. But $l(A^{-1}B) = 16$, and this is inconsistent with Proposition 1.6 (2)(*ii*) of [44]: this value is neither $l(B) - l(A)$, nor of the form $l(A) + l(B) - 2\Delta$ for some $\Delta < 8$. On the other hand, this does agree with Proposition 3.2.1 (3)(*iii*) if $\Delta' = 4$. Proposition 1.6 of [44] has also been referred to in some other papers (for instance, [19] and [23]), but our correction does not appear to affect the results that depend on it.

We conclude this section with an application of Proposition 3.2.1 to finding the translation length of the commutator $[A, B] = A^{-1}B^{-1}AB$ of two hyperbolic isometries A and B of a simplicial tree. This extends the formulae given (in the context of isometries of $\mathbb{R}$-trees) in Lemma 3.4 and Remark 3.5 of [16].

Proposition 3.2.2. *Let A and B be hyperbolic isometries of a simplicial tree such that AB (and hence also BA) is hyperbolic. Then precisely one of the following holds:*

(1) *The axes of A and B do not intersect, are distance k apart and*

$$l([A, B]) = 2l(A) + 2l(B) + 4k.$$

(2) *The axes of A and B intersect along a path of length $\Delta = \Delta(A, B) \geq 0$ and*

$$l([A, B]) = \begin{cases} 2l(A) + 2l(B) - 2\Delta & \text{if } \Delta < l(A) + l(B) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The general method of proof is as follows: We first apply Proposition 3.2.1 to the isometries A and B to determine how the axes of AB and BA (and hence $(BA)^{-1}$) interact. Since $[A, B] = (BA)^{-1}AB$, we can again apply Proposition 3.2.1, in this case to the axes of $(BA)^{-1}$ and AB, in order to evaluate $l([A, B])$.

In case (1), the axes of A and B do not intersect. If P denotes the path of minimum distance k between the axes of A and B, then it follows from Proposition 3.2.1 (1) that $l(AB) = l(BA) = l(A) + l(B) + 2k$, and the axes of $(BA)^{-1}$ and AB intersect with the same orientation exactly along this path P; see Figure 3.1. Proposition 3.2.1 (2) (applied to the axes of $(BA)^{-1}$ and AB) then gives

$$l([A, B]) = l(BA) + l(AB) = 2l(A) + 2l(B) + 4k.$$

For case (2), the axes of A and B intersect along a (possibly infinite) path P of length $\Delta = \Delta(A, B)$. If $\Delta \geq l(A) + l(B)$, then it is straightforward to find a vertex x on the path P such that $ABx = BAx$, so $[A, B]$ is elliptic. Hence, for the remainder of the proof, we may assume that the axes of A and B intersect along a path P of length $\Delta < l(A) + l(B)$.

If the orientations of the axes of A and B agree, then Proposition 3.2.1 (2) gives $l(AB) = l(BA) = l(A) + l(B)$. Moreover, the axes of $(BA)^{-1}$ and AB intersect with opposite orientations exactly along the path P; see the right-hand diagram of Figure 3.2. Since $\Delta((BA)^{-1}, AB) = \Delta < \min\{l(BA), l(AB)\}$, it follows from Proposition 3.2.1 (3)(i) (applied to the axes of $(BA)^{-1}$ and AB) that

$$l([A, B]) = l(AB) + l(BA) - 2\Delta = 2l(A) + 2l(B) - 2\Delta.$$

If the orientations of the axes of A and B do not agree, then we consider the various subcases given in Proposition 3.2.1 (3). If $\Delta < \min\{l(A), l(B)\}$, then subcase (3)($i$) of Proposition 3.2.1 implies that $l(BA) = l(AB) = l(A) + l(B) - 2\Delta$, and the axes of $(BA)^{-1}$ and AB do not intersect and are distance Δ apart; see Figure 3.3. Thus Proposition 3.2.1 (1) (applied to the axes of $(BA)^{-1}$ and AB) gives

$$l([A, B]) = l(AB) + l(BA) + 2\Delta = 2l(A) + 2l(B) - 2\Delta.$$

Similarly, if $\Delta > \min\{l(A), l(B)\}$, then Proposition 3.2.1 (3)(ii) implies that $l(AB) = l(BA) = |l(A) - l(B)|$. Since AB is hyperbolic, $l(A) \neq l(B)$. Therefore the axes of $(BA)^{-1}$ and AB either do not intersect and are distance $2\min\{l(A), l(B)\} - \Delta$ apart (if $\Delta < 2\min\{l(A), l(B)\}$) or intersect with opposite orientations exactly along a subpath of P which has length $\Delta - 2\min\{l(A), l(B)\}$ (if $\Delta \geq 2\min\{l(A), l(B)\}$); see the left- and right-hand diagrams of Figure 3.4 respectively.

In the first situation, $\Delta < 2\min\{l(A), l(B)\}$, and it follows from Proposition 3.2.1 (1) (applied to the axes of $(BA)^{-1}$ and AB) that

$$l([A,B]) = l(AB) + l(BA) + 2(2\min\{l(A), l(B)\} - \Delta) = 2l(A) + 2l(B) - 2\Delta.$$

On the other hand, if $\Delta \geq 2\min\{l(A), l(B)\}$, then (since $\Delta < l(A) + l(B)$) we have $\Delta((BA)^{-1}, AB) = \Delta - 2\min\{l(A), l(B)\} < \min\{l(BA), l(AB)\}$. It follows from Proposition 3.2.1 (3)(i) (applied to the axes of $(BA)^{-1}$ and AB) that

$$l([A,B]) = l(AB) + l(BA) - 2(\Delta - 2\min\{l(A), l(B)\}) = 2l(A) + 2l(B) - 2\Delta.$$

Finally, we consider the case that $\Delta = \min\{l(A), l(B)\}$. In this situation, since AB is hyperbolic, Proposition 3.2.1 (3)(iii) gives $l(AB) = l(BA) = |l(A) - l(B)| - 2\Delta'$ for some $0 \leq \Delta' < \frac{|l(A)-l(B)|}{2}$. Moreover, the axes of $(BA)^{-1}$ and AB do not intersect and are distance $\Delta + 2\Delta'$ apart; see Figure 3.6. Proposition 3.2.1 (1) (applied to the axes of $(BA)^{-1}$ and AB) then implies

$$l([A,B]) = l(AB) + l(BA) + 2(\Delta + 2\Delta') = 2\max\{l(A), l(B)\} = 2l(A) + 2l(B) - 2\Delta,$$

which completes the proof. $\square$

3.3 Deciding whether a two-generated subgroup of $\mathrm{SL}_2(K)$ is discrete and free

We conclude this chapter by presenting a practical algorithm which, given any two-generated subgroup $G \leq \mathrm{SL}_2(K)$, determines after finitely many steps whether or not G is both discrete and free of rank two. The key idea is to perform Nielsen transformations on the generators of G until this produces either an elliptic element or two hyperbolic elements satisfying the Ping Pong Lemma.

We begin by showing that the translation length formulae in the previous section can be used to determine whether or not two hyperbolic elements of $\mathrm{SL}_2(K)$ generate a subgroup which is both discrete and free of rank two. This relies on the fact that two hyperbolic isometries of a tree generate a free group when the intersection between their axes is sufficiently small. This observation has been made in Lemma 2.6 of [16] (in the context of $\mathbb{R}$-trees) and Lemma 3.2 of [54] (in the context of Λ-trees). Here we use our version of the Ping Pong Lemma to give a similar, yet stronger, result in the context of simplicial trees.

Proposition 3.3.1. *Let G be a metrisable topological group acting continuously by isometries and without inversions on a simplicial tree T. Suppose that $A, B \in G$ are hyperbolic, and that their axes are either disjoint or intersect along a path of length $0 \leq \Delta(A,B) < \min\{l(A), l(B)\}$. Then the subgroup $\langle A, B\rangle \leq G$ is both discrete and free of rank two.*

Proof. First of all, if the axes of A and B are disjoint, then there is a unique path P of minimum distance between the two axes. Suppose that this path is between a vertex p' on the axis of A and a vertex q' on the axis of B. Choose vertices p and q (on the axes of A and B respectively) so that the interior of the path between p and Ap contains p', and the interior of the path between q and Bq contains q'; see the left-hand diagram of Figure 3.7. (Note that if either A or B has translation length one, then it may be necessary to subdivide each edge of T at its midpoint in order to find such vertices.)

On the other hand, if the axes of A and B intersect along a common subpath P of length $\Delta(A,B) < \min\{l(A), l(B)\}$, then choose vertices p and q (on the axes of A and B respectively) such that the interiors of the paths between p and Ap, and between q and Bq, both contain P; see the right-hand diagram of Figure 3.7 for the case when the axes of A and B have the same orientation. (Note that if either A or B has translation length $\Delta(A,B)+1$, then it may be necessary to subdivide each edge of T at its midpoint in order to find such vertices.)

In either case, define X_1^+ (respectively X_1^-) to be the maximal subtree of T containing all vertices on the axis of A from Ap onwards (respectively, up to and

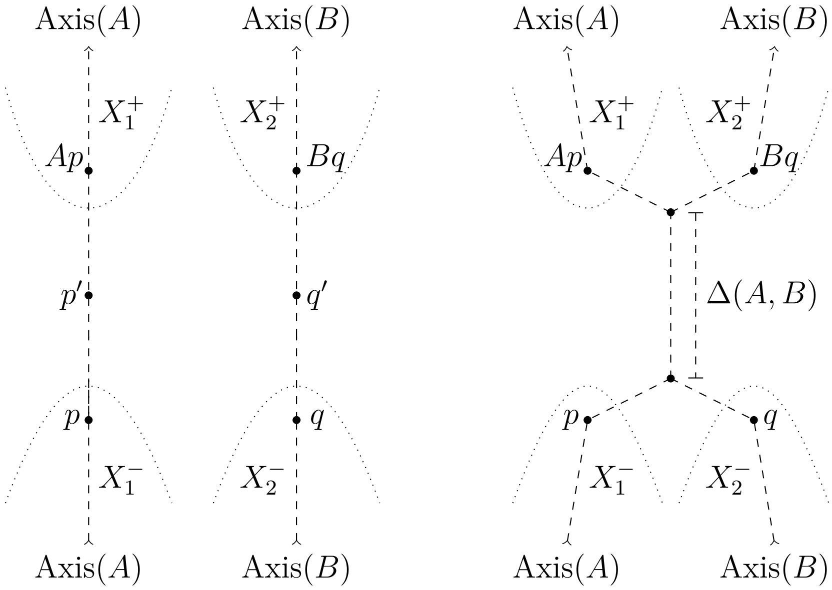

Figure 3.7: Applying the Ping Pong Lemma to a pair of hyperbolic isometries of a tree.

including p), with respect to the direction of translation, but no other vertices on the axis of A. Similarly define X_2^+ (respectively X_2^-) as the maximal subtree containing all vertices on the axis of B from Bq onwards (respectively, up to and including q), but no other vertices on the axis of B. Then X_1^+, X_1^-, X_2^+ and X_2^- are non-empty, closed and pairwise disjoint subsets that do not cover T. Moreover, Proposition 3.1.6 implies that $A(T\backslash X_1^-) \subseteq X_1^+$, $A^{-1}(T\backslash X_1^+) \subseteq X_1^-$, $B(T\backslash X_2^-) \subseteq X_2^+$ and $B^{-1}(T\backslash X_2^+) \subseteq X_2^-$. The conclusion then follows from the Ping Pong Lemma. □

Corollary 3.3.2. *Let G be a metrisable topological group acting continuously by isometries and without inversions on a simplicial tree. If $A, B \in G$ are hyperbolic and $|l(A) - l(B)| < \min\{l(AB), l(A^{-1}B)\}$, then A and B satisfy the hypotheses of the Ping Pong Lemma and the subgroup $\langle A, B\rangle \leq G$ is both discrete and free of rank two.*

Proof. We consider the cases given in Proposition 3.2.1. If the axes of A and B do not intersect, then

$$l(AB) = l(A^{-1}B) \geq l(A) + l(B) > |l(A) - l(B)|.$$

If the axes of A and B intersect along a path of length $\Delta(A,B) < \min\{l(A), l(B)\}$, then

$$\min\{l(AB), l(A^{-1}B)\} = l(A) + l(B) - 2\Delta(A,B) > |l(A) - l(B)|.$$

Otherwise the axes of A and B intersect along a path of length $\Delta(A,B) \geq \min\{l(A), l(B)\}$ and

$$\min\{l(AB), l(A^{-1}B)\} \leq |l(A) - l(B)|.$$

Hence $|l(A) - l(B)| < \min\{l(AB), l(A^{-1}B)\}$ if and only if the axes of A and B either do not intersect, or intersect along a path of length $0 \leq \Delta(A,B) < \min\{l(A), l(B)\}$. By Proposition 3.3.1, this implies that $\langle A, B\rangle \leq G$ is discrete and free of rank two. $\square$

We now present a practical algorithm (see [14, Algorithm 4.1]) that takes as input a two-generated subgroup of $\mathrm{SL}_2(K)$ and determines whether or not it is both discrete and free of rank two. Note that (with the same method of proof as Proposition 2.1.4) it can be shown that an n-generated subgroup $G \leq \mathrm{SL}_2(K)$ is discrete and free of rank n if and only if the corresponding subgroup $\overline{G} \leq \mathrm{PSL}_2(K)$ is discrete and free of rank n. Thus our algorithm can also be applied to two-generated subgroups of $\mathrm{PSL}_2(K)$, by taking representatives of the generators in $\mathrm{SL}_2(K)$.

Recall that Proposition 3.1.7 gives the translation length of a matrix in $\mathrm{SL}_2(K)$, and that $\mathrm{SL}_2(K)$ is a metrisable topological group which acts continuously by isometries and without inversions on the Bruhat-Tits tree T_v.

Algorithm 3.3.3. *Let K be a non-archimedean local field. Given $A, B \in \mathrm{SL}_2(K)$, we proceed as follows: If $G = \langle A, B\rangle \leq \mathrm{SL}_2(K)$ is both discrete and free of rank two, then the algorithm will return true and output a generating pair for G which satisfies the hypotheses of the Ping Pong Lemma, and otherwise it will return false.*

(1) *Set* $X = A$*,* $Y = B$*. If* $l(X) = 0$*, or* $l(Y) = 0$*, then return false.*

(2) *If* $l(X) > l(Y)$*, then swap* X *and* Y*.*

(3) *Compute* $m = \min\{l(XY), l(X^{-1}Y)\}$*.*

(4) *If* $m = 0$*, then return false.*

(5) *If* $m \leq l(Y) - l(X)$*, then replace* Y *by the element from* $\{XY, X^{-1}Y\}$ *which has translation length* m *and return to* (2)*.*

(6) *Return true and the pair* (X, Y)*.*

Theorem 3.3.4. *Algorithm 3.3.3 terminates after finitely many steps and produces the correct output.*

Proof. If at any point the algorithm encounters an elliptic element, then it follows from Corollary 3.1.9 that G is not both discrete and free. So suppose that the algorithm only ever encounters hyperbolic elements. Then it must reach step (5). If $m > l(Y) - l(X)$, then G is discrete and free by Corollary 3.3.2, and the elements X and Y satisfy the hypotheses of the Ping Pong Lemma. Hence the algorithm is correct.

On the other hand, if $m \leq l(Y) - l(X)$, then the algorithm performs a Nielsen transformation and outputs a new pair of generators for G on which to run the algorithm. If this sequence of Nielsen transformations never terminates, then there is an infinite sequence $(x_n, y_n) = (l(X_n), l(Y_n))$ of integral translation length pairs with the property that $0 < x_n \leq y_n$ for all $n \in \mathbb{N}$, and which is decreasing in each component; such a sequence must converge. Moreover, for each pair (X_n, Y_n) of generators of G, we are in either case (3)(*ii*), or the first subcase of (3)(*iii*), of Proposition 3.2.1. Therefore step (5) replaces (x_n, y_n) by $(x_n, y_n - x_n - k_n)$, where k_n is either 0 or $2\Delta'$ (as given in cases (3)(*ii*) and (*iii*) of Proposition 3.2.1 respectively). In particular, this implies that $x_{n+1} + y_{n+1} = y_n - k_n$ for all $n \in \mathbb{N}$. After rearranging, and taking limits, it follows that $\lim_{n\to\infty} x_n = -\lim_{n\to\infty} k_n \leq 0$. This is a contradiction since each x_n is a positive integer, and hence this algorithm must eventually terminate. □

Recall that one of the key steps in Algorithm 2.2.3 involved the commutator $[A, B] = A^{-1}B^{-1}AB$ of the generating pair $A, B \in \mathrm{SL}_2(\mathbb{R})$: if $\mathrm{tr}([A, B]) \in (-2, 2]$, then the algorithm returned false, and otherwise two subcases were considered depending on whether $\mathrm{tr}([A, B]) > 2$ or $\mathrm{tr}([A, B]) \leq -2$. In the case of Algorithm 3.3.3, one could analagously consider $l([A, B])$, but the value of this does not affect the remainder of the algorithm. In particular, if $l([A, B]) = 0$, then one could immediately return false at step (1).

In fact, Corollary 3.3.2 could be rephrased in terms of the translation length of the commutator. Indeed, it follows from Proposition 3.2.2 and Proposition 3.3.1 that, if A and B are hyperbolic elements of a group G (which acts continuously by isometries and without inversions on a simplicial tree) whose product AB is hyperbolic and $l([A, B]) > 2\max\{l(A), l(B)\}$, then the subgroup $\langle A, B\rangle \leq G$ is both discrete and free of rank two. However, replacing either A or B by $[A, B]$ is not a Nielsen transformation, and hence this alternate version of Corollary 3.3.2 is not as useful algorithmically.

We conclude this chapter by discussing the implementation of Algorithm 3.3.3, and giving some examples which compare and contrast it with the discrete and free algorithm for two-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$ from [17].

In terms of implementing Algorithm 3.3.3 in a computational package such as MAGMA, the software needs to be able to perform matrix multiplications over K, and compute traces and valuations. Since each non-zero element of K can be expressed uniquely in the form $a = \sum_{i=N}^{\infty} a_i\pi^i$ for some integer $N = v(a)$ with $a_N \neq 0$, and some uniformiser π (see Equation (3.1)), computing valuations and performing both addition and multiplication over K is straightforward. However, there is a clear obstacle in the computational storage space needed for elements of K with an infinite expression of the above form. This can theoretically be overcome by storing elements of K in terms of the data $\{\pi; a_N, a_{N+1}, \ldots, a_M\}$ up to some appropriate integer M. Indeed, given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix},$$

one iteration of Algorithm 3.3.3 first requires computing $l(A) = -2\min\{0, v(a+d)\}$ and $l(B) = -2\min\{0, v(e+h)\}$. Since any non-negative valuation gives a translation length of 0, calculating these accurately requires storing the entries of A and B only up to the coefficient of π^0 (in other words, $M = 0$ will suffice). On the other hand, when $0 < l(A) \leq l(B)$, the first iteration of Algorithm 3.3.3 will also require computing $l(AB) = -2\min\{0, v(ae+bg+cf+dh)\}$ and $l(A^{-1}B) = -2\min\{0, v(de-bg-cf+ah)\}$. Storing the entries of A and B up to the coefficient of $\pi^{-\min\{0,v(a),v(b),\ldots,v(h)\}}$ is sufficient to compute these valuations accurately. It follows inductively that storing the entries of A and B up to the coefficient of $\pi^{-r\min\{0,v(a),v(b),\ldots,v(h)\}}$ is enough to correctly apply r iterations of Algorithm 3.3.3. Hence, given any two matrices $A, B \in \mathrm{SL}_2(K)$ as above, choosing large enough M (compared with $-\min\{0, v(a), v(b), \ldots, v(h)\}$) allows the algorithm to run correctly. If at any point the number of iterations exceeds $\frac{M}{-\min\{0,v(a),v(b),\ldots,v(h)\}}$, then a higher bound M will need to be chosen and the algorithm restarted.

The examples we discuss below avoid this issue entirely for the case where $K = \mathbb{Q}_p$ for some prime p. By restricting our interest to pairs of matrices in $\mathrm{SL}_2(\mathbb{Q})$, we can perform matrix multiplication and compute traces in the usual sense, and then consider p-adic valuations separately. In this particular case, it is interesting to view the subgroups generated as subgroups of both $\mathrm{SL}_2(\mathbb{Q}_p)$ and $\mathrm{SL}_2(\mathbb{R})$, and then compare the properties of each. For instance, we showed in the previous chapter that

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

generate a discrete and free subgroup of $\mathrm{SL}_2(\mathbb{R})$, whereas the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

do not. However, neither of these pairs of matrices generate a discrete and free subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ since any matrix of the form $\begin{bmatrix} 1 & * \\ * & 1 \end{bmatrix}$ is elliptic.

One iteration of Algorithm 3.3.3 also shows that, for any prime $p \neq 2$, the matrices

$$A = \begin{bmatrix} p & p-1 \\ \frac{-1}{p} & \frac{1}{p^2} \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{2}{p^4} & p^3 \\ \frac{1}{p^3} & p^4 \end{bmatrix}$$

generate a subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ which is discrete and free of rank two, because $l(A) = 4, l(B) = 8$ and $\min\{l(AB), l(A^{-1}B)\} = 6$. Using the same matrices as input for Algorithm 2.2.3 shows that they do not generate a free and discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ (this follows since $\mathrm{tr}(AB) = \frac{p+1}{p^3} < 2$, so AB is conjugate to a rotation matrix). On the other hand, for any prime $p \neq 2$, the matrices

$$A = \begin{bmatrix} p & p-1 \\ \frac{-1}{p} & \frac{1}{p^2} \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{2}{p^3} & p^4 \\ \frac{1}{p^4} & p^3 \end{bmatrix}$$

generate subgroups of both $\mathrm{SL}_2(\mathbb{Q}_p)$ and $\mathrm{SL}_2(\mathbb{R})$ which are discrete and free of rank two. This follows respectively from Corollary 3.3.2 (since $l(A) = 4, l(B) = 6$ and $\min\{l(AB), l(A^{-1}B)\} = 8$) and Theorem 2.2.2 ($i$) (since, after replacing A by A^{-1}, we have $\mathrm{tr}(A), \mathrm{tr}(B) \geq 2$ and $\mathrm{tr}(A^{-1}B) = -p^3 + p + \frac{2}{p^2} + \frac{1}{p^3} - \frac{1}{p^4} \leq -2$).

Each of these examples requires only one iteration of Algorithm 3.3.3, but this is certainly not always the case. Indeed, given a prime $p \neq 2$ and a positive integer r, it requires $r + 2$ iterations of Algorithm 3.3.3 to show that

$$A = \begin{bmatrix} p^3 & 0 \\ 0 & \frac{1}{p^3} \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{2}{p^{3r+1}} & p^3 \\ \frac{1}{p^3} & p^{3r+1} \end{bmatrix}$$

generate a discrete and free subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$.

Chapter 4

Discrete and free three-generated subgroups of $\mathrm{SL}_2(K)$

Building upon the methods used in the previous chapter, we now show how the translation length of the product of three hyperbolic elements of $\mathrm{SL}_2(K)$ can be used to determine whether or not these elements satisfy the hypotheses of the Ping Pong Lemma. This leads to a generalisation of Algorithm 3.3.3: given a three-generated subgroup $G \leq \mathrm{SL}_2(K)$, we give an algorithm that determines after finitely many steps whether or not G is both discrete and free of rank three.

4.1 Translation length conditions

As in the case of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$, finitely generated subgroups of $\mathrm{SL}_2(K)$ on three or more generators have not been extensively studied. The following theorem of Weidmann (which we present in the form of [1, Theorem 4]) is applicable but non-constructive, since it does not give an explicit method to determine if no such elliptic element exists.

Theorem 4.1.1. *Let $G = \langle g_1, \ldots, g_n \rangle$ be a finitely generated group acting by isometries and without inversions on a simplicial tree. Then either G is free of rank n, or there is a Nielsen-equivalent generating set $(g_1', \ldots, g_n')$ of G with some g_i' elliptic.*

Proof. See Theorem 7 of [55], and set each $S_i = \varnothing$. $\square$

Note that generalisations of Theorem 4.1.1 exist for groups acting by isometries on any hyperbolic metric space, in the sense that any such group generated by n elements is either free of rank n or contains an element of 'small' translation length; see both [4] and [28] for independent proofs of this.

If $n = 2$ and $g_1, g_2 \in \mathrm{SL}_2(K)$, then Algorithm 3.3.3 can be used to decide between the two possible outcomes of Theorem 4.1.1. In particular, if $\langle g_1, g_2 \rangle \leq \mathrm{SL}_2(K)$ is not free of rank two, then one can track the Nielsen transformations performed in Algorithm 3.3.3 to obtain a specific elliptic element (as a word in g_1 and g_2) which causes the algorithm to return false.

In the following section, we introduce a practical method of deciding between the two possible outcomes of Theorem 4.1.1 in the case that $n = 3$ and $g_1, g_2, g_3 \in \mathrm{SL}_2(K)$. We first show in this section how translation length can be used to determine whether or not three hyperbolic elements of $\mathrm{SL}_2(K)$ satisfy the hypotheses of the Ping Pong Lemma, and hence if they generate a subgroup that is both discrete and free of rank three.

Definition 4.1.2. Let g_i and g_j be hyperbolic isometries of a simplicial tree, with axes denoted by γ_i and γ_j. Following the notation used in [1], we define the *projection of γ_i onto γ_j* to be

$$\mathrm{Proj}_{\gamma_j}(\gamma_i) = \{x \in \gamma_j : d(x, \gamma_i) = d(\gamma_i, \gamma_j)\}.$$

Note that $\mathrm{Proj}_{\gamma_j}(\gamma_i)$ is either the unique vertex of γ_j that is closest to γ_i (when γ_i and γ_j do not intersect), or the path $\gamma_i \cap \gamma_j$ (when γ_i and γ_j intersect). This gives the following reformulation of the Ping Pong Lemma, in the context of simplicial trees:

Proposition 4.1.3. *Let G be a metrisable topological group acting continuously by isometries and without inversions on a simplicial tree. Suppose that $g_1, \ldots, g_n \in G$ are hyperbolic, and that for each $1 \leq j \leq n$ there is a subpath $P_j \subseteq \gamma_j$ of length $\Delta_j < l(g_j)$ such that*

$$\bigcup_{i \neq j} \mathrm{Proj}_{\gamma_j}(\gamma_i) \subseteq P_j.$$

Then $g_1, \ldots, g_n$ satisfy the hypotheses of the Ping Pong Lemma, and the subgroup $\langle g_1, \ldots, g_n \rangle \leq G$ is both discrete and free of rank n.

Proof. After subdividing each edge of the tree T at its midpoint, if necessary, for each $1 \leq j \leq n$ choose a vertex $p_j \in \gamma_j$ such that the interior of the path between p_j and $g_j p_j$ contains P_j. Define X_j^+ (respectively X_j^-) to be the maximal subtree of T containing all vertices of γ_j from $g_j p_j$ onwards (respectively, up to and including p_j) with respect to the direction of translation, but no other vertices of γ_j. Then the subtrees $X_1^{\pm}, \ldots, X_n^{\pm}$ satisfy the hypotheses of the Ping Pong Lemma, which completes the proof. See also [36, Proposition 1.6]. $\square$

We now show how, in the case that $n = 3$, translation length can be used to determine whether or not the hypotheses of Proposition 4.1.3 (and hence of the Ping Pong Lemma) are satisfied. Given hyperbolic isometries g_1, g_2 and g_3 of a simplicial tree, with axes denoted by γ_1, γ_2 and γ_3, we define m_{123} to be the minimum translation length of all products of the form $g_1^{\pm 1} g_2^{\pm 1} g_3^{\pm 1}$ or $g_2^{\pm 1} g_1^{\pm 1} g_3^{\pm 1}$ and their cyclic permutations. Since conjugation and inversion preserves translation length, we have

$$m_{123} = \min \left\{ \begin{array}{l} l(g_1 g_2 g_3), l(g_1^{-1} g_2 g_3), l(g_1 g_2^{-1} g_3), l(g_1^{-1} g_2^{-1} g_3), \\ l(g_2 g_1 g_3), l(g_2^{-1} g_1 g_3), l(g_2 g_1^{-1} g_3), l(g_2^{-1} g_1^{-1} g_3) \end{array} \right\}.$$

Theorem 4.1.4. *Let G be a metrisable topological group acting continuously by isometries and without inversions on a simplicial tree. Suppose that $g_1, g_2, g_3 \in G$ are hyperbolic elements which satisfy both of the following conditions:*

(i) $m_{123} = l(g_1 g_2 g_3)$;

(ii) $\min\{l(g_i g_j), l(g_i^{-1} g_j)\} > |l(g_i) - l(g_j)|$ *for all distinct $i, j \in \{1, 2, 3\}$.*

Then g_1, g_2, g_3 satisfy the hypotheses of Proposition 4.1.3 if and only if

$$m_{123} > \max\{|l(g_1 g_2) - l(g_3)|, |l(g_3 g_1) - l(g_2)|, |l(g_2 g_3) - l(g_1)|\}. \qquad (*)$$

Proof. Firstly, recall from Corollary 3.3.2 that condition (ii) implies that each pair of axes γ_i and γ_j either do not intersect and $l(g_ig_j) = l(g_i^{-1}g_j) = l(g_i) + l(g_j) + 2d(\gamma_i, \gamma_j)$, or intersect along a path of length $\Delta(\gamma_i, \gamma_j) < \min\{l(g_i), l(g_j)\}$. If the axes γ_i and γ_j intersect with opposite orientations, then $l(g_ig_j) = l(g_i) + l(g_j) - 2\Delta(\gamma_i, \gamma_j)$, and $l(g_ig_j) = l(g_i) + l(g_j)$ otherwise.

We split the proof of the theorem into five cases, depending on how the axes γ_1, γ_2 and γ_3 interact. In each case, we use Proposition 3.2.1 to choose relative orientations of the axes such that $g_1g_2g_3$ has minimal translation length among all products of the pairs of elements $(g_1^{\pm 1}g_2^{\pm 1}, g_3)$ and $(g_2^{\pm 1}g_1^{\pm 1}, g_3)$, as this ensures that $m_{123} = l(g_1g_2g_3)$.

In the first case, suppose that none of the axes intersect. If the shortest path between each pair of axes does not intersect the remaining axis, then it follows from Figure 3.1 (applied to g_1 and g_2) that the axis of any product of the form $g_1^{\pm 1}g_2^{\pm 1}$ or $g_2^{\pm 1}g_1^{\pm 1}$ is the same distance $k = d(\gamma_1, \gamma_3) + d(\gamma_2, \gamma_3) - d(\gamma_1, \gamma_2)$ from γ_3; see the left-hand figure of Figure 4.1. Note that g_1, g_2, g_3 satisfy the hypotheses of Proposition 4.1.3, and applying Proposition 3.2.1 (1) to the elements g_1g_2 and g_3 gives

$$m_{123} = l(g_1g_2) + l(g_3) + 2k.$$

Thus $m_{123} > |l(g_1g_2) - l(g_3)|$ and, by symmetry, condition $(*)$ holds.

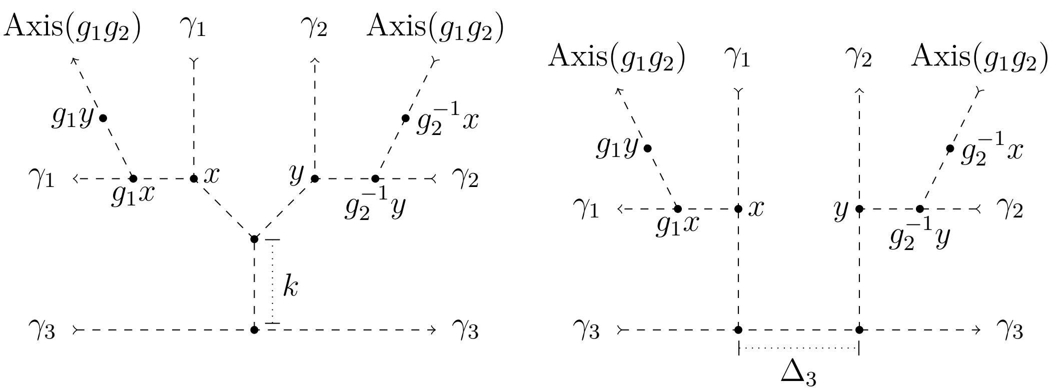

Figure 4.1: None of the axes γ_1, γ_2 and γ_3 intersect.

Therefore we may suppose, without loss of generality, that the shortest path between γ_1 and γ_2 intersects γ_3. It follows from Figure 3.1 (applied to g_1 and g_2) that the axis of any product of the form $g_1^{\pm 1}g_2^{\pm 1}$ or $g_2^{\pm 1}g_1^{\pm 1}$ intersects γ_3 along the same subpath. If we additionally suppose that g_3 translates γ_1 towards γ_2, then the axis of any product of the form $g_1^{\pm 1}g_2^{\pm 1}$ intersects γ_3 with opposite orientations, and this ensures that $m_{123} = l(g_1g_2g_3)$; see the right-hand diagram of Figure 4.1. Now consider the length Δ_3 of the path $\mathrm{Axis}(g_1g_2) \cap \gamma_3$, and note that $\Delta_3 < l(g_1g_2)$. Observe that g_1, g_2, g_3 satisfy the hypotheses of Proposition 4.1.3 if and only if $\Delta_3 < l(g_3)$. If $\Delta_3 < l(g_3)$, then applying Proposition 3.2.1 (3)(i) to the elements g_1g_2 and g_3 gives

$$\begin{aligned} m_{123} &= l(g_1g_2) + l(g_3) - 2\Delta_3 \\ &= l(g_3g_1) + l(g_2) + 2d(\gamma_2, \gamma_3) \\ &= l(g_2g_3) + l(g_1) + 2d(\gamma_1, \gamma_3), \end{aligned}$$

and it follows that condition ($*$) holds. On the other hand, if $\Delta_3 \geq l(g_3)$, then Proposition 3.2.1 (3)(ii) and (3)(iii) (applied to g_1g_2 and g_3) give $m_{123} \leq |l(g_1g_2) - l(g_3)|$, whereby ($*$) cannot hold.

In the second case, we suppose that exactly two of the axes γ_1, γ_2 and γ_3 intersect; without loss of generality, this is γ_1 and γ_3. If the vertex $\mathrm{Proj}_{\gamma_3}(\gamma_2)$ lies on $\gamma_1 \cap \gamma_3$, then it bisects $\gamma_1 \cap \gamma_3$ into two subpaths and we may further assume that g_1 translates the shorter of these subpaths towards the longer one. It follows from Figure 3.1 (applied to g_1 and g_2) that, among all products of the form $g_1^{\pm 1}g_2^{\pm 1}$ and $g_2^{\pm 1}g_1^{\pm 1}$, the axes of $g_1g_2^{\pm 1}$ and $g_2^{\pm 1}g_1^{-1}$ intersect γ_3 along the longest possible subpath. If, in addition, γ_1 and γ_3 are oppositely oriented, then this ensures that $m_{123} = l(g_1g_2g_3)$; see the left-hand diagram of Figure 4.2. It is clear that g_1, g_2, g_3 satisfy the hypotheses of Proposition 4.1.3, since $\Delta(\gamma_1, \gamma_3) < \min\{l(g_1), l(g_3)\}$. Moreover, if δ_3 denotes the length of the path $\mathrm{Axis}(g_1g_2) \cap \gamma_3$, then $\delta_3 \leq \Delta(\gamma_1, \gamma_3) < \min\{l(g_1g_2), l(g_3)\}$ and applying

Proposition 3.2.1 (3)(i) to the elements g_1g_2 and g_3 gives

$$\begin{aligned} m_{123} &= l(g_1g_2) + l(g_3) - 2\delta_3 \\ &= l(g_3g_1) + l(g_2) + 2d(\gamma_1, \gamma_2) + 2\Delta(\gamma_1, \gamma_3) - 2\delta_3 \\ &= l(g_2g_3) + l(g_1) - 2\delta_3. \end{aligned}$$

Since $\delta_3 < l(g_2g_3)$, it follows that condition $(*)$ holds.

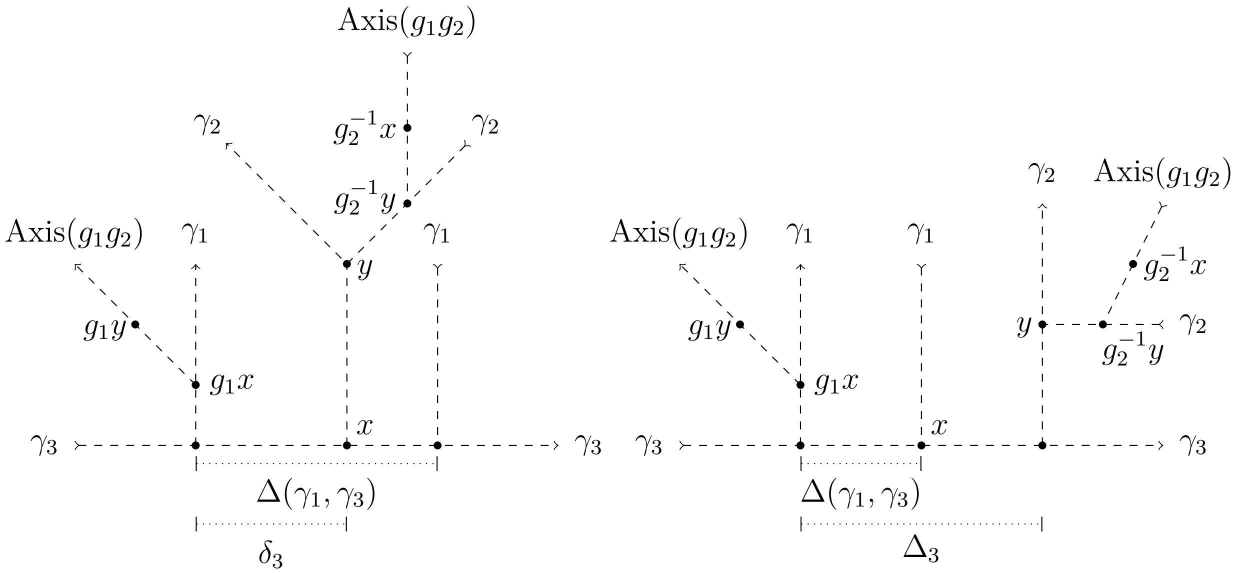

Figure 4.2: One pair of the axes γ_1, γ_2 and γ_3 intersect.

Therefore we may suppose that the vertex $\mathrm{Proj}_{\gamma_3}(\gamma_2)$ does not lie on $\gamma_1 \cap \gamma_3$. We may also assume that g_3 translates γ_1 towards γ_2 and that γ_1 and γ_3 are oppositely oriented; see the right-hand diagram of Figure 4.2. It follows from Figure 3.1 (applied to g_1 and g_2) that, among all products of the form $g_1^{\pm 1}g_2^{\pm 1}$ and $g_2^{\pm 1}g_1^{\pm 1}$, the axes of g_1g_2 and $g_1g_2^{-1}$ intersect γ_3 with opposite orientations along the longest possible subpath, and this ensures that $m_{123} = l(g_1g_2g_3)$. Consider the length Δ_3 of the path $\mathrm{Axis}(g_1g_2) \cap \gamma_3$, and note that $\Delta_3 < l(g_1g_2)$. Observe that g_1, g_2, g_3 satisfy the hypotheses of Proposition 4.1.3 if and only if $\Delta_3 < l(g_3)$. If $\Delta_3 < l(g_3)$, then applying

Proposition 3.2.1 (3)(i) to the elements g_1g_2 and g_3 gives

$$\begin{aligned} m_{123} &= l(g_1g_2) + l(g_3) - 2\Delta_3 \\ &= l(g_3g_1) + l(g_2) + 2d(\gamma_2, \gamma_3) \\ &= l(g_2g_3) + l(g_1) - 2\Delta(\gamma_1, \gamma_3), \end{aligned}$$

and since $\Delta(\gamma_1, \gamma_3) < \min\{l(g_1), l(g_2g_3)\}$, it follows that condition $(*)$ holds. On the other hand, if $\Delta_3 \geq l(g_3)$, then Proposition 3.2.1 (3)(ii) and (3)(iii) (applied to g_1g_2 and g_3) give $m_{123} \leq |l(g_1g_2) - l(g_3)|$, whereby $(*)$ cannot hold.

In the third case, we suppose that exactly two pairs of the three axes γ_1, γ_2 and γ_3 intersect. Without loss of generality, we can assume that γ_3 intersects both γ_1 and γ_2. Suppose additionally that g_3 translates γ_1 towards γ_2, and that γ_3 intersects both of these axes with opposite orientations; see Figure 4.3. It follows from Figure 3.1 (applied to g_1 and g_2) that, among all products of the form $g_1^{\pm1}g_2^{\pm1}$ and $g_2^{\pm1}g_1^{\pm1}$, the axis of g_1g_2 intersects γ_3 with opposite orientations along the longest possible subpath, and this ensures that $m_{123} = l(g_1g_2g_3)$. Again consider the length Δ_3 of the path $\mathrm{Axis}(g_1g_2) \cap \gamma_3$, and note that $\Delta_3 < l(g_1g_2)$. Observe that g_1, g_2, g_3 satisfy the hypotheses of Proposition 4.1.3 if and only if $\Delta_3 < l(g_3)$. If $\Delta_3 < l(g_3)$, then applying Proposition 3.2.1 (3)(i) to the elements g_1g_2 and g_3 gives

$$\begin{aligned} m_{123} &= l(g_1g_2) + l(g_3) - 2\Delta_3 \\ &= l(g_3g_1) + l(g_2) - 2\Delta(\gamma_2, \gamma_3) \\ &= l(g_2g_3) + l(g_1) - 2\Delta(\gamma_1, \gamma_3), \end{aligned}$$

and since $\Delta(\gamma_2, \gamma_3) < \Delta_3 - \Delta(\gamma_1, \gamma_3) < l(g_3g_1)$ and $\Delta(\gamma_1, \gamma_3) < \Delta_3 - \Delta(\gamma_2, \gamma_3) < l(g_2g_3)$, it follows that condition $(*)$ holds. If, however, $\Delta_3 \geq l(g_3)$, then Proposition 3.2.1 (3)(ii) and (3)(iii) (applied to g_1g_2 and g_3) give $m_{123} \leq |l(g_1g_2) - l(g_3)|$, whereby $(*)$ cannot hold.

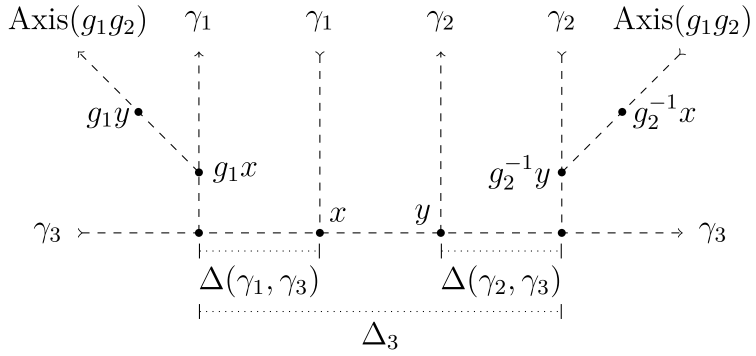

Figure 4.3: Two pairs of the axes γ_1, γ_2 and γ_3 intersect.

Finally, we consider the situation where all three pairs of the axes γ_1, γ_2 and γ_3 intersect. By Helley's Theorem, $\gamma_1 \cap \gamma_2 \cap \gamma_3 \neq \varnothing$. This leads to two possible cases, depending on whether or not one of the axes contains all three of the paths $\gamma_1 \cap \gamma_2, \gamma_1 \cap \gamma_3$ and $\gamma_2 \cap \gamma_3$.

In the fourth case, we suppose that one axis contains each of $\gamma_1 \cap \gamma_2, \gamma_1 \cap \gamma_3$ and $\gamma_2 \cap \gamma_3$; without loss of generality, we can assume that this axis is γ_3 and that γ_1 and γ_2 are both oppositely oriented to γ_3. Note that (as opposed to the previous cases) γ_1 and γ_2 agree in orientation, so $l(g_1g_2g_3)$ is at most $2\Delta(\gamma_1, \gamma_2)$ larger than it would be if the orientation of either γ_1 or γ_2 was reversed. If we were to reverse the orientation of either γ_1 or γ_2, then (by inspecting the right-hand diagram of Figure 3.2 and Figure 3.3) the overlap between the axis of g_1g_2 and γ_3 would be decreased by at least $\Delta(\gamma_1, \gamma_2)$, and consequently $l(g_1g_2g_3)$ would be increased by at least $2\Delta(\gamma_1, \gamma_2)$. Therefore, choosing the orientations of γ_1 and γ_2 so that they disagree with the orientation of γ_3, but agree with each other, helps to ensure that $m_{123} = l(g_1g_2g_3)$.

Let us first consider the subcase where one of the paths $\gamma_1 \cap \gamma_3$ or $\gamma_2 \cap \gamma_3$ is contained in the other. Without loss of generality, we may suppose that $\gamma_1 \cap \gamma_3$ contains $\gamma_2 \cap \gamma_3$ (and hence also $\gamma_1 \cap \gamma_2$); see the top diagram of Figure 4.4. Then $(\gamma_1 \cap \gamma_3) \backslash (\gamma_2 \cap \gamma_3)$ is the disjoint union of two subpaths and we may further assume that g_1 translates the shorter of these subpaths towards the longer one. It follows from the right-hand diagram of Figure 3.2 (applied to g_1 and g_2) that, among all products of the form $g_1^{\pm 1} g_2^{\pm 1}$ and $g_2^{\pm 1} g_1^{\pm 1}$, the axis of g_1g_2 intersects γ_3 with opposite orientations along the longest possible subpath, and this ensures that $m_{123} = l(g_1g_2g_3)$. Moreover, g_1, g_2, g_3 satisfy the hypotheses of Proposition 4.1.3 since $\Delta(\gamma_1,\gamma_2) = \Delta(\gamma_2,\gamma_3) < \min\{l(g_1), l(g_2), l(g_3)\}$ and $\Delta(\gamma_1,\gamma_3) < \min\{l(g_1), l(g_3)\}$. Now consider the length δ_3 of the path $\mathrm{Axis}(g_1g_2) \cap \gamma_3$, and note that $\delta_3 \leq \Delta(\gamma_1,\gamma_3) < \min\{l(g_1g_2), l(g_3)\}$. Applying Proposition 3.2.1 (3)(i) to the elements g_1g_2 and g_3 gives

$$\begin{aligned} m_{123} &= l(g_1g_2) + l(g_3) - 2\delta_3 \\ &= l(g_3g_1) + l(g_2) + 2\Delta(\gamma_1,\gamma_3) - 2\delta_3 \\ &= l(g_2g_3) + l(g_1) + 2\Delta(\gamma_2,\gamma_3) - 2\delta_3. \end{aligned}$$

Since $\delta_3 - \Delta(\gamma_2,\gamma_3) < l(g_3) - \Delta(\gamma_2,\gamma_3) < l(g_2g_3)$, it follows that condition $(*)$ holds.

In the other subcase, we assume that neither $\gamma_1 \cap \gamma_3$ nor $\gamma_2 \cap \gamma_3$ contains the other. We can further assume that g_3 translates γ_1 towards γ_2; see the bottom diagram of Figure 4.4. It follows from the right-hand diagram of Figure 3.2 (applied to g_1 and g_2) that, among all products of the form $g_1^{\pm1}g_2^{\pm1}$ and $g_2^{\pm1}g_1^{\pm1}$, the axis of g_1g_2 intersects γ_3 with opposite orientations along the longest possible subpath, and this ensures that $m_{123} = l(g_1g_2g_3)$. Consider the length Δ_3 of the path $\mathrm{Axis}(g_1g_2) \cap \gamma_3$, and note that $\Delta_3 < l(g_1g_2)$. Observe that g_1, g_2, g_3 satisfy the hypotheses of Proposition 4.1.3 if and only if $\Delta_3 < l(g_3)$. If $\Delta_3 < l(g_3)$, then applying Proposition 3.2.1 (3)(i) to the elements g_1g_2 and g_3 gives

$$\begin{aligned} m_{123} &= l(g_1g_2) + l(g_3) - 2\Delta_3 \\ &= l(g_3g_1) + l(g_2) + 2\Delta(\gamma_1,\gamma_2) - 2\Delta(\gamma_2,\gamma_3) \\ &= l(g_2g_3) + l(g_1) + 2\Delta(\gamma_1,\gamma_2) - 2\Delta(\gamma_1,\gamma_3). \end{aligned}$$

Since $\Delta(\gamma_2,\gamma_3) - \Delta(\gamma_1,\gamma_2) = \Delta_3 - \Delta(\gamma_1,\gamma_3) < l(g_3) - \Delta(\gamma_1,\gamma_3) < l(g_3g_1)$, and similarly $\Delta(\gamma_1,\gamma_3) - \Delta(\gamma_1,\gamma_2) < l(g_2g_3)$, it follows that condition $(*)$ holds. If, however, $\Delta_3 \geq l(g_3)$, then Proposition 3.2.1 (3)(ii) and (3)(iii) (applied to g_1g_2 and g_3) give $m_{123} \leq |l(g_1g_2) - l(g_3)|$, whereby $(*)$ cannot hold.

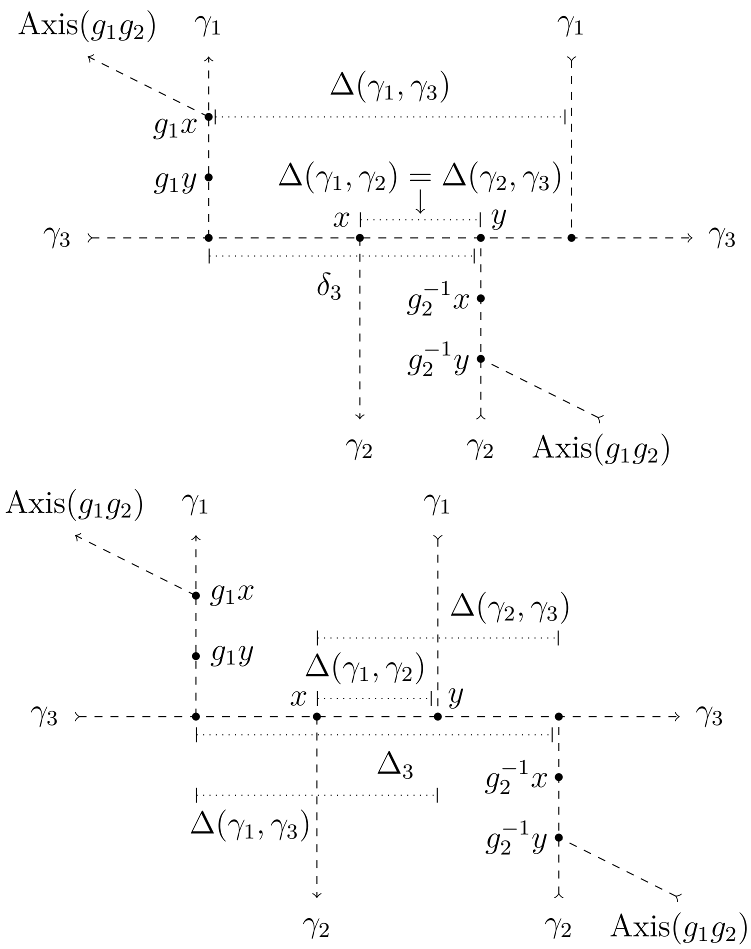

Figure 4.4: All three axes γ_1, γ_2 and γ_3 intersect, and the axis γ_3 contains each of $\gamma_1 \cap \gamma_2$, $\gamma_1 \cap \gamma_3$ and $\gamma_2 \cap \gamma_3$.

For the fifth and final case, we suppose that all three axes γ_1, γ_2 and γ_3 intersect with each other, but none of them contains each of the paths $\gamma_1 \cap \gamma_2, \gamma_1 \cap \gamma_3$ and $\gamma_2 \cap \gamma_3$. Denote the length of $\mathrm{Proj}_{\gamma_1}(\gamma_2) \cup \mathrm{Proj}_{\gamma_1}(\gamma_3)$ by $\Delta_1 = \Delta(\gamma_1, \gamma_2) + \Delta(\gamma_1, \gamma_3)$. Similarly, denote the length of $\mathrm{Proj}_{\gamma_2}(\gamma_1) \cup \mathrm{Proj}_{\gamma_2}(\gamma_3)$ by $\Delta_2 = \Delta(\gamma_1, \gamma_2) + \Delta(\gamma_2, \gamma_3)$, and the length of $\mathrm{Proj}_{\gamma_3}(\gamma_1) \cup \mathrm{Proj}_{\gamma_3}(\gamma_2)$ by $\Delta_3 = \Delta(\gamma_1, \gamma_3) + \Delta(\gamma_2, \gamma_3)$. Note that $\gamma_1 \cap \gamma_2 \cap \gamma_3$ is a single vertex and $\Delta(\gamma_1, \gamma_2), \Delta(\gamma_1, \gamma_3), \Delta(\gamma_2, \gamma_3) > 0$, for otherwise we would be in the previous case.

Suppose that all three axes are oppositely oriented, and that g_3 translates γ_1 towards γ_2. It follows from Figure 3.3 (applied to g_1 and g_2) that, among all products of the form $g_1^{\pm 1} g_2^{\pm 1}$ and $g_2^{\pm 1} g_1^{\pm 1}$, the axis of $g_1 g_2$ intersects γ_3 with opposite orientations along the longest possible subpath, and this ensures that $m_{123} = l(g_1g_2g_3)$. Observe that g_1, g_2, g_3 satisfy the hypotheses of Proposition 4.1.3 if and only if $\Delta_r < l(g_r)$ for all $r \in \{1, 2, 3\}$; see the top diagram of Figure 4.5, under the additional assumption that $\Delta_3 < l(g_3)$. If this does occur, then $\mathrm{Axis}(g_1g_2) \cap \gamma_3$ has length precisely Δ_3, where $\Delta_3 < l(g_1g_2)$. Applying Proposition 3.2.1 (3)(i) to the elements g_1g_2 and g_3 then gives

$$m_{123} = l(g_1g_2) + l(g_3) - 2\Delta_3.$$

Thus $m_{123} > |l(g_1g_2) - l(g_3)|$ and, by symmetry, condition $(*)$ holds.

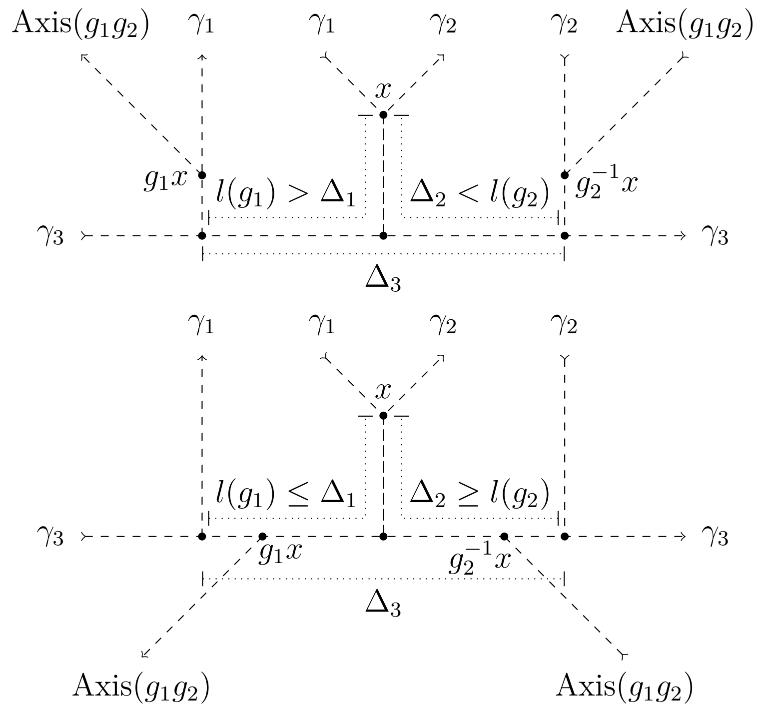

Figure 4.5: The three axes γ_1, γ_2 and γ_3 intersect with each other, but with none of them containing each of $\gamma_1 \cap \gamma_2$, $\gamma_1 \cap \gamma_3$ and $\gamma_2 \cap \gamma_3$.

On the other hand, suppose that $\Delta_r \geq l(g_r)$ for some $r \in \{1, 2, 3\}$. If $\Delta_r \geq l(g_r)$ holds for precisely one value of $r \in \{1, 2, 3\}$, then without loss of generality let us suppose that it holds for $r = 3$ only; see the top diagram of Figure 4.5, under the additional assumption that $\Delta_3 \geq l(g_3)$. Here $\mathrm{Axis}(g_1g_2) \cap \gamma_3$ has length precisely Δ_3, so it follows from Proposition 3.2.1 (3)(ii) and (3)(iii) (applied to g_1g_2 and g_3) that

$m_{123} \le |l(g_1g_2) - l(g_3)|$ and hence condition $(*)$ does not hold. If $\Delta_r \ge l(g_r)$ holds for at least two values of $r \in \{1,2,3\}$, then without loss of generality let us suppose that it holds for $r = 1$ and 2; see the bottom diagram of Figure 4.5. Here $\mathrm{Axis}(g_1g_2) \cap \gamma_3$ has length (at least) $l(g_1g_2)$, so it again follows from Proposition 3.2.1 (3)(ii) and (3)(iii) that condition $(*)$ does not hold. This proves the theorem. □

We conclude this section by showing that if we strengthen condition (ii) of Theorem 4.1.4, then the hypotheses of Proposition 4.1.3 are satisfied (and hence g_1, g_2, g_3 generate a discrete and free subgroup) in all but a small number of cases.

Corollary 4.1.5. *Let G be a metrisable topological group acting continuously by isometries and without inversions on a simplicial tree. Suppose that $g_1, g_2, g_3 \in G$ are hyperbolic elements which satisfy both of the following conditions:*

(i) $m_{123} = l(g_1g_2g_3)$;

(ii) $\max\{l(g_i), l(g_j)\} \le \min\{l(g_ig_j), l(g_i^{-1}g_j)\} \le \min\{l(g_kg_ig_k^{-1}g_j), l(g_kg_i^{-1}g_k^{-1}g_j)\}$ *for all distinct* $i, j, k \in \{1,2,3\}$.

Then precisely one of the following holds:

(a) g_1, g_2, g_3 *satisfy the hypotheses of Proposition 4.1.3 and* $H = \langle g_1, g_2, g_3 \rangle \le G$ *is discrete and free of rank three;*

(b) $m_{123} \le |l(g_ig_j) - l(g_k)|$ *for some* $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$ *and the axes* γ_i, γ_j *and* γ_k *interact as in Figure 4.6.*

Proof. First observe that the left-hand inequality of (ii) implies condition (ii) of Theorem 4.1.4. Moreover, by Proposition 3.2.1, the right-hand inequality of (ii) ensures that replacing g_i by $g_kg_ig_k^{-1}$ (which has the effect of replacing γ_i by $g_k \cdot \gamma_i$) does not strictly decrease the distance between γ_i and γ_j (when $\gamma_i \cap \gamma_j = \varnothing$) or does not strictly increase the length of $\gamma_i \cap \gamma_j$ (when $\gamma_i \cap \gamma_j \ne \varnothing$). We now consider the same five cases given in the proof of Theorem 4.1.4, but under these strengthened conditions. Note that it follows from Theorem 4.1.4 that conclusions (a) and (b) are mutually exclusive.

Let us first assume that at most two pairs of the axes γ_1, γ_2 and γ_3 intersect. If no two of the axes intersect and the shortest path between each pair of axes does not intersect the remaining axis, or if exactly two of the axes intersect and their path of intersection contains the closest point of both axes to the third axis, then it is clear that (a) holds; see the left-hand diagrams of Figure 4.1 and Figure 4.2.

Hence we can further assume that g_3 translates γ_1 towards γ_2 and, if γ_3 intersects either γ_1 or γ_2, then it does so with opposite orientations; see the right-hand diagrams of Figures 4.1 and 4.2, and Figure 4.3. Recall from the proof of Theorem 4.1.4 that this ensures $m_{123} = l(g_1g_2g_3)$. In each of these cases, if the length Δ_3 of the path $\mathrm{Axis}(g_1g_2) \cap \gamma_3$ is at least $l(g_3)$, then replacing g_1 by $g_3g_1g_3^{-1}$ either strictly reduces the distance between γ_1 and γ_2, or causes them to intersect when they did not previously. This is a contradiction to the right-hand inequality of (ii). Thus $\Delta_3 < l(g_3)$, and it follows that (a) holds in this case too.

We may now suppose that all of the axes γ_1, γ_2 and γ_3 intersect with each other. If one of the paths $\gamma_1 \cap \gamma_2, \gamma_1 \cap \gamma_3$ or $\gamma_2 \cap \gamma_3$ contains each of the other two, then (a) holds; see the top diagram of Figure 4.4. Otherwise, we can assume that g_3 translates γ_1 towards γ_2, and that γ_3 is oppositely oriented to both γ_1 and γ_2. Recall from the proof of Theorem 4.1.4 that this ensures $m_{123} = l(g_1g_2g_3)$. We consider two cases, depending on whether or not $\gamma_1 \cap \gamma_2$ is contained in γ_3.

In the former case, we assume that $\gamma_1 \cap \gamma_2$ is contained in γ_3; see the bottom diagram of Figure 4.4. If the length Δ_3 of the path $\mathrm{Axis}(g_1g_2) \cap \gamma_3$ satisfies $\Delta_3 < l(g_3)$, then (a) holds. Hence suppose that $\Delta_3 \geq l(g_3)$. Then $l(g_3) \leq 2\max\{\Delta(\gamma_1, \gamma_3), \Delta(\gamma_2, \gamma_3)\}$, with equality occurring if and only if $\Delta(\gamma_1, \gamma_3) = \Delta(\gamma_2, \gamma_3) = \frac{1}{2}l(g_3)$ and $\Delta(\gamma_1, \gamma_2) = 0$. If these equalities do not hold, then either $l(g_3g_1) = l(g_1) + l(g_3) - 2\Delta(\gamma_1, \gamma_3)$ is strictly less than $l(g_1)$, or $l(g_2g_3) = l(g_2) + l(g_3) - 2\Delta(\gamma_2, \gamma_3)$ is strictly less than $l(g_2)$, both of which contradict the left-hand inequality of (ii). Thus γ_1, γ_2 and γ_3 must interact as in the top diagram of Figure 4.6 with $(i, j, k) = (1, 2, 3)$. Note that $m_{123} \leq |l(g_ig_j) - l(g_k)|$, since $\mathrm{Axis}(g_ig_j) \cap \gamma_k$ has length at least $\Delta_k = l(g_k)$, and hence (b) holds.

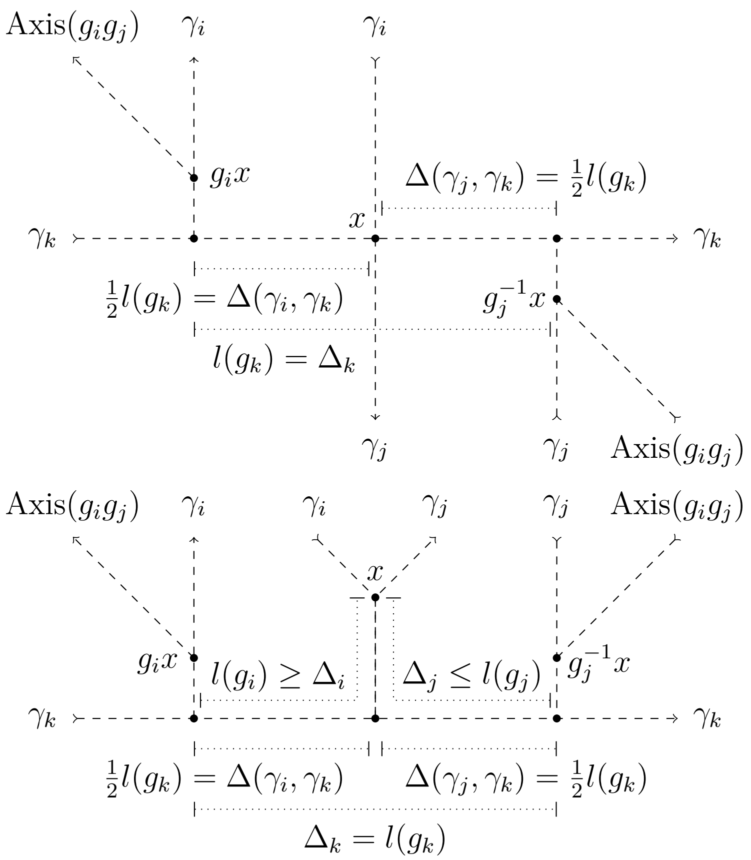

Figure 4.6: The two cases that may occur if conditions (i) and (ii) of Corollary 4.1.5 are satisfied, but $m_{123} \leq |l(g_ig_j) - l(g_k)|$ for some $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$.

In the latter case, we suppose that $\gamma_1 \cap \gamma_2$ is not contained in γ_3. Recall from the proof of Theorem 4.1.4 that we defined three lengths Δ_1, Δ_2 and Δ_3, and if $\Delta_r < l(g_r)$ for all $r \in \{1,2,3\}$, then (a) holds; see the top diagram of Figure 4.5, under the additional assumption that $\Delta_3 < l(g_3)$. Hence suppose without loss of generality that $\Delta_3 \geq l(g_3)$. Then $l(g_3) \leq 2\max\{\Delta(\gamma_1,\gamma_3), \Delta(\gamma_2,\gamma_3)\}$, with equality occurring if and only if $\Delta(\gamma_1,\gamma_3) = \Delta(\gamma_2,\gamma_3) = \frac{1}{2}l(g_3)$. As before, this gives a contradiction to the left-hand inequality of (ii) unless the specified equalities hold. This same argument holds for g_1 and g_2, therefore we must have $\Delta_1 \leq l(g_1), \Delta_2 \leq l(g_2)$ and $\Delta_3 = l(g_3)$, so γ_1, γ_2 and γ_3 interact as in the bottom diagram of Figure 4.6 with $(i,j,k) = (1,2,3)$. Note that $m_{123} \leq |l(g_ig_j) - l(g_k)|$, since $\mathrm{Axis}(g_ig_j) \cap \gamma_k$ has length (at least) $l(g_k)$, and hence (b) holds. This completes the proof. $\square$

4.2 Deciding whether a three-generated subgroup of $\mathrm{SL}_2(K)$ is discrete and free

We conclude this chapter by presenting a practical algorithm which, given any three-generated subgroup $G \leq \mathrm{SL}_2(K)$, determines after finitely many steps whether or not G is both discrete and free of rank three. As with Algorithm 3.3.3, the key idea is to perform Nielsen transformations in a 'translation length minimising' manner. In this case, this eventually gives an elliptic element, or three hyperbolic elements generating G and satisfying conditions (i) and (ii) of Corollary 4.1.5. If conclusion (b) of Corollary 4.1.5 holds for these three elements, then we demonstrate that further Nielsen transformations can be performed to ensure that the algorithm terminates.

Algorithm 4.2.1. *Let K be a non-archimedean local field. Given $A, B, C \in \mathrm{SL}_2(K)$, we proceed as follows: If $G = \langle A, B, C\rangle \leq \mathrm{SL}_2(K)$ is discrete and free of rank three, then the algorithm will return true and output a generating triple satisfying the hypotheses of the Ping Pong Lemma, and otherwise it will return false.*

(1) *Set $g_1 = A, g_2 = B$ and $g_3 = C$. If $l(g_i) = 0$ for some $i \in \{1, 2, 3\}$, then return false.*

(2) *For $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$ do the following:*

   (i) *Compute $m_{ij} = \min\{l(g_i g_j), l(g_i^{-1} g_j)\}$. If $m_{ij} = 0$, then return false.*

   (ii) *If $m_{ij} < \max\{l(g_i), l(g_j)\}$, then replace an element of $\{g_i, g_j\}$ with maximal translation length by the element of $\{g_i g_j, g_i^{-1} g_j\}$ that has translation length m_{ij} and return to (2).*

   (iii) *If $m_{ij} > \min\{l(g_k g_i g_k^{-1} g_j), l(g_k g_i^{-1} g_k^{-1} g_j)\}$ where $k \neq i, j$, then replace g_i by $g_k g_i g_k^{-1}$ and return to (2).*

   (iv) *If $m_{ij} > \min\{l(g_i g_k g_j g_k^{-1}), l(g_i^{-1} g_k g_j g_k^{-1})\}$ where $k \neq i, j$, then replace g_j by $g_k g_j g_k^{-1}$ and return to (2).*

(3) *Compute* $m_{123} = \min \left\{ \begin{array}{l} l(g_1g_2g_3), l(g_1^{-1}g_2g_3), l(g_1g_2^{-1}g_3), l(g_1^{-1}g_2^{-1}g_3), \\ l(g_2g_1g_3), l(g_2^{-1}g_1g_3), l(g_2g_1^{-1}g_3), l(g_2^{-1}g_1^{-1}g_3) \end{array} \right\}$.
If $m_{123} = 0$*, then return false.*

(4) *Relabel* g_1, g_2 *and* g_3 *so that* $m_{123} = l(g_1g_2g_3)$.

(5) *If* $m_{123} \leq |l(g_ig_j) - l(g_k)|$ *for some* $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$*, then replace* g_j *by* g_jg_k *and return to* (2).

(6) *Return true and the triple* (g_1, g_2, g_3).

Theorem 4.2.2. *Algorithm 4.2.1 terminates after finitely many steps and produces the correct output.*

Proof. We first prove that the algorithm is correct. If at any point the algorithm returns false, then G is not discrete and free of rank three by Corollary 3.1.9. Otherwise, step (2) performs Nielsen transformations to give a generating triple for G that satisfies condition (ii) of Corollary 4.1.5. If step (6) is reached, then g_1, g_2 and g_3 additionally satisfy $m_{123} = l(g_1g_2g_3)$ and conclusion (a) of Corollary 4.1.5 holds. Hence G is both discrete and free of rank three.

We now prove that the algorithm terminates after finitely many steps. At each step, we consider the integer sum

$$S_{123} = l(g_1) + l(g_2) + l(g_3).$$

Note that Step (2)(ii) strictly reduces S_{123}, and hence this step cannot be performed indefinitely. On the other hand, Steps (2)(iii) and (iv) replace an element by its conjugate, which does not change the value of S_{123}. However, if Step (2) iterated itself infinitely many times, then m_{ij} would eventually reach 0 and the algorithm would return false. Thus Step (2) terminates after finitely many steps, giving a generating triple for G which satisfies condition (ii) of Corollary 4.1.5.

The only other recursive step is Step (5). If $m_{123} \leq |l(g_ig_j) - l(g_k)|$ for some $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, then it follows from Corollary 4.1.5 (b) that the axes γ_i, γ_j

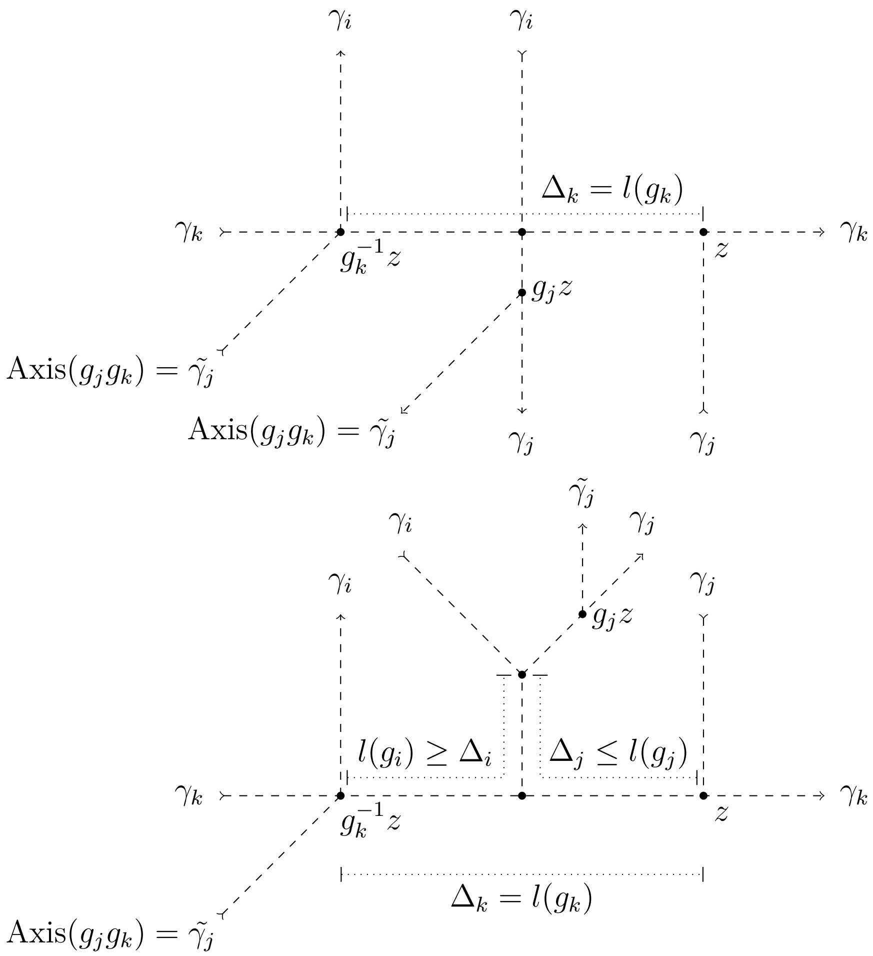

Figure 4.7: How the axis $\tilde{\gamma}_j = \mathrm{Axis}(g_jg_k)$ interacts with γ_i and γ_k in the situations depicted by Figure 4.6.

and γ_k interact as in one of the two diagrams in Figure 4.6. In either situation, replacing g_j by $\tilde{g}_j = g_jg_k$ (which has translation length $l(\tilde{g}_j) = l(g_j) + l(g_k) - 2\Delta(\gamma_j, \gamma_k) = l(g_j)$) does not change S_{123}, but changes the configuration of the axes. In particular, it follows from Figure 3.3 (applied to g_j and g_k) that $\tilde{\gamma}_j = \mathrm{Axis}(g_jg_k)$ interacts with γ_i and γ_k as depicted in one of the two diagrams in Figure 4.7.

In the top diagram of Figure 4.7 (in which γ_i and γ_j intersect at a single vertex), the right-hand inequality of Corollary 4.1.5 (*ii*) ensures that $\tilde{\gamma}_j$ cannot intersect γ_i anywhere other than $\gamma_i \cap \gamma_k$, for otherwise replacing g_i by $g_kg_ig_k^{-1}$ causes γ_i and γ_j to intersect along a non-trivial path. Hence the axes $\gamma_i, \tilde{\gamma}_j$ and γ_k satisfy the hypotheses of Proposition 4.1.3, and one further iteration of Algorithm 4.2.1 would return true.

In the bottom diagram of Figure 4.7, it is possible for $\tilde{\gamma}_j$ to intersect γ_i further along (with respect to the direction of translation) than $\gamma_i \cap \gamma_k$, and so the new triple $(g_i, \tilde{g}_j, g_k)$ might not necessarily satisfy the hypotheses of Proposition 4.1.3. If this new triple does satisfy the hypotheses of Proposition 4.1.3, then one further iteration of Algorithm 4.2.1 will return true. Otherwise $\gamma_i \cap \tilde{\gamma}_j$ is too large: it must be at least $\min\{l(g_i), l(\tilde{g}_j)\}$, and hence $\min\{l(g_i\tilde{g}_j), l(g_i^{-1}\tilde{g}_j)\} \leq |l(g_i) - l(\tilde{g}_j)| < \max\{l(g_i), l(\tilde{g}_j)\}$. Therefore the sum S_{123} will be strictly decreased on the next iteration of step $(2)(ii)$, and so the algorithm must eventually terminate. □

Note that if a three-generated subgroup $G \leq \mathrm{SL}_2(K)$ is not free, then (by keeping track of the Nielsen transformations performed at each step) Algorithm 4.2.1 can be used to explicitly find an elliptic element as a word in the generators of G. This gives a constructive method of deciding between the two possible outcomes of Theorem 4.1.1, in the case that $n = 3$ and $g_1, g_2, g_3 \in \mathrm{SL}_2(K)$.

By the same reasoning as detailed in Chapter 3, Algorithm 4.2.1 can also be applied to three-generated subgroups of $\mathrm{PSL}_2(K)$, by taking representatives of the generators in $\mathrm{SL}_2(K)$. Furthermore, Algorithm 4.2.1 can be implemented in a computational package such as MAGMA, so long as elements of K are stored in terms of the data $\{\pi; a_N, a_{N+1}, \ldots, a_M\}$ up to some appropriate integer M, and the number of enumerations of the algorithm is closely monitored; this follows from a similar argument to the discussion at the end of Chapter 3.

The author expects that the techniques used in this chapter should generalise to give an algorithm that can decide whether or not any finitely-generated subgroup of $\mathrm{SL}_2(K)$ is both discrete and free. If G is generated by elements $g_1, \ldots, g_n \in \mathrm{SL}_2(K)$ (for some $n \geq 4$), and every triple (g_i, g_j, g_k) of generators satisfies the hypotheses of Proposition 4.1.3, then G is both discrete and free of rank n. At this stage, however, some further analysis is required to consider the potential effects of the Nielsen transformations specified in step (5) of Algorithm 4.2.1 on the interaction with axes not involved in the relevant triple (g_i, g_j, g_k).

We conclude this chapter by giving some examples to illustrate Algorithm 4.2.1. Recall from Chapter 1 that, for $\alpha, \beta, \gamma \in \mathbb{C}$, the subgroup

$$F_{\alpha,\beta,\gamma} = \left\langle \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}, \begin{bmatrix} 1-\gamma & -\gamma \\ \gamma & 1+\gamma \end{bmatrix} \right\rangle$$

of $\mathrm{SL}_2(\mathbb{C})$ is free of rank three whenever $|\alpha|, |\beta|, |\gamma| \geq 3$. If one instead views these generating matrices as elements of $\mathrm{SL}_2(\mathbb{Q}_p)$ (with $\alpha, \beta, \gamma \in \mathbb{Q}_p$), then the corresponding subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ is not both discrete and free as each generating matrix is elliptic; see the discussion following Algorithm 3.3.3. Thus any of these matrices would return false at step (1) of Algorithm 4.2.1.

On the other hand, recall the matrices

$$X = \begin{bmatrix} 7^3 & 0 \\ 0 & \frac{1}{7^3} \end{bmatrix} \text{ and } Y = \begin{bmatrix} \frac{2}{7^7} & 7^3 \\ \frac{1}{7^3} & 7^7 \end{bmatrix}$$

from the discussion following Proposition 3.2.1 (or set $p = 7$ and $r = 2$ in the example at the end of Chapter 3). Then Algorithm 3.3.3 shows that $H = \langle X, Y \rangle \leq \mathrm{SL}_2(\mathbb{Q}_7)$ is both discrete and free of rank two. Moreover, if $A = XY, B = X^2Y^2$ and $C = X^3Y^3$, then the subgroup $G = \langle A, B, C \rangle \leq H$ is both discrete and free of rank three. We will show how this can be verified by Algorithm 4.2.1.

Indeed, the first step of Algorithm 4.2.1 sets $g_1 = A, g_2 = B$ and $g_3 = C$. Since

$$A = \begin{bmatrix} \frac{2}{7^4} & 7^6 \\ \frac{1}{7^6} & 7^4 \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{4}{7^8} + 7^6 & 2 \cdot 7^2 + 7^{16} \\ \frac{2}{7^{16}} + \frac{1}{7^2} & \frac{1}{7^6} + 7^8 \end{bmatrix},$$

we have $l(g_1) = 8, l(g_2) = 16$ and $m_{12} = l(g_1^{-1}g_2) = 8$, so step (2)($ii$) of Algorithm 4.2.1 replaces g_2 by $g_1^{-1}g_2$. This gives $m_{12} = l(g_1^{-1}g_2) = 12$, which is not reduced by conjugating either g_1 or g_2 by g_3. This completes step (2) for $(i, j) = (1, 2)$, and at this point $(g_1, g_2, g_3) = (A, A^{-1}B, C)$.

We next consider the pair (g_1, g_3), where $l(g_1) = 8, l(g_3) = 32$ and $m_{13} = l(g_1^{-1}g_3) = 16$. On the next iteration of Algorithm 4.2.1, step (2)(ii) then replaces g_3 by $g_1^{-1}g_3$. This gives $m_{13} = l(g_1g_3) = l(g_1^{-1}g_3) = 32$, but conjugating g_3 by g_2 reduces this value and so g_3 is subsequently replaced by $g_2g_3g_2^{-1}$. This completes step (2) for $(i, j) = (1, 3)$, and at this point $(g_1, g_2, g_3) = (A, A^{-1}B, A^{-1}BA^{-1}CB^{-1}A)$.

Finally, we consider step (2) for the pair (g_2, g_3). Note that $l(g_2) = 8, l(g_3) = 16$ and $m_{23} = l(g_2^{-1}g_3) = 8$. Here, step (2)$(ii)$ replaces g_3 by $g_2^{-1}g_3$ and it follows that $m_{23} = l(g_2^{-1}g_3) = 12$, which is not reduced by conjugating either g_2 or g_3 by g_1. This gives $(g_1, g_2, g_3) = (A, A^{-1}B, A^{-1}CB^{-1}A)$. However, this replacement alters the interaction between g_1 and g_3: $l(g_1) = l(g_3) = 8$ and $m_{13} = l(g_1^{-1}g_3) = 24$, but conjugating g_3 by g_2 reduces the value of m_{13}. Hence we make one final replacement of g_3 by $g_2g_3g_2^{-1}$, giving $(g_1, g_2, g_3) = (A, A^{-1}B, A^{-1}BA^{-1}CB^{-1}AB^{-1}A)$ as a triple of elements that generates G and satisfies condition (ii) of Corollary 4.1.5.

Moreover, $l(g_1) = l(g_2) = l(g_3) = 8$, $m_{12} = l(g_1^{-1}g_2) = m_{23} = l(g_2^{-1}g_3) = 12$ and $m_{13} = l(g_1g_3) = l(g_1^{-1}g_3) = 16$. By Proposition 3.2.1, this implies that $\gamma_1 \cap \gamma_3$ is a single vertex, while $\gamma_1 \cap \gamma_2$ and $\gamma_2 \cap \gamma_3$ are both paths of length two. Such a configuration satisfies the hypotheses of Proposition 4.1.3, and hence the Ping Pong Lemma. This is detected in the final steps of the algorithm: replacing g_2 by g_2^{-1} gives $m_{123} = l(g_1g_2g_3) = 16$, which is larger than $|l(g_1g_2) - l(g_3)| = |l(g_2g_3) - l(g_1)| = 4$ and $|l(g_3g_1) - l(g_2)| = 8$. Thus the algorithm reaches step (6) and returns true.

Chapter 5

Generalisations and applications

In this final chapter, we discuss some generalisations and applications of both Algorithm 3.3.3 and Algorithm 4.2.1. We show that both these algorithms hold more generally in the context of two or three-generated subgroups of the isometry group of a locally finite simplicial tree, when equipped with the topology of pointwise convergence and a method of computing translation lengths. We also discuss applications of these algorithms to the constructive membership problem.

5.1 Isometry groups of locally finite simplicial trees

Given any proper metric space X (for instance, a locally finite simplicial tree), the isometry group $\mathrm{Isom}(X)$ (viewed as a subspace of X^X, the space of all continuous maps $X \to X$ equipped with the product topology) is a metrisable topological group; see [15, Lemmas 5.B.3 and 5.B.5]. This topology is often known as the *topology of pointwise convergence*, in the sense that a sequence (f_i) in $\mathrm{Isom}(X)$ converges to $f \in \mathrm{Isom}(X)$ if and only if the sequence $(f_i(x))$ converges to $f(x)$ for each $x \in X$. Note that the group $\mathrm{PSL}_2(K)$, as a subgroup of the isometry group $\mathrm{PGL}_2(K)$ of the corresponding Bruhat-Tits tree, inherits the topology of pointwise convergence, and this coincides with the standard topology on $\mathrm{PSL}_2(K)$ used in this thesis (that is, the quotient topology inherited from $\mathrm{SL}_2(K)$).

In the setting of isometry groups, the topology of pointwise convergence is equivalent to the well-known *compact-open topology*; see [15, Lemmas 5.B.1 and 5.B.2]. The pointwise convergence property of these equivalent topologies leads to an analogue of Corollary 3.1.9 for the isometry group of a locally finite simplicial tree. By subdividing each edge of such a tree at its midpoint, if necessary, every element of the corresponding isometry group can be assumed to act without inversions.

Proposition 5.1.1. *Let T be a locally finite simplicial tree, and suppose that the subgroup $G \leq \mathrm{Isom}(T)$ is both discrete (with respect to the topology of pointwise convergence) and free. Then G contains no elliptic elements.*

Proof. Suppose that G contains some elliptic element g, which fixes a vertex p of T. There are only finitely many vertices adjacent to p, and g acts to permute these. This implies that there is some integer n_1 for which g^{n_1} fixes p and all of its adjacent vertices. One continues inductively to obtain a sequence (g^{n_i}) of elements of $\mathrm{Isom}(T)$, where g^{n_i} fixes all vertices at distance at most i from p. But then $(g^{n_i}(x))$ converges to x for each vertex x of T, and so (g^{n_i}) converges to the identity. Thus either g has finite order, or G is not discrete. □

For any proper metric space X, the natural map $\mathrm{Isom}(X) \times X \to X$ (given by $(g, x) \mapsto gx$) is continuous; see [15, Lemma 5.B.4 (2)]. This implies that Corollary 3.3.2, Theorem 4.1.4 and Corollary 4.1.5 can also be applied to the isometry group of a locally finite simplicial tree, when equipped with the topology of pointwise convergence. Thus we have the following generalisation of Algorithm 3.3.3 and Algorithm 4.2.1:

Algorithm 5.1.2. *Let T be a locally finite simplicial tree, and let $\mathrm{Isom}(T)$ be its isometry group, equipped with the topology of pointwise convergence and a method of computing translation lengths. Given $A, B \in \mathrm{Isom}(T)$ (respectively $A, B, C \in \mathrm{Isom}(T)$), we proceed through steps (1) to (6) of Algorithm 3.3.3 (respectively Algorithm 4.2.1). If $G = \langle A, B\rangle \leq \mathrm{Isom}(T)$ (respectively $G = \langle A, B, C\rangle \leq \mathrm{Isom}(T)$) is discrete and free of rank two (respectively three), then the algorithm will return true and output* *a generating set for G which satisfies the hypotheses of the Ping Pong Lemma, and otherwise it will return false.*

Theorem 5.1.3. *Algorithm 5.1.2 terminates after finitely many steps and produces the correct output.*

Proof. The only difference from the proofs of Theorem 3.3.4 and Theorem 4.2.2 is that, if the algorithm encounters an elliptic element, then G cannot be both discrete and free by Proposition 5.1.1, instead of Corollary 3.1.9. □

Algorithm 5.1.2 can be applied, for instance, to certain amalgamated free products. Suppose that $\Gamma = H *_C K$ is the amalgamated free product of groups H and K over some subgroup C which has finite index in both H and K. It is well-known that, given fixed transversals T_H and T_K of right coset representatives of C in H and K respectively, each element $g \in \Gamma$ has a unique normal form

$$g = cx_1 \dots x_n$$

for some integer $n \geq 0$, where $c \in C$ and, for each $i \geq 1$, either $x_i \in T_H$ and $x_{i+1} \in T_K$, or vice versa. Moreover, Γ acts faithfully by isometries and without inversions on a locally finite simplicial tree T, with vertices given by cosets of the form gH or gK, and edges given by cosets gC (where $g \in \Gamma$); see [53, Chapter I, Section 4].

Consider the shortest normal form $cx_1 \dots x_{n_0}$ of all conjugates of g in Γ. Such a form is *cyclically reduced* in the sense that either $n_0 \in \{0, 1\}$, or x_1 and x_{n_0} lie in different transversals. If n_0 is equal to 0 or 1, then g is conjugate into either A or B and hence $l(g) = 0$. On the other hand, if $n_0 > 1$, then $l(g) = n_0$, and it follows from Lemma 2.25 of [3] and Proposition 1.7 of [44] that this is an even integer. Thus, given such a group Γ, and a method of computing a cyclically reduced normal form of each element (which exist because the transversals T_H and T_K are finite), Algorithm 5.1.2 can be applied to determine whether or not any two or three-generated subgroup of Γ is both discrete and free.

5.2 The constructive membership problem

We conclude this thesis with an application of Algorithm 3.3.3 and Algorithm 4.2.1 to the *constructive membership problem.* Given a group H and elements $g_1, \ldots, g_n, h \in H$, this involves determining whether or not h is an element of $G = \langle g_1, \ldots, g_n \rangle \leq H$, and if it is, then finding a word in $g_1, \ldots, g_n$ that evaluates to h.

In [17], it is discussed how Algorithm 2.2.3 can be applied to solve the constructive membership problem for two-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$ which are both discrete and free of rank two. This uses the following notion:

Definition 5.2.1. Given a group G acting on a topological space X, a *fundamental domain* is an open set $D \subseteq X$ (with closure in X denoted by $\overline{D}$) such that both of the following conditions hold:

(i) $\bigcup_{g \in G} g\overline{D} = X$;

(ii) $gD \cap hD = \varnothing$ for all distinct $g, h \in G$.

In the proof of Proposition 3.3.1, given a metrisable topological group G (acting continuously by isometries and without inversions on a simplicial tree T) and two hyperbolic elements $A, B \in G$, whose axes are either disjoint or intersect along a sufficiently short path, we found vertices p and q (on the axes of A and B respectively) and considered their images Ap and Bq in order to construct subtrees $X_1^+, X_1^-, X_2^+, X_2^- \subseteq T$ satisfying the hypotheses of the Ping Pong Lemma; see Figure 3.7. Note that, in each case, if D_A is defined as the interior of the path between p and Ap (which is isometric to an open interval in $\mathbb{R}$ with integral endpoints, and is hence open in T), then D_A is a fundamental domain for the action of $\langle A \rangle$ on $\mathrm{Axis}(A)$. Similarly the open set D_B, defined as the interior of the path between q and Bq, is a fundamental domain for the action of $\langle B \rangle$ on $\mathrm{Axis}(B)$.

If the axes of A and B do not intersect, then define D as the union of D_A and D_B with the path between p' and q'; otherwise, define $D = D_A \cup D_B$. In either case, the union of images of $\overline{D}$ under the action of $\langle A, B \rangle$ forms a subtree $S \subseteq T$ for which D

is a fundamental domain for the action of $\langle A, B\rangle$ on S; see the proof of [16, Lemma 2.6] for further details. Then $D = T\backslash(X_1^+ \cup X_2^- \cup X_2^+ \cup X_2^-)$, where $X_1^+, X_1^-, X_2^+, X_2^-$ are as in Figure 3.7 and S plays the role of T. Moreover, it follows from the proof of Proposition 3.3.1 that there is at least one vertex in D.

More generally, recall the proof of Proposition 4.1.3, and note that for each $1 \leq j \leq n$ the interior D_j of the path between p_j and $g_j p_j$ is a fundamental domain for the action of $\langle g_j\rangle$ on γ_j. Let D be the smallest subtree of T containing $D_1 \cup \cdots \cup D_n$. Then the union of images of $\overline{D}$ under the action of $\langle g_1, \ldots, g_n\rangle$ forms a subtree $S \subseteq T$ with the property that D is a fundamental domain for the action of $\langle g_1, \ldots, g_n\rangle$ on S. Also $D = T\backslash(X_1^+ \cup X_1^- \cup \cdots \cup X_n^+ \cup X_n^-)$, where each X_j^+ and X_j^- is as in the proof of Proposition 4.1.3 and S plays the role of T. Again there is at least one vertex in D.

These observations imply that there is an algorithm to solve the constructive membership problem for two- or three-generated subgroups of either $\mathrm{SL}_2(K)$, or the isometry group of a locally finite simplicial tree T (equipped with the topology of pointwise convergence and a method of computing translation lengths), which are both discrete and free. Indeed, given such a subgroup $G = \langle A, B\rangle$ (respectively $G = \langle A, B, C\rangle$), one can first run the appropriate algorithm (either Algorithm 3.3.3, Algorithm 4.2.1 or Algorithm 5.1.2) to obtain generators $g_1 = g_1(A, B)$ and $g_2 = g_2(A, B)$ (respectively $g_1 = g_1(A, B, C), g_2 = g_2(A, B, C)$ and $g_3 = g_3(A, B, C)$) of G satisfying the hypotheses of the Ping Pong Lemma. Then, following the discussion above, one can also find a fundamental domain $D = T\backslash(X_1^+ \cup X_1^- \cup \cdots \cup X_n^+ \cup X_n^-)$ for the action of $\langle g_1, g_2\rangle$ (respectively $\langle g_1, g_2, g_3\rangle$) on a relevant subtree T. The following algorithm, which is essentially Algorithm 1 of [17], can then be applied:

Algorithm 5.2.2. *Let H be a metrisable topological group which acts continuously on a topological space X. Suppose that $g_1, \ldots, g_n \in H$ and $X_1^{\pm}, \ldots, X_n^{\pm} \subseteq X$ satisfy the hypotheses of the Ping Pong Lemma, and that $D = X\backslash(X_1^+ \cup X_1^- \cup \cdots \cup X_n^+ \cup X_n^-) \neq \varnothing$ is a fundamental domain for the action of $G = \langle g_1, \ldots, g_n\rangle \leq H$ on X. Choose a point $z' \in D$ and an element $h \in H$. If $h \in G$, then the algorithm will return true and* *output a word $w = w(x_1, \ldots, x_n)$ (where $x_1, \ldots, x_n$ are abstract elements generating a free group F of rank n) such that $w(g_1, \ldots, g_n) = h$, and otherwise it will return false.*

(1) *Set $w = 1 \in F$ and $z = Cz'$.*

(2) *While $z \notin D$:*

  (i) *if $z \in X_i^+$, then replace z by $g_i^{-1}z$ and w by wx_i;*

  (ii) *if $z \in X_i^-$, then replace z by $g_i z$ and w by wx_i^{-1}.*

(3) *If $w(g_1, \ldots, g_n) = h$ and $z = z'$, then return true and the word $w = w(x_1, \ldots, x_n)$, and otherwise return false.*

This algorithm is correct and terminates after finitely many steps, as in the proof following Algorithm 1 in [17]. In our context (where G is a two- or three-generated subgroup of either $\mathrm{SL}_2(K)$ or the isometry group of a locally finite simplicial tree), this gives another practical algorithm which can be implemented, so long as there is a method to determine whether or not a vertex lies in the fundamental domain D and, if it does not, then which of the subtrees $X_1^{\pm}, \ldots, X_n^{\pm}$ it belongs to. This is possible in our setting, since Proposition 3.1.6 implies that for any hyperbolic isometry g of a simplicial tree T, and any vertex p of T (for instance, in the Bruhat-Tits tree T_v, one could take p to represent the standard lattice $\mathcal{O}^2$), the midpoint of the path between p and gp lies on the axis of g. Hence one can find a specific vertex on each of the relevant axes and, after translating these vertices along each axis by appropriate powers of the generating elements and comparing distances between them, one can determine the vertices lying on each axis. This gives a method of distinguishing between vertices in $X_1^{\pm}, \ldots, X_n^{\pm}$ and D.

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