LARGE TIME BEHAVIOR OF THE *A PRIORI* BOUNDS FOR THE SOLUTIONS TO THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATIONS WITH SOFT POTENTIALS

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Abstract. We consider the spatially homogeneous Boltzmann equation for regularized soft potentials and Grad's angular cutoff. We prove that uniform (in time) bounds in $L^1((1+|v|^s)dv)$ and H^k norms, $s, k \geq 0$ hold for its solution. The proof is based on the mixture of estimates of polynomial growth in time of those norms together with the quantitative results of relaxation to equilibrium in L^1 obtained by the so-called "entropy-entropy production" method in the context of dissipative systems with slowly growing *a priori* bounds [14].

Mathematics Subject Classification (2000): 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05].

Keywords: Boltzmann equation; spatially homogeneous; soft potentials; moment bounds; regularity bounds; uniform in time.

Contents

1. Introduction

This note is devoted to the study of the asymptotic behavior of solutions to the spatially homogeneous Boltzmann equation in the case of regularized soft potentials with Grad's angular cutoff.

More precisely, we are concerned with the evolution of suitable norms which measure the asymptotic tail behavior (when $|v| \to +\infty$) of the distribution, and its smoothness. We shall prove bounds on the $L^1((1+|v|^q)dv)$ moments (resp. H^k norms) of the distribution which are uniform with respect to time, provided that the initial datum belongs to $L^1((1+|v|^{q_0})dv) \cap H^{k_0}$ with q_0, k_0 big enough.

The Boltzmann equation (Cf. [3] and [4]) describes the behavior of a dilute gas when the only interactions taken into account are binary collisions. In the case when the distribution function is assumed to be independent on the position x, we obtain the so-called spatially homogeneous Boltzmann equation, which reads

$$\frac{\partial f}{\partial t}(t,v) = Q(f,f)(t,v), \qquad v \in \mathbb{R}^N, \quad t \geq 0, \tag{1.1}$$

where $N \geq 2$ is the dimension. In equation (1.1), Q is the quadratic Boltzmann collision operator, defined by the bilinear form

$$Q(g,f)(v) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(|v-v_*|, \cos\theta)\,(g'_* f' - g_* f)\; dv_*\, d\sigma,$$

where we have used the shorthands $f = f(v)$, $f' = f(v')$, $g_* = g(v_*)$ and $g'_* = g(v'_*)$. Moreover, v' and v'_* are parametrized by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\,\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\,\sigma.$$

Finally, $\theta \in [0,\pi]$ is the deviation angle between $v'-v'_*$ and $v-v_*$ defined by $\cos\theta = (v'-v'_*)\cdot(v-v_*)/|v-v_*|^2$, and B is the Boltzmann collision kernel determined by physics (related to the cross-section $\Sigma(v-v_*,\sigma)$ by the formula $B = |v-v_*|\,\Sigma$). We also denote

$$Q^+(g,f)(v) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(|v-v_*|, \cos\theta)\, g'_* f'\, dv_*\, d\sigma$$

the positive part of Q, and

$$L(g)(v) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(|v-v_*|, \cos\theta)\, g_*\, dv_*\, d\sigma$$

the linear operator appearing in the loss part of Q.

Boltzmann's collision operator has the fundamental properties of conserving mass, momentum and energy

$$\int_{\mathbb{R}^N} Q(f,f)\,\phi(v)\,dv = 0, \qquad \phi(v) = 1, v, |v|^2, \tag{1.2}$$

and satisfying Boltzmann's H theorem, which writes (at the formal level)

$$-\frac{d}{dt}\int_{\mathbb{R}^N} f \log f\, dv = -\int_{\mathbb{R}^N} Q(f,f)\log(f)\, dv \geq 0.$$

Boltzmann's H theorem implies that (when $B > 0$ a.e.) any equilibrium distribution function has the form of a Maxwellian distribution

$$M(\rho,u,T)(v) = \frac{\rho}{(2\pi T)^{N/2}} \exp\left(-\frac{|u-v|^2}{2T}\right),$$

where $\rho \geq 0$, $u \in \mathbb{R}^N$, $T > 0$ are the density, mean velocity and temperature of the gas, defined by

$$\rho = \int_{\mathbb{R}^N} f(v)\, dv, \quad u = \frac{1}{\rho} \int_{\mathbb{R}^N} v f(v)\, dv, \quad T = \frac{1}{N\rho} \int_{\mathbb{R}^N} |u - v|^2 f(v)\, dv,$$

and determined by the mass, momentum and energy of the initial datum thanks to the conservation properties (1.2). As a result of the process of entropy production pushing towards local equilibrium combined with the constraints (1.2), solutions are expected to converge to a unique Maxwellian equilibrium.

This suggests for uniform bounds in time on the decay (in the v variable) and smoothness of the distribution $f = f(t, v)$. The main idea of this paper is to quantify this idea in a situation where the uniform bounds are not obvious : for so-called soft potentials.

More precisely, we shall consider the following assumptions on the collision kernel B:

(H1) It takes the following tensorial form (with Φ, b nonnegative functions)

$$B(|v - v_*|, \cos\theta) = \Phi(|v - v_*|)\, b(\cos\theta).$$

(H2) The kinetic part Φ is C^∞ and satisfies the bounds

$$\forall\, z \in \mathbb{R}^N, \quad c_\Phi\, (1 + |z|)^\gamma \leq \Phi(|z|) \leq C_\Phi\, (1 + |z|)^\gamma,$$

$$\forall\, z \in \mathbb{R}^N,\ p \in \mathbb{N}^*, \quad |\Phi^{(p)}(|z|)| \leq C_{\Phi,p},$$

with $\gamma \in (-2, 0]$, and $c_\Phi, C_\Phi, C_{\Phi,p} > 0$.

(H3) The angular part $\sigma \mapsto b(u \cdot \sigma)$ is integrable on $\mathbb{S}^{N-1}$, and it satisfies the bound from below

$$\forall\, \theta \in [0, \pi], \quad b(\cos\theta) \geq b_0$$

for some constant $b_0 > 0$.

This includes the so-called "mollified" soft potentials with Grad's angular cutoff assumption (the word "mollified" is related to the singularity for small relative velocities). It does not include the very soft potentials (that is the case when $\gamma \in (-N, -2]$).

We shall systematically use the notations ($s \in \mathbb{R}$, $p \in [1, +\infty)$, $k \in \mathbb{N}$)

$$\|f\|^p_{L^p_s} := \int_{\mathbb{R}^N} |f(v)|^p\, (1 + |v|^2)^{ps/2}\, dv,$$

and

$$\|f\|^2_{H^k_s} := \sum_{0 \leq |i| \leq k} \|\partial^i f\|^2_{L^2_s},$$

where ∂^i denotes the partial derivative related to the multi-index i.

The Cauchy theory for equation (1.1) under assumptions (H1)-(H2)-(H3) is already known and is particularly simple (the collision operator is bounded). Using the arguments of Arkeryd [1], one can construct global nonnegative solutions in L^1_2. Uniqueness (in this class) follows from the boundedness of the operator (as a bilinear function in L^1_2).

As far as hard potentials (that is, $\gamma \in (0,1]$) or Maxwell molecules (that is, $\gamma = 0$) are concerned, the propagation of the L^1 moments (that is, the L^1_s norms for $s > 2$) was proven in [6] and [9]. Moreover, the bounds were shown there to be uniform with respect to time. It was later noticed that for hard potentials those moments appear even if they don't initially exist, under reasonable assumptions (Cf. [5], and the improvements in [18, 19, 10]).

Still for hard potentials (with angular cutoff), uniform in time estimates of L^p norms or H^k norms were first obtained in [7, 8] and [20], and later simplified and systematically studied in [11].

In the case of (mollified) soft potentials (with angular cutoff), polynomially growing bounds on the L^1 moments were first obtained in [5] and later extended to the case of the Landau equation in [15, Part I, Appendix B] and [14]. Polynomially growing bounds on the L^p norms were also obtained in [14].

This paper is devoted to the obtention of uniform in time bounds on L^1 moments and H^k norm in the setting of (mollified) soft potentials (with angular cutoff), where only polynomially growing bounds exist, as we just explained.

We now state our main result.

Theorem 1.1. *Let $s > 2$ and $k \geq 0$ be given, together with an initial datum $0 \leq f_{in} \in L^1_2(\mathbb{R}^N)$. We consider the unique solution $f(t,v) \geq 0$ in L^1_2 to equation* (1.1) *under assumptions (H1)-(H2)-(H3). Then*

(i) *there exists $q_0 > 0$ (depending on s, but not on k) such that if $f_{in} \in L^1_{2s} \cap L^2_{q_0}$, the associated solution $f = f(t,\cdot)$ satisfies*
$$\sup_{t \geq 0} \|f(t,\cdot)\|_{L^1_s} \leq C(s)$$
for some explicit bound $C(s) > 0$;

(ii) *there is $s_0 > 0$ and $k' \geq k$ (both depend on k) such that if $f_{in} \in L^1_{s_0} \cap H^{k'}$, the associated solution $f = f(t,\cdot)$ satisfies*
$$\sup_{t \geq 0} \|f(t,\cdot)\|_{H^k} \leq C(k)$$
for some explicit bound $C(k) > 0$.

Remarks:

1. In both points (i) and (ii) of this theorem, the assumptions on the initial datum are most probably not optimal, and are likely to be relaxed, up to technical refinements in the proofs (for example, in point (i), the weighted L^2 space can be replaced by some weighted L^p space for any $p > 1$). We do not try here to look for such optimal assumptions, since we are more interested in showing how to obtain the uniform bounds. Note however that the sole assumption $f_{in} \in \cap_{s>0} L^1_s$ is probably not sufficient to propagate uniformly the L^1_s norm for $s > 2$, and we conjecture that it may be possible to construct some counter-examples in the same spirit as those constructed in [2] in order to disprove Cercignani's conjecture for Maxwell molecules interactions.

2. We then note that the assumptions on the collision kernel can also certainly be relaxed. We conjecture that all derivatives on the kinetic part of the cross section are not really needed (probably one is enough), and that the angular part need not really be bounded below. However, our proof depends strongly on the angular cutoff, and one would need original extra arguments to treat the non cutoff case. It also does not work for very soft potentials (see the remark at the end of the proof).

3. We think that our proof could be adapted to the Landau kernel with soft potential without too many changes. However, too soft potentials like the Coulomb potential might not be reachable.

4. When f_{in} belongs to $\mathcal{S}(\mathbb{R}^N)$ the Schwartz space of rapidly decaying C^∞ function, then $f(t,\cdot) \in \mathcal{S}(\mathbb{R}^N)$ and the corresponding seminorms are bounded uniformly with respect to time. This is obtained thanks to Sobolev inequalities and standard interpolations between L^1_s and H^k. In particular, uniform bounds of the form

$$\forall\, t \geq 0,\ \forall\, v \in \mathbb{R}^N, \quad f(t,v) \leq C\,(1+|v|)^{-q}$$

are available.

5. A rough calculation shows that for point (i) of this theorem, $q_0 = 26$ is sufficient in the case when $N = 3$ and $\gamma = -1$.

2. Proof of slowly increasing bounds

In this section, we recall results on the slowly increasing polynomial bounds on the moments and L^p norms of the solutions of equation (1.1) from [5, 14], and we extend them to deal with the H^k norms.

Estimates of linear growth in time on the moments were obtained in [5] in the case $\gamma > -1$, and sketched in [16] and [14] for $\gamma > -2$. We give here a precise statement together with a short proof.

Proposition 2.1. *Let $s > 2$. Then for any initial datum $f_{in} \in L^1_s$, the unique associated solution $f = f(t, \cdot)$ to equation* (1.1) *under assumptions (H1)-(H2)-(H3) satisfies the bounds*

$$\forall\, t \geq 0, \quad \|f(t,\cdot)\|_{L^1_s} \leq C_0(s)\,(1+t),$$

for some explicit constant $C_0(s) > 0$ depending only on the mass and L^1_s norm of f_{in}.

Proof of Proposition 2.1. We compute the time derivative of the s-th L^1 moment of f thanks to the pre-postcollisional change of variable (see [17, Chapter 1, Section 4.5]):

$$\frac{d}{dt}\|f(t,\cdot)\|_{L^1_s} = \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} f\, f_*\, b\, \left(|v'_*|^s + |v'|^s - |v_*|^s - |v|^s\right)\, \Phi(|v - v_*|)\, dv\, dv_*\, d\sigma.$$

Using then Povzner's inequality (Cf. [19] for instance), we get (for some $C_+, K_- > 0$)

$$\int_{\mathbb{S}^{N-1}} \left(|v'_*|^s + |v'|^s - |v_*|^s - |v|^s\right)\, b(\cos\theta)\, d\sigma \\ \leq C_+ \left(|v|^{s-2}|v_*|^2 + |v_*|^{s-2}|v|^2\right) - K_- \left(|v|^s + |v_*|^s\right).$$

Hence, using assumption (H2), for some $K_0 > 0$,

$$\frac{d}{dt}\|f(t,\cdot)\|_{L^1_s} \leq C_1 + C_2\, \|f(t,\cdot)\|_{L^1_{s-2}} - K_0\, \|f(t,\cdot)\|_{L^1_{s+\gamma}}. \tag{2.1}$$

We conclude by using an interpolation of $\|f(t,\cdot)\|_{L^1_{s+\gamma}}$ between $\|f(t,\cdot)\|_{L^1}$ and $\|f(t,\cdot)\|_{L^1_{s-2}}$. We see that

$$\frac{d}{dt}\|f(t,\cdot)\|_{L^1_s} \leq C_3(s), \tag{2.2}$$

so that

$$\|f(t,\cdot)\|_{L^1_s} \leq C_4(s)\,(1+t). \tag{2.3}$$

□

We now take care of the smoothness. The following result is a straightforward consequence of [14, Corollary 9.1] and general methods developed in [11]. It essentially says that the control of the regularity in our context can be obtained by the control of the moments.

Proposition 2.2. *Let $1 < p < +\infty$ (resp. $k \in \mathbb{N}^*$). Let us consider $0 \leq f_{in} \in L^1_2$ an initial datum and $f = f(t,\cdot)$ the unique associated solution to equation* (1.1) *under* *assumptions (H1)-(H2)-(H3). Then, there are $C, s, \alpha > 0$ depending on p (resp. $C', s', \alpha' > 0$ depending on k) such that the following* a priori *estimates hold*

$$\begin{cases} \dfrac{d}{dt}\|f(t,\cdot)\|_{L^p} \le C\,\|f(t,\cdot)\|^{\alpha}_{L^1_s}, \\ \dfrac{d}{dt}\|f(t,\cdot)\|_{H^k} \le C'\,\|f(t,\cdot)\|^{\alpha'}_{L^2_{s'}}. \end{cases}$$

Proof of Proposition 2.2. Concerning the first *a priori* bound on the L^p norm, it is proven in [14, Proposition 9]. The proof is based on the regularity property of the gain part Q^+ of the collision operator in the following form (see [20, 11])

$$\|Q^+(g,f)\|_{H^{(N-1)/2}_s} \le C\big(\|f\|_{L^1_{1+2s}}\|g\|_{L^2_{1+s}} + \|f\|_{L^2_{1+s}}\|g\|_{L^1_{1+2s}}\big).$$

Then using that (for any derivative ∂)

$$\partial Q^+(g,f) = Q^+(\partial g, f) + Q^+(g, \partial f)$$

thanks to the translation invariance, to Cauchy-Schwartz type inequalities like

$$\|f\|_{L^1_s} \le C\,\|f\|_{L^2_{s+q}}$$

for some $C, q > 0$, and to some classical interpolation in the H^k spaces, we deduce that

$$\|Q^+(f,f)\|_{H^{k+(N-1)/2}_s} \le C\|f\|_{L^2_{s+w}}\|f\|_{H^k_{s+w}} \tag{2.4}$$

for any $s, k \ge 0$ and some $C, w > 0$. Now let us consider the time derivative of the square of the L^2 norm of $\partial^k f$ for some multi-index k with $|k| \ge (N-1)/2$. We get

$$\begin{aligned} \frac{d}{dt}\|\partial^k f\|^2_{L^2} &\le C\,\|Q^+(f,f)\|_{H^k_{-\gamma}}\|\partial^k f\|_{L^2_\gamma} \\ &\quad + C\left(\sum_{0<l\le k}\big\|(\partial^l L(f))(\partial^{k-l} f)\big\|_{L^2_{-\gamma}}\right)\|\partial^k f\|_{L^2_\gamma} - K\,\|\partial^k f\|^2_{L^2_\gamma}, \end{aligned}$$

where we have $L(f) = C_b\,(\Phi * f)$ (C_b is the L^1 norm of b on the sphere $\mathbb{S}^{N-1}$). The back term comes from the classical lower bound

$$L(f) \ge K\,(1+|v|)^\gamma$$

for some constant $K > 0$ depending on the mass and entropy of the initial datum (see [1] for instance).

Then on the one hand equation (2.4) yields

$$\|Q^+(f,f)\|_{H^k_{-\gamma}} \le C_+\,\|f\|_{L^2_w}\|f\|_{H^{k-(N-1)/2}_w}$$

for some explicit constants $C_+, w > 0$, and then by interpolation

$$\|Q^+(f,f)\|_{H^k_{-\gamma}} \le C_+ \, \|f\|^{1+2\theta_+}_{L^2_{w_+}} \|f\|^{1-2\theta_+}_{H^k_\gamma}$$

for some explicit constants $C_+, w_+, \theta_+ > 0$. On the other hand the convolution structure of $L(f)$ together with the smoothness assumption on Φ in (H2) yields easily

$$\left(\sum_{0<l\le k} \left\| (\partial^l L(f))(\partial^{k-l} f) \right\|_{L^2_{-\gamma}} \right) \le C \, \|f\|_{L^2_w} \|f\|_{H^{k-1}_w}$$

for some constants $C, w > 0$ and thus by interpolation

$$\left(\sum_{0<|l|\le k} \left\| (\partial^l L(f))(\partial^{k-l} f) \right\|_{L^2_{-\gamma}} \right) \le C_L \, \|f\|^{1+2\theta_L}_{L^2_{w_L}} \|f\|^{1-2\theta_L}_{H^k_\gamma}$$

for some explicit constants $C_L, w_L, \theta_L > 0$ (depending on k). Thus if $\theta_0 = \min\{\theta_+, \theta_L\}$, we easily obtain for some $\bar{w} > 0$,

$$\frac{d}{dt} \|f\|^2_{H^k} \le C_0 \, \|f\|^{1+2\theta_0}_{L^2_{\bar{w}}} \|f\|^{2-2\theta_0}_{H^k_\gamma} - K \, \|f\|^2_{H^k_\gamma}.$$

Finally, we use the inequality

$$\forall \, X \ge 0, \quad A \, X^{1-\delta} - K X \le C \, A^{1/\delta}$$

(with $C \equiv C(K,\delta)$) to conclude the proof. □

3. Proof of uniform bounds

In this section, we combine the results of Section 2 with the quantitative results of convergence to equilibrium obtained in [14]. We conclude in this way the proof of Theorem 1.1.

Let us recall the quantitative result of trend to equilibrium we shall use. We denote by $M = M(\rho, u, T)$ the Maxwellian with parameters ρ, u, T corresponding to the initial datum.

Proposition 3.1. *Let us consider an initial datum $0 \le f_{in} \in L^1_2$ and $\tau > 0$. Then there exists $q_0 > 0$ such that if $f_{in} \in L^2_{q_0}$, the unique associated solution $f = f(t, \cdot)$ of equation* (1.1) *under assumptions (H1)-(H2)-(H3) satisfies*

$$\forall \, t \ge 0, \quad \|f(t,\cdot) - M\|_{L^1} \le C_1 \, (1+t)^{-\tau}$$

for some explicit bound $C_1 > 0$ depending only on τ, ρ, and the $L^2_{q_0}$ norm of f_{in}.

Proof of Proposition 3.1. This result is a particular case of more general results in [14] (see Proposition 6 in this paper). Indeed [14, Theorem 11] implies the conclusion of Proposition 3.1 as soon as f_{in} satisfies a lower bound of the form $f_{in} \geq K_0 \, e^{-A_0 |v|^2}$. This assumption can be relaxed thanks to [12, Theorem 5.1], which shows that this lower bound appears immediately under the assumption we have on the initial datum (in particular the assumption of finite entropy for [12, Theorem 5.1] is implied by $f_{in} \in L^2_{q_0}$). $\square$

Now we can conclude the proof of Theorem 1.1 by gathering this proposition with the results of Section 2.

Proof of point (i): Assume that $f_{in} \in L^1_{2s} \cap L^2_{q_0}$. On the one hand, from Proposition 2.1, the unique associated solution satisfies

$$\forall\, t \geq 0, \quad \|f(t,\cdot)\|_{L^1_{2s}} \leq C_0 \, (1+t).$$

On the other hand, from Proposition 3.1 (with $\tau = 1$), it satisfies

$$\forall\, t \geq 0, \quad \|f(t,\cdot) - M\|_{L^1} \leq C_1 \, (1+t)^{-1}.$$

We deduce that for any $t \geq 0$,

$$\begin{aligned}
\|f(t,\cdot)\|_{L^1_s} &\leq \|M\|_{L^1_s} + \|f(t,\cdot) - M\|_{L^1_s} \\
&\leq \|M\|_{L^1_s} + \|f(t,\cdot) - M\|_{L^1}^{1/2} \|f(t,\cdot) - M\|_{L^1_{2s}}^{1/2} \\
&\leq \|M\|_{L^1_s} + C_1^{1/2} (1+t)^{-1/2} \big(\|f(t,\cdot)\|_{L^1_{2s}} + \|M\|_{L^1_{2s}}\big)^{1/2} \\
&\leq \|M\|_{L^1_s} + C_1^{1/2} (1+t)^{-1/2} \big(C_0 \, (1+t) + \|M\|_{L^1_{2s}}\big)^{1/2} \\
&\leq C(s) < +\infty.
\end{aligned}$$

This concludes the proof of point (i).

Proof of point (ii): First let us prove the uniform bound in the case $k = 0$. In fact we shall prove uniform bounds on any L^p norms, $1 < p < +\infty$. From Proposition 2.2 we have for any $p \in (1, +\infty)$

$$\frac{d}{dt} \|f(t,\cdot)\|_{L^p} \leq C \, \|f(t,\cdot)\|_{L^1_s}^{\alpha} \tag{3.1}$$

for some explicit $C, s, \alpha > 0$ (depending on p).

We assume enough L^1 moments bounded on the initial datum, and enough derivatives in L^2. Then, thanks to Sobolev inequalities, the initial datum is in L^p with $p > 2$. By standard interpolations, the initial datum has enough moments bounded in L^2. As a consequence, we can use point (i), and obtain that $\|f(t,\cdot)\|_{L^1_s}$ is uniformly bounded for all t. Using once again enough derivatives in L^2 of the initial datum and Sobolev inequalities, we get L^p bounds (for any $p \in]1,+\infty[$) on the initial datum. Consequently, (3.1) yields

$$\|f(t,\cdot)\|_{L^p} \le C_0(p)\,(1+t) \tag{3.2}$$

for some explicit constant $C_0(p) > 0$.

Then for any $p \in (1,+\infty)$ (using Proposition 3.1 and (3.2) for $2p$ instead of p)

$$\begin{aligned}
\|f(t,\cdot)\|_{L^p} &\le \|M\|_{L^p} + \|f(t,\cdot) - M\|_{L^p} \\
&\le \|M\|_{L^p} + \|f(t,\cdot) - M\|_{L^1}^{1/(2p-1)} \|f(t,\cdot) - M\|_{L^{2p}}^{1-1/(2p-1)} \\
&\le \|M\|_{L^p} + C_1^{1/(2p-1)} (1+t)^{-2(p-1)/(2p-1)} \big(\|f(t,\cdot)\|_{L^{2p}} + \|M\|_{L^{2p}}\big)^{1-1/(2p-1)} \\
&\le \|M\|_{L^p} + C_1^{1/(2p-1)} (1+t)^{-2(p-1)/(2p-1)} \big(C_0(2p)\,(1+t) + \|M\|_{L^{2p}}\big)^{1-1/(2p-1)} \\
&\le C(p) < +\infty.
\end{aligned}$$

Let us now assume that $k \ge 1$. From Proposition 2.2, we have

$$\frac{d}{dt}\|f(t,\cdot)\|_{H^k} \le C'\,\|f(t,\cdot)\|_{L^2_{s'}}^{\alpha'}$$

for some explicit constants $C', s', \alpha' > 0$ (depending on k). From the previous study, by assuming enough L^1 moments and H^k bounds on the initial datum, we can assume that $\|f(t,\cdot)\|_{L^p}$ is uniformly bounded for all t. Using then point (i) and a standard interpolation, we see that $\|f(t,\cdot)\|_{L^2_{s'}}$ is uniformly bounded for all t. Hence for any $k \ge 1$, we have

$$\|f(t,\cdot)\|_{H^k} \le C_0(k)\,(1+t) \tag{3.3}$$

for some explicit constant $C_0(k) > 0$.

Then for any k, using Proposition 3.1 with $\tau = 1$ and (3.3) with $2k + (N+1)/2$ instead of k and the continuous embedding $L^1(\mathbb{R}^N) \hookrightarrow H^{-(N+1)/2}(\mathbb{R}^N)$, we have

$$\begin{aligned}
\|f(t,\cdot)\|_{H^k} &\le \|M\|_{H^k} + \|f(t,\cdot) - M\|_{H^k} \\
&\le \|M\|_{H^k} + \|f(t,\cdot) - M\|_{H^{-(N+1)/2}}^{1/2} \|f(t,\cdot) - M\|_{H^{2k+(N+1)/2}}^{1/2} \\
&\le \|M\|_{H^k} + C_1^{1/2} (1+t)^{-1/2} \big(\|f(t,\cdot)\|_{H^{2k+(N+1)/2}} + \|M\|_{H^{2k+(N+1)/2}}\big)^{1/2} \\
&\le \|M\|_{H^k} + C_1^{1/2} (1+t)^{-1/2} \big(C_0\,(1+t) + \|M\|_{H^{2k+(N+1)/2}}\big)^{1/2} \\
&\le C(k) < +\infty.
\end{aligned}$$

This concludes the proof.

Remarks:

1. Our analysis does not work for (mollified) very soft potentials. What happens then is that (if we denote by m_s the s-th moment in L^1 of f),

$$\frac{d}{dt} m_s \leq C_0 + C_1\, m_{s-a} m_a - K\, m_{s+\gamma}$$

for all $a \in [0, s]$, so that (in dimension $N = 3$ with $-3 < \gamma < -2$)

$$m_s(t) \leq C\, \left(1 + t^{s/2-1}\right).$$

However, this estimate doesn't seem sufficient to obtain any rate of convergence to equilibrium. A rough calculation shows that an estimate in $t^{\lambda s}$ instead of $t^{s/2}$ (with $\lambda < 1/2$) could be the minimum required in order to get some rate of convergence to equilibrium with the "entropy-entropy production" method. Note however that for the Landau kernel for (mollified) very soft potentials (although not for the limiting Coulomb case) such estimates are available (see [14]), suggesting that our method applies as well for this model.

2. We conclude with a last remark: once bounds which are uniform in time have been proven, they can be used in order to prove directly the rate of convergence toward equilibrium like in [13] (that is, without entering the details of the method of "slowly growing *a priori* estimates" devised by G. Toscani and C. Villani in [14]). Note however that in order to get the bounds on moments which are uniform in time, this method (of "slowly growing *a priori* estimates") is used, so it really seems unavoidable.

Acknowledgment: Support by the European network HYKE, funded by the EC as contract HPRN-CT-2002-00282, is acknowledged.

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