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Cut-off for lamplighter chains on tori: dimension interpolation and Phase transition

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Abstract

Given a finite, connected graph G, the lamplighter chain on G is the lazy random walk $X^{\diamond}$ on the associated lamplighter graph $\mathsf{G}^{\diamond} = \mathbb{Z}_2 \wr \mathsf{G}$. The mixing time of the lamplighter chain on the torus $\mathbb{Z}_n^d$ is known to have a cutoff at a time asymptotic to the cover time of $\mathbb{Z}_n^d$ if $d = 2$, and to half the cover time if $d \geq 3$. We show that the mixing time of the lamplighter chain on $\mathsf{G}_n(a) = \mathbb{Z}_n^2 \times \mathbb{Z}_{a \log n}$ has a cutoff at $\psi(a)$ times the cover time of $\mathsf{G}_n(a)$ as $n \to \infty$, where ψ is an explicit weakly decreasing map from $(0, \infty)$ onto $[1/2, 1)$. In particular, as $a > 0$ varies, the threshold continuously interpolates between the known thresholds for $\mathbb{Z}_n^2$ and $\mathbb{Z}_n^3$. Perhaps surprisingly, we find a phase transition (non-smoothness of ψ) at the point $a_* = \pi r_3(1 + \sqrt{2})$, where high dimensional behavior ($\psi(a) = 1/2$ for all $a \geq a_*$) commences. Here r_3 is the effective resistance from 0 to ∞ in $\mathbb{Z}^3$.

Keywords Wreath product · Lamplighter walk · Mixing time · Cutoff · Uncovered set

Mathematics Subject Classification 60J10 · 60D05 · 37A25

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1 Introduction

1.1 Setup

Suppose that G is a finite, connected graph with vertices $V(\mathsf{G})$ and edges $E(\mathsf{G})$, respectively. Each vertex $(\underline{f}, x)$ of the *wreath product* $\mathsf{G}^\diamond = \mathbb{Z}_2 \wr \mathsf{G}$ consists of a $\{0, 1\}$-labeling $\underline{f}$ of $V(\mathsf{G})$ and $x \in V(\mathsf{G})$. There is an edge between $(\underline{f}, x)$ and $(\underline{g}, y)$ if and only if $\{x, y\} \in E(\mathsf{G})$ and $f_z = g_z$ for all $z \notin \{x, y\}$. Recall that the transition kernel of the *lazy random walk* X on G is

$$P(x, y) := \mathbf{P}_x[X_1 = y] = \begin{cases} \frac{1}{2} & \text{if} \quad x = y, \\ \frac{1}{2d(x)} & \text{if} \quad \{x, y\} \in E(\mathsf{G}), \end{cases} \tag{1.1}$$

where $d(x)$ is the degree of $x \in V(\mathsf{G})$ and $\mathbf{P}_x$ denotes the law under which $X_0 = x$. The *lamplighter chain* $X^\diamond$ is the lazy random walk on $\mathsf{G}^\diamond$. Explicitly, it moves from $(\underline{f}, x)$ by

1. picking y adjacent to x in G according to P, then
2. if $y \neq x$, updating each of the values of f_x and f_y independently according to the uniform measure on $\mathbb{Z}_2$ (with f_z unchanged for all $z \notin \{x, y\}$).

We refer to f_x as the state of the lamp at x. If $f_x = 1$ (resp. $f_x = 0$) we say that the lamp at x is on (resp. off); this is the source of the name "lamplighter." Note that the projection of $X^\diamond$ to G evolves as a lazy random walk on G. It is easy to see that the unique stationary distribution of $X^\diamond$ is given by the product of the (unique) stationary distribution of $P(\cdot, \cdot)$ and the uniform measure over the $\{0, 1\}$-labelings of $V(\mathsf{G})$. See Fig. 1 for an illustration of the lamplighter chain.

The purpose of this work is to determine the asymptotics of the total variation mixing time of the lamplighter chain on a particular sequence of graphs. In order to state our main results precisely and put them into context, we will first review some basic terminology from the theory of Markov chains. Suppose that μ, ν are measures on a finite probability space. The *total variation distance* between μ, ν is given by

$$\|\mu - \nu\|_{\mathrm{TV}} = \max_A |\mu(A) - \nu(A)| = \frac{1}{2} \sum_x |\mu(x) - \nu(x)|. \tag{1.2}$$

The δ-*total variation mixing time* of a transition kernel Q on a graph H with stationary distribution $\pi(\cdot)$ is given by

$$t_{\mathrm{mix}}(\mathsf{H}, \delta) = \min \left\{ t \geq 0 : \max_{x \in V(\mathsf{H})} \|Q^t(x, \cdot) - \pi(\cdot)\|_{\mathrm{TV}} \leq \delta \right\}. \tag{1.3}$$

Throughout, we let $t_{\mathrm{mix}}(\mathsf{H}) = t_{\mathrm{mix}}(\mathsf{H}, \frac{1}{2e})$. Lazy random walk $\widehat{X}$ on a family of graphs (H_n) is said to exhibit *cutoff* if

$$\lim_{n \to \infty} \frac{t_{\mathrm{mix}}(\mathsf{H}_n, \delta)}{t_{\mathrm{mix}}(\mathsf{H}_n, 1 - \delta)} = 1 \quad \text{for all} \quad \delta > 0. \tag{1.4}$$

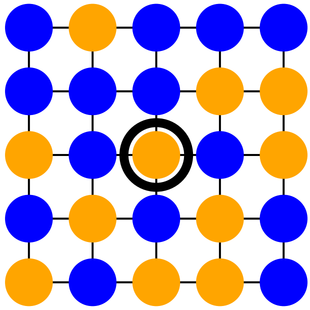

Fig. 1 Shown is a lamplighter configuration on $\mathbb{Z}_5^2$ (without the wraparound edges). The state of the lamps is indicated by the colors. The circle gives the position of the underlying random walker (color figure online)

For each $x \in V(\mathsf{H})$ let $\tau_x = \min\{k \geq 0 : \widehat{X}_k = x\}$ be the hitting time of x. With $\mathbf{E}_x$ the expectation associated with $\mathbf{P}_x$, the *maximal hitting time* of H is given by

$$t_{\text{hit}} = t_{\text{hit}}(\mathsf{H}) = \max_{x,y \in V(\mathsf{H})} \mathbf{E}_y[\tau_x]$$

and the *cover time* of H is

$$t_{\text{cov}} = t_{\text{cov}}(\mathsf{H}) = \max_{y \in V(\mathsf{H})} \mathbf{E}_y \left[\max_{x \in V(\mathsf{H})} \tau_x \right].$$

1.2 Related work

The mixing time of $\mathsf{G}^\diamond$ was first studied by Häggström and Jonasson [13] in the case of the complete graph K_n and the one-dimensional cycle $\mathbb{Z}_n$. Their work implies a total variation cutoff with threshold $\frac{1}{2}t_{\text{cov}}(\mathsf{K}_n)$ in the former case and that there is no cutoff in the latter. The connection between $t_{\text{mix}}(\mathsf{G}^\diamond)$ and $t_{\text{cov}}(\mathsf{G})$ is explored further in [23] (see also the account given in [19, Chapter 19]), in addition to developing the relationship between $t_{\text{hit}}(\mathsf{G})$ and the relaxation time (i.e., inverse spectral gap) of $\mathsf{G}^\diamond$, and the relationship between exponential moments of the size of the uncovered set $\mathcal{U}(t)$ of G at time t and the uniform, i.e., ℓ_∞-mixing time of $\mathsf{G}^\diamond$. In particular, it is shown in [23, Theorem 1.3] that if (G_n) is a sequence of graphs with $|V(\mathsf{G}_n)| \to \infty$ and $t_{\text{hit}}(\mathsf{G}_n) = o(t_{\text{cov}}(\mathsf{G}_n))$ then

$$\frac{1}{2}(1 + o(1))t_{\text{cov}}(\mathsf{G}_n) \leq t_{\text{mix}}(\mathsf{G}_n^\diamond) \leq (1 + o(1))t_{\text{cov}}(\mathsf{G}_n) \quad \text{as} \quad n \to \infty. \tag{1.5}$$

Related bounds on the order of magnitude of the uniform mixing time and the relaxation with generalized lamps were obtained respectively in [15,16].

By combining the results of [1,10], it is observed in [23] that $t_{\text{mix}}((\mathbb{Z}_n^2)^\diamond)$ has a threshold at $t_{\text{cov}}(\mathbb{Z}_n^2)$. Thus, (1.5) gives the best *universal* bounds, since K_n attains the lower bound and $\mathbb{Z}_n^2$ attains the upper bound. In [21], it is shown that $t_{\text{mix}}((\mathbb{Z}_n^d)^\diamond) \sim \frac{1}{2}t_{\text{cov}}(\mathbb{Z}_n^d)$ when $d \geq 3$ and more generally that $t_{\text{mix}}(\mathsf{G}_n^\diamond) \sim \frac{1}{2}t_{\text{cov}}(\mathsf{G}_n)$ whenever (G_n) is a sequence of graphs with $|V(\mathsf{G}_n)| \to \infty$ satisfying certain uniform local transience assumptions. This prompted the question [21, Section 7] of whether for each $\gamma \in (\frac{1}{2}, 1)$

there exists a (natural) family of graphs (G_n) such that $t_{\mathrm{mix}}(\mathsf{G}_n^\diamond) \sim \gamma t_{\mathrm{cov}}(\mathsf{G}_n)$ as $n \to \infty$. In this work we give an affirmative answer to this question by analyzing the lamplighter chain on a thin 3D torus.

Cutoff for lazy random walks on $\mathsf{G}_n^\diamond$ is further examined in [7] for a large class of fractal graphs G_n. They show that cutoff never occurs for strongly recurrent G_n (namely of spectral dimension $d_s < 2$), while the sufficient conditions of [21] for cutoff at $\frac{1}{2}t_{\mathrm{cov}}(\mathsf{G}_n)$, apply for transient G_n (i.e. having $d_s > 2$). However, such universality seem to not hold in the setting of $d_s = 2$, namely for the fractal analog of the 2D and thin 3D torus considered here.

1.3 Main results

Fix $a > 0$. We consider the mixing time for the SRW $X_k^\diamond$, $k \in \mathbb{N}$, on the lamplighter graph $(\mathsf{G}_n(a))^\diamond$ for the 3D thin tori $\mathsf{G}_n(a) = (V_n, E_n) = \mathbb{Z}_n^2 \times \mathbb{Z}_h$ of size $n \times n \times h$, where $h = [a \log n]$. From the main result of [9] we know that the cover time of the 2D projection of SRW on $\mathsf{G}_n(a)$ to $\mathbb{Z}_n^2$ is given by

$$t_{\mathrm{cov}}^{\square} := \frac{3}{2} t_{\mathrm{cov}}\left(\mathbb{Z}_n^2\right) \quad \text{where} \quad t_{\mathrm{cov}}\left(\mathbb{Z}_n^2\right) := \frac{4}{\pi} n^2 (\log n)^2 (1 + o(1))$$

(where the factor $\frac{3}{2}$ is due to the lazy steps of walk in the h-direction, which occur with probability $\frac{1}{3}$). Let

$$\phi := \pi r_3 a \tag{1.6}$$

where r_3 denotes the resistance $0 \leftrightarrow \infty$ for the SRW in $\mathbb{Z}^3$. That is,

$$r_3 = \frac{1}{6q} \quad \text{where} \quad q = \mathbf{P}_0[T_0 = \infty], \tag{1.7}$$

and T_0 denotes the return time to zero by SRW in $\mathbb{Z}^3$ (see [19, Proposition 9.5] for the relation (1.7) and an explicit formula for q). In Sect. 2, we use the recent development which relates cover time with the extremes of Gaussian fields, see [6], to establish the following theorem.

Theorem 1.1 *The cover time $t_{\mathrm{cov}}(a, n)$ of $\mathsf{G}_n(a)$ by SRW is given by*

$$t_{\mathrm{cov}}(a, n) = (1 + o(1)) C(a, n), \quad \text{as } n \to \infty$$

where

$$C(a, n) := (1 + 2\phi) t_{\mathrm{cov}}^{\square} \tag{1.8}$$

and ϕ is as in (1.6).

Remark 1.2 One expects the cutoff threshold transition from 2D to 3D behavior to occur when $t_{\text{cov}}(\mathsf{G}_n(a))/t_{\text{cov}}(\mathbb{Z}_n^2) = O(1)$, while depending on the height multiplier a. By Theorem 1.1 the correct scaling for this is $\log n$ (which as shown in Sect. 2, has to do with the decay rate of $\text{Diam}_{R_{\text{eff}}}(\mathbb{Z}_n^2)$, see (2.12)).

Our main result establishes cutoff for SRW $\{X_k^\diamond\}$ on the lamplighter graph $(\mathsf{G}_n(a))^\diamond$ and determines its location as a function of the height parameter a.

Theorem 1.3 *Total-variation cut-off occurs for $\{X_k^\diamond\}$ on $\mathsf{G}_n(a)$ at $\Psi(\phi)t_{\text{cov}}^{\square}$, where*

$$\Psi(\phi) := \begin{cases} \left(1 + (1 - \frac{1}{\sqrt{2}})\phi\right)^2, & \text{if } \phi \le \sqrt{2} + 1, \\ \frac{1+2\phi}{2}, & \text{if } \phi > \sqrt{2} + 1. \end{cases} \tag{1.9}$$

In particular, $t_{\text{mix}} = (\Psi(\phi) + o(1))t_{\text{cov}}^{\square}$.

Comparing Theorems 1.1 and 1.3 we see that the ratio between the mixing time of $\{X_k^\diamond\}$ and the cover time $C(a, n)$ of the base graph by the SRW $\{X_k\}$, monotonically interpolates between the fraction of the cover time necessary to mix in two dimensions (ratio 1) [9,23] and the fraction in three dimensions (ratio 1/2) [21]. This gives an affirmative answer to the first question posed in [21, Section 7]. See Fig. 2 for a plot of the quantities from Theorem 1.3 and how they relate to the bounds (1.5).

We note in passing that for all $\phi > 0$ the value of $t_{\text{mix}}/t_{\text{cov}}^{\square} \to \Psi(\phi)$ is bounded away from its trivial bound 1. The latter corresponds to the mixing time for the lamplighter graph on the 2D torus of side length n that corresponds to the base sub-graph $(x_1, x_2, 1)$ of $\mathsf{G}_n(a)$ (which as shown in [23] coincides with the cover time $t_{\text{cov}}^{\square}(1 + o(1))$ for the corresponding (lazy) 2D projected SRW). However, when $\phi \ge \sqrt{2} + 1$ asymptotically t_{mix} matches the elementary bound $t_{\text{mix}} \ge \frac{(1+o(1))}{2}C(a, n)$ (see (1.8), and [19, Lemmas 19.3 and 19.4]), which applies for the lamplighter chain on any base graph having maximal hitting time which is significantly smaller than the corresponding cover time.

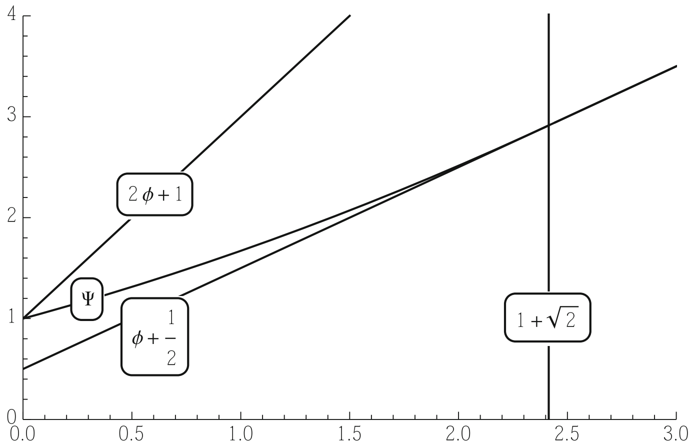

Fig. 2 The function Ψ from (1.9) which gives the asymptotic ratio of $t_{\text{mix}}/t_{\text{cov}}^{\square}$. Also shown are the bounds of $2\phi + 1$ and $\phi + \frac{1}{2}$ on $t_{\text{mix}}/t_{\text{cov}}^{\square}$; recall (1.5). The lower bound is attained by Ψ starting at $\phi = 1 + \sqrt{2}$

Remark 1.4 It is possible to adapt the proof of Theorem 1.3 so that it will yield a similar conclusion in the setting of a more general 3-dimensional lattice confined to a thin slab of size $n \times n \times h$.

Remark 1.5 Clearly, $X_t^{\diamond}$ is not mixed for as long as the uncovered set $\mathcal{U}(t)$ of X exhibits some non-trivial systematic geometric structure that makes the corresponding lamp states distinguishable from the uniform marking of $V(\mathsf{G})$ by i.i.d. fair coin flips. Further, the uniformity of $\mathcal{U}(t)$ typically determines the threshold t for mixing time of $X^{\diamond}$, and indeed our work contributes to the literature on the geometric structure of the last visited points by the SRW (see [3,4,9,10,21,22]).

Remark 1.6 By the reasoning of Remark 1.5, up to technical issues, we expect that $t_{\text{mix}}(\mathsf{G}_n^{\diamond})$ is $\gamma t_{\text{cov}}(\mathsf{G}_n)(1+o(1))$ for some $\gamma \in (1/2, 1)$, provided that:

- The Green's functions $G_n(x, y)$ for G_n are bounded above on the diagonal. (This should prevent clustering in $\mathcal{U}(\gamma t_{\text{cov}}(\mathsf{G}_n))$ for γ sufficiently close to 1.)
- The decay of $G_n(x, y)$ in terms of the distance between x and y is non-uniform in n. (This should lead to clustering in $\mathcal{U}(\gamma t_{\text{cov}}(\mathsf{G}_n))$ beyond $\gamma = \frac{1}{2}$, while [21] show that a uniform decay rate results in the threshold at $\frac{1}{2} t_{\text{cov}}(\mathsf{G}_n)$.)

One interesting family of graphs G_n of this type is given by the infinite cluster for super-critical Bernoulli percolation restricted to a thin slab of size $n \times n \times h$.

1.4 Outline of the proof of Theorem 1.3

Fixing $s \geq 1$, for any $\rho, z \in [0, 1]$, the functions

$$b_\rho(z) = 1 - \rho - \frac{s(1-z)^2}{1-\rho}, \quad \alpha_\rho(z) = \frac{sz^2}{\frac{\rho}{2} + \phi}, \tag{1.10}$$

control the structure of $\mathcal{U}(st_{\text{cov}}^{\square})$. Specifically, for any $\rho \in [0, 1]$ we associate with each $x \in V_n$ a type $z \in [0, 1]$ according to the number of excursions of the SRW, by time $st_{\text{cov}}^{\square}$, across the 2D cylindrical annulus of radii Mhn^{ρ} and M^2hn^{ρ}, centered at the 2D projection of x. Our parameters are such that for $n \to \infty$ followed by $M \to \infty$, WHP about $n^{2b_\rho(z)+o(1)}$ of the $n^{2(1-\rho)+o(1)}$ such annuli are of z-type and points $x \in V_n$ whose 2D projection is not far from the center of such z-type annulus, are unvisited by the SRW with probability $n^{-\alpha_\rho(z)+o(1)}$. Further, in Sect. 3.1 we confirm the following representation of $\Psi(\phi)$.

Lemma 1.7 *For $s \geq 1$ and $\rho, z \in [0, 1]$ let $b_\rho(z)$, $\alpha_\rho(z)$ be as in (1.10), with the convention that $b_1(z) = -\infty \mathbf{1}_{\{z \neq 1\}}$. Then, $\Psi(\cdot)$ of (1.9) emerges from the following variational problem:*

$$\Psi(\phi) = \inf\{s \geq 1 : \forall \rho, z \in [0, 1],\ b_\rho(z) \geq 0 \implies \alpha_\rho(z) \geq \rho\} \tag{1.11}$$

$$= \sup\{s \geq 1 : \exists \rho, z \in [0, 1],\ \textit{such that } b_\rho(z) \geq 0 \textit{ and } \alpha_\rho(z) \leq \rho\}. \tag{1.12}$$

Calling a z-type *ρ-admissible* if $b_\rho(z) > 0$, we know from (1.12) that for any $s < \Psi(\phi)$ there exist $\rho \in (0, 1)$ and ρ-admissible $z' \in (0, 1)$ with $\alpha_\rho(z') < \rho$. By continuity, the same applies for L large enough and $\rho_k = k/L$ with $k := [\rho L]$. Using this approximation, we show in Sect. 6 that the maximum discrepancy at time $st^{\square}_{\text{cov}}$ between "off-lamps" and "on-lamps" over a certain large enough (and spatially well separated) collection $\mathsf{A}_{2\text{D},k}$ of 2D disjoint cylinders of radii hn^{ρ_k}, far exceeds its value under the invariant (uniform) law for the SRW $\{X^{\diamond}_{\cdot}\}$. This statistics distinguishes between the law of the lamplighter chain at time $st^{\square}_{\text{cov}}$ and its stationary law, thereby yielding the stated lower bound on $t_{\text{mix}} = t_{\text{mix}}(\mathsf{G}_n(a)^{\diamond})$.

In contrast, by the dual variational problem (1.11), for $s > \Psi(\phi)$, if $b_\rho(z) \geq 0$ then the discrepancy of about $n^{-\alpha_\rho(z)}$ between the fractions of "off-lamps" and "on-lamps" within each such annulus, is buried under the inherent noise level of $n^{-\rho}$. Thus, all such statistics agree with the stated upper bound $t_{\text{mix}} \leq \Psi(\phi)t^{\square}_{\text{cov}}$. As explained in Sect. 3, to actually upper bound t_{mix}, one needs to control exponential moments of the size of $\mathcal{U}(st^{\square}_{\text{cov}})$ (more precisely, the size of the intersection of the unvisited sites by two independent random walks), which is the main technical challenge here. This is carried out by carefully estimating the number of excursions within consecutive annuli. Specifically, utilizing Hölder's inequality it suffices to separately consider each z-type and to do so on a certain sparse sub-lattice A of V_n, where at $\rho = 0$ the Bernoulli($n^{-\alpha_\rho(z)}$) variables corresponding to z-type unvisited sites in A are approximately independent even in terms of tail probabilities.

At any $\rho > 0$ the corresponding Bernoulli($n^{-\alpha_\rho(z)}$) variables are no longer asymptotically independent. To circumvent this problem, we group the vertices of A into nested, growing cylindrical annuli, centered at sub-lattices $\mathsf{A}_{2\text{D},k}$ that correspond to $\rho_k = k/L$, $k = 0, 1, \ldots, L$. Then, for each vertex/base point, the excursion counts across different scale annuli define a type profile $\underline{z} \in [0, 1]^{L+1}$ (that coincide at $k = 0$ with its z_0-type). We characterize the collection of all *possible* excursion count profiles by a careful extension of the concept of ρ-admissible z-types to that of admissible $\underline{z}$-types. The bulk of this article is thus about controlling the exponential moment of the number of unvisited sites per fixed admissible $\underline{z}$-type. Taking first $n \to \infty$, then $M \to \infty$ and finally $L \to \infty$, this is done in Sects. 3–5 via estimates on modified Green functions and utilizing stochastic domination to employ large deviation tail estimates for sums of i.i.d. variables.

We note in passing that while lower bounding t_{mix} we find that the most likely way to have z-type at the $O(h)$ size 2D annulus corresponding to $\rho = 0$, is via the profile $z(\rho) = 1 - (1 - \rho)(1 - z)$. However, we also show in Sect. 6 that such profiles are *highly unlikely* for the set $\mathcal{U}(st^{\square}_{\text{cov}})$. Thus, for a sharp upper bound on t_{mix} one must control the large deviations of *all admissible* $z(\cdot)$-type profiles.

2 Cover time for the thin torus: proof of Theorem 1.1

The Gaussian Free Field (in short GFF), on finite, connected graph $\mathsf{G} = (V, E)$, with respect to some fixed $v_0 \in V$, is the stochastic process $\{\eta_u\}_{u \in V}$ with $\eta_{v_0} = 0$, whose density with respect to Lebesgue measure on $V \backslash \{v_0\}$ is proportional to

$$\exp\left(-\frac{1}{4}\sum_{u\sim v}|\eta_u-\eta_v|^2\right), \tag{2.1}$$

where we used $u\sim v$ to denote $\{u,v\}\in E$. An important connection between GFF and the SRW on G is the following identity (see for example, [14, Theorem 9.17]):

$$\mathbf{E}\left[(\eta_u-\eta_v)^2\right]=R_{\mathrm{eff}}(u,v). \tag{2.2}$$

Here $R_{\mathrm{eff}}(u,v)$ is the effective resistance between $u,v\in V$ in the electrical network associated with G by placing a unit resistor on each edge $\{u,v\}\in E$ (and we sometimes use $R^{\mathsf{G}}_{\mathrm{eff}}(u,v)$ to emphasize the underlying graph G, in case of possible ambiguity).

Our proof of Theorem 1.1 relies on the following relation between the cover time $t_{\mathrm{cov}}(\mathsf{G})$ of G by SRW and the maximum of the corresponding GFF.

Theorem 2.1 [6, Theorem 1.1] *Consider a sequence of graphs $\mathsf{G}_n=(V_n,E_n)$ of uniformly bounded maximal degrees, such that $t_{\mathrm{hit}}(\mathsf{G}_n)=o(t_{\mathrm{cov}}(\mathsf{G}_n))$ as $n\to\infty$. For each n, let $\{\eta_v\}_{v\in V_n}$ denote a* GFF *on G_n with $\eta_{v_0^n}=0$ for certain $v_0^n\in V_n$. Then, as $n\to\infty$,*

$$t_{\mathrm{cov}}(\mathsf{G}_n)=(1+o(1))|E_n|\left(\mathbf{E}\left[\left\{\sup_{v\in V_n}\eta_v\right\}\right]\right)^2. \tag{2.3}$$

In light of the preceding theorem, the key to the proof of Theorem 1.1 is an estimate on the expected supremum for the associated GFF. To this end, we start with few estimates of effective resistances assuming familiarity with the connection between random walks and electric flows (see for example [20, Chapter 2]).

Lemma 2.2 *Let $\{X_n\}$ denote the* SRW *on the graph $\mathsf{G}=(V,E)$ started at some $o\in V$, independent of a Geometric random variable T. Then, there exists a current flow $\theta=\{\theta_{u,v}:\{u,v\}\in E\}$ with unit current source at o, current $p_v:=\mathbf{P}[X_T=v]$ reaching each $v\in V$, and the Dirichlet energy bound*

$$\mathcal{D}(\theta):=\sum_{(u,v)\in E}\theta^2_{u,v}\le\tfrac{1}{d_o}\mathbf{E}\left[\sum_{n=0}^{T}\mathbf{1}_{\{X_n=o\}}\right].$$

Proof Let $t=\mathbf{P}[T\ge 1]\in(0,1)$. Set $\overline{\mathcal{L}}(v):=\frac{1}{d_v}\mathbf{E}[\sum_{n=0}^{T}\mathbf{1}_{\{X_n=v\}}]$ and $N(u,v):=\sum_{n=0}^{T-1}\mathbf{1}_{\{X_n=u,X_{n+1}=v\}}$, for each $u,v\in V$. Then, due to the memory-less property of Geometric random variables, clearly

$$p_v=\mathbf{1}_{v=o}+\sum_{u:u\sim v}(\mathbf{E}[N(u,v)]-\mathbf{E}[N(v,u)])=\mathbf{1}_{v=o}+t\sum_{u:u\sim v}(\overline{\mathcal{L}}(u)-\overline{\mathcal{L}}(v)).$$

Thus, the current flow $\theta^\star_{u,v} := t(\overline{\mathcal{L}}(u) - \overline{\mathcal{L}}(v))$ on $(u,v) \in E$, together with external unit current into o, results with current p_v reaching each $v \in V$. Furthermore,

$$\sum_{(u,v)\in E} \left(\theta^\star_{u,v}\right)^2 = \tfrac{t^2}{2} \sum_{(u,v)\in E} (\overline{\mathcal{L}}(u) - \overline{\mathcal{L}}(v))^2 \le t \sum_{u\in V} (\overline{\mathcal{L}}(u) \sum_{v:v\sim u} (\overline{\mathcal{L}}(u) - \overline{\mathcal{L}}(v)))$$
$$\le t\overline{\mathcal{L}}(o) \sum_{v:v\sim o} (\overline{\mathcal{L}}(o) - \overline{\mathcal{L}}(v)) \le \overline{\mathcal{L}}(o),$$

since $t\sum_{v:v\sim u}(\overline{\mathcal{L}}(u) - \overline{\mathcal{L}}(v)) = -p_u \le 0$ for all $u \ne o$, and is at most one at $u = o$. □

We will also need the following claim.

Lemma 2.3 *For any graph* $\mathsf{G} = (V, E)$*, let* R *be the diameter for the effective resistance (of the* SRW*, namely with unit edge weights). Consider a collection of numbers* $\{\rho_v : v \in V\}$ *such that* $\sum_{v\in V} \rho_v = 0$ *and* $\frac{1}{2}\sum_{v\in V} |\rho_v| = 1$*, and let* Θ *denote the collection of all flows on* G *such that at any vertex* v *the difference between out-going and in-coming flow is* ρ_v*. Then,*

$$\min_{\theta\in\Theta}\{\mathcal{D}(\theta)\} \le R.$$

Proof Let $V^+ = \{v \in V : \rho_v \ge 0\}$ and $V^- = V\backslash V^+$. We define a function $w : V^+ \times V^- \mapsto [0,\infty)$ by $w(v,u) = |\rho_v \rho_u|$. By assumption on ρ, we see that

$$\sum_{u\in V^-} w(v,u) = \rho_v \text{ for all } v \in V^+ \text{ and } \sum_{u\in V^+} w(u,v) = -\rho_v \text{ for all } v \in V^-.$$

So in particular we have $\sum_{v\in V^+, u\in V^-} w(v,u) = 1$. For $(v,u) \in V^+ \times V^-$, let $\theta^{v,u}$ be an electric current which sends unit amount of flow from v to u (so in particular $\mathcal{D}(\theta^{v,u}) \le R_{\text{eff}}(v,u)$). Denoting $\theta := \sum_{v\in V^+,u\in V^-} w(v,u)\theta^{v,u}$, by our construction of $w(\cdot,\cdot)$ we see that $\theta \in \Theta$. It remains to bound the Dirichlet energy of θ. By Cauchy-Schwarz inequality, we get that

$$\mathcal{D}(\theta) = \sum_{e\in E} \theta_e^2 = \sum_{e\in E} \left(\sum_{v\in V^+,u\in V^-} w(v,u)\theta_e^{v,u} \right)^2 \le \sum_{e\in E} \sum_{v\in V^+,u\in V^-} w(v,u) \left(\theta_e^{v,u}\right)^2$$
$$\le \sum_{v\in V^+,u\in V^-} w(v,u)\mathcal{D}(\theta^{v,u}) \le R,$$

completing the proof of the lemma. □

Lemma 2.4 *With* $R_{\text{eff}}(\cdot,\cdot)$ *denoting effective resistances on* $\mathsf{G}_n(a) = (V_n, E_n)$*, we have that for all* $x, x' \in V_n$*,*

$$R_{\text{eff}}(x,x') \le 2r_3 + \tfrac{1}{a\pi} + o(1). \tag{2.4}$$

Furthermore, for $x = (y, 0)$ *and* $x' = (y', 0)$ *where* $y, y' \in \mathbb{Z}^2$ *and* $\|y - y'\|_{\mathbb{Z}_n^2} \geq 2a \log n$*, we have*

$$R_{\text{eff}}(x, x') = 2r_3 + \frac{1}{\pi a \log n} \left(\log \|y - y'\|_{\mathbb{Z}_n^2} \right) + o(1). \tag{2.5}$$

Proof Fixing arbitrary $x, x' \in V_n$ we establish (2.4) upon constructing a flow of $1 + o(1)$ current from x to x' whose Dirichlet energy is at most $2r_3 + 1/(a\pi) + o(1)$. To this end, for $\{X_n\}$ a SRW on $\mathsf{G}_n(a)$ and an independent Geometric random variable T of mean $(\log n)^4$, let $p_v = \mathbf{P}_x[X_T = v]$ for $v \in V_n$, and $p_{[i]} := \sum_{v \in \mathbb{Z}_n^2 \times \{i\}} p_v$ (namely, the probability that the "vertical" coordinate of X_T is at $i \in \mathbb{Z}_h$). We claim that

$$\frac{1}{6} \mathbf{E}_x \left[\sum_{t=0}^{T} \mathbf{1}_{\{X_t = x\}} \right] = r_3 + o(1). \tag{2.6}$$

In order to see the lower bound in (2.6), we note that the random walk is the same as a random walk in $\mathbb{Z}^3$ in the first $h = [a \log n]$ steps, during which period the expected number of visits accumulated at x is already $6(r_3 + o(1))$. Setting $N = (\log n)^5$, since $\mathbf{E}(T \mathbf{1}_{T \geq N}) \to 0$, we get the matching upper bound upon showing that

$$\mathbf{E}_x \left[\sum_{t=h}^{N} \mathbf{1}_{\{X_t = x\}} \right] = o(1). \tag{2.7}$$

To this end, with A denoting the event that simultaneously for all $h \leq t \leq N$, the number of vertical steps made by the SRW up to time t is in the range $(t/10, t/2)$, we clearly have that $\mathbf{P}[A^c] \leq (\log n)^{-r}$ for any r finite and all n large enough. Therefore

$$\mathbf{E}_x \left[\sum_{t=h}^{N} \mathbf{1}_{\{X_t = x\}} \right] \leq N\mathbf{P}[A^c] + \mathbf{E}_x \left[\sum_{t=h}^{N} \mathbf{1}_{\{X_t = x,\, A\}} \right] = o(1) + \sum_{t=h}^{N} \frac{O(1)}{\sqrt{\log n}} \frac{O(1)}{t} = o(1),$$

with the term $\frac{O(1)}{\sqrt{\log n}}$ upper bounding the probability of the SRW returning at time t to its starting height (referring to its vertical coordinate), and $O(1/t)$ bounding the probability of its 2D projection returning to the starting point, respectively (we obtain their independence upon conditioning on the number of vertical steps the SRW made up to time t). Combined with (2.7), this completes the verification of (2.6).

Now, by (2.6) and Lemma 2.2, there exists a unit current flow $\theta^{(x)}$ out of x, with current inflow of p_v into each $v \in V_n$ and

$$\mathcal{D}(\theta^{(x)}) = \sum_{(u,v) \in E_n} \left(\theta_{u,v}^{(x)} \right)^2 \leq r_3 + o(1). \tag{2.8}$$

Setting $p'_v := \mathbf{P}_{x'}[X_T = v]$ and $p'_{[i]} := \sum_{v \in \mathbb{Z}_n^2 \times \{i\}} p'_v$, we have by the same reasoning a unit current flow $\theta^{(x')}$ out of x', with current inflow p'_v into each $v \in V_n$ and

$$\mathcal{D}\left(\theta^{(x')}\right) \leq r_3 + o(1). \tag{2.9}$$

Furthermore, it is clear that with probability $1 - o(h^{-4/3})$ we have $T \geq h^{5/2}$, and thus by time T the vertical component of $\{X_t\}$ is so nearly uniformly distributed that (here we use the fact that the mixing time for a cycle of size k is $O(k^2)$ and we apply this fact to the random walks started at x and x' separately)

$$\max_i \left|hp_{[i]} - 1\right| = o(1) = \max_i \left|hp'_{[i]} - 1\right|. \tag{2.10}$$

Next, fixing $i \in \mathbb{Z}_h$ set $\rho_i, \rho'_i \in [0, 1]$ such that

$$\rho_i p_{[i]} = \rho'_i p'_{[i]} = \min\left\{p_{[i]}, p'_{[i]}\right\}$$

so there exist zero-net current flows on the sub-graph $\mathbb{Z}_n^2 \times \{i\}$ of $\mathsf{G}_n(a)$, with outflow $\rho_i p_v$ and inflow $\rho'_i p'_v$ at each $v \in \mathbb{Z}_n^2 \times \{i\}$. Let θ^i denote the flow of minimal Dirichlet energy among all such current flows and $|\theta^i| = \frac{1}{2} \sum_{v \in \mathbb{Z}_n^2 \times \{i\}} |\rho_i p_v - \rho'_i p'_v|$ its total flow. Then, by Lemma 2.3 we have that

$$\mathcal{D}(\theta^i) \leq |\theta^i|^2 \mathrm{Diam}_{R_{\mathrm{eff}}}\left(\mathbb{Z}_n^2\right),$$

where $\mathrm{Diam}_{R_{\mathrm{eff}}}(\mathbb{Z}_n^2)$ is the diameter for the resistance metric in the torus $\mathbb{Z}_n^2$. Note that

$$\sum_i (\theta^i)^2 \leq \max_i |\theta^i| \sum_i |\theta^i| \leq \max_i |\theta^i|,$$

and that thanks to (2.10),

$$|\theta^i| \leq \frac{1}{2} \sum_{v \in \mathbb{Z}_n^2 \times \{i\}} |\rho_i| p_v + |\rho'_i| p'_v = \min\left\{p_{[i]}, p'_{[i]}\right\} \leq \frac{1 + o(1)}{h}. \tag{2.11}$$

Combining the three preceding inequalities we obtain that

$$\sum_i \mathcal{D}(\theta^i) \leq \frac{1 + o(1)}{h} \mathrm{Diam}_{R_{\mathrm{eff}}}\left(\mathbb{Z}_n^2\right).$$

Combined with the standard estimate

$$\mathrm{Diam}_{R_{\mathrm{eff}}}\left(\mathbb{Z}_n^2\right) \leq \frac{1 + o(1)}{\pi} \log n \tag{2.12}$$

(see, e.g, [5, Lemma 3.4]), we arrive at

$$\sum_i \mathcal{D}(\theta^i) \le \frac{1+o(1)}{h} \mathrm{Diam}_{R_{\mathrm{eff}}}\left(\mathbb{Z}_n^2\right) \le \frac{1}{a\pi}(1+o(1)). \tag{2.13}$$

Consider now the current flow $\theta^\star$ from x to x' obtained by combining $\theta^{(x)}$ with the union of all flows $\{\theta^i, i \in \mathbb{Z}_h\}$ and the current flow $-\theta^{(x')}$. The net amount of current reaching sub-graph $\mathbb{Z}_n^2 \times \{i\}$ is then $p_{[i]} - p'_{[i]}$, so by (2.10) the flow from x to x' via $\theta^\star$ is 1+o(1), whereas by (2.8), (2.9) and (2.13), its Dirichlet energy is at most

$$\mathcal{D}\left(\theta^{(x)}\right) + \sum_i \mathcal{D}(\theta^i) + \mathcal{D}\left(\theta^{(x')}\right) \le 2r_3 + \tfrac{1}{a\pi} + o(1),$$

completing the proof of the upper bound (2.4).

For the lower bound, we let Q_x and $\mathsf{Q}_{x'}$ be cubes of side-length $\log\log n$ centered around x and x', respectively. Let $\mathsf{G}_{a,n}$ be the graph obtained by identifying $\partial\mathsf{Q}_x$ (also $\partial\mathsf{Q}_{x'}$) as a single vertex, as well as identifying $\{(z,i) : 1 \le i \le h\}$ as a single vertex for each $z \in \mathbb{Z}_n^2$. By Rayleigh monotonicity principle, we see that

$$R_{\mathrm{eff}}(x,x') \ge R_{\mathrm{eff}}(x, \partial\mathsf{Q}_x) + R_{\mathrm{eff}}(x', \partial\mathsf{Q}_{x'}) + R_{\mathrm{eff}}^{\mathsf{G}_{a,n}}(\partial\mathsf{Q}_x, \partial\mathsf{Q}_{x'}).$$

It is clear that $R_{\mathrm{eff}}(x, \partial\mathsf{Q}_x) = R_{\mathrm{eff}}(x', \partial\mathsf{Q}_{x'}) = r_3 + o(1)$. In addition, by the triangle inequality we see that

$$\begin{aligned} R_{\mathrm{eff}}^{\mathsf{G}_{a,n}}(\partial\mathsf{Q}_x, \partial\mathsf{Q}_{x'}) &\ge R_{\mathrm{eff}}^{\mathsf{G}_{a,n}}(x,x') - R_{\mathrm{eff}}^{\mathsf{G}_{a,n}}(x, \partial\mathsf{Q}_x) - R_{\mathrm{eff}}^{\mathsf{G}_{a,n}}(x', \partial\mathsf{Q}_{x'}) \\ &= \frac{1}{h}\left(R_{\mathrm{eff}}^{\mathbb{Z}_n^2}(y,y') - 2R_{\mathrm{eff}}^{\mathbb{Z}_n^2}(o, \partial\widetilde{\mathsf{Q}}_o)\right) \\ &= \tfrac{1}{\pi a \log n}\left(\log \|y-y'\|_{\mathbb{Z}_n^2}\right) + o(1), \end{aligned}$$

where $\widetilde{\mathsf{Q}}_o$ is a 2D box of side-length $\log\log n$ centered around o, and the last equality follows for example from [5, Lemma 3.4]. Altogether, this gives the desired lower bound on the effective resistance. □

The following lemma is useful in comparing the maxima of two Gaussian processes (see for example [12, Corollary 2.1.3]).

Lemma 2.5 (Sudakov–Fernique) *Let J be an arbitrary finite index set and let $\{\eta_j\}_{j\in J}$ and $\{\xi_j\}_{j\in J}$ be two centered Gaussian processes such that*

$$\mathbf{E}(\eta_j - \eta_k)^2 \ge \mathbf{E}(\xi_j - \xi_k)^2, \ \textit{for all } j,k \in J. \tag{2.14}$$

Then $\mathbf{E}[\max_{j\in J} \eta_j] \ge \mathbf{E}[\max_{j\in J} \xi_j]$.

We are now ready to estimate the maximum of the GFF on the thin torus.

Lemma 2.6 *Let* $\{\eta_v : v \in V_n\}$ *be a* GFF *on* $\mathsf{G}_n(a)$ *with* $\eta_{v_0} = 0$. *Then,*

$$\mathbf{E}\left[\max_{v\in V_n} \eta_v\right] = 2\sqrt{r_3 + \tfrac{1}{2a\pi} + o(1)}\sqrt{\log n}.$$

Proof We first prove the upper bound. By (2.2) and Lemma 2.4, we get that

$$\sup_{u,v\in V_n} \{\mathrm{Var}(\eta_u - \eta_v)\} := 2\sigma_n^2 \le 2r_3 + \tfrac{1}{a\pi} + o(1).$$

Thus, for i.i.d. centered Gaussian variables $\{X_u : u \in V_n\}$ of variance σ_n^2 we have by Lemma 2.5 that

$$\mathbf{E}\left[\max_{u\in V_n} \eta_u\right] \le \mathbf{E}\left[\max_{u\in V_n} X_u\right]. \tag{2.15}$$

Note that

$$\mathbf{E}\left[\max_{u\in V_n} X_u\right] \le \int_0^\infty \left[\left(\sum_{u\in V_n} \mathbf{P}(X_u \ge r)\right) \wedge 1\right] dr. \tag{2.16}$$

Further, for a centered Gaussian variable Y of variance σ^2 we have

$$\mathbf{P}(Y \ge r) \le \mathrm{e}^{-\frac{r^2}{2\sigma^2}}, \quad \forall r \ge 0.$$

Combined with (2.16) it yields that $\mathbf{E}[\max_{u\in V_n} X_u] \le 2\sigma_n\sqrt{\log n}(1 + o(1))$, so from (2.15) and the bound on σ_n we deduce the stated upper bound on $\mathbf{E}[\max_{u\in V_n} \eta_u]$.

For the lower bound, we employ a comparison argument. Let A be a 2D box of side-length $n/(8h)$, and let $\{\xi_v : v \in \mathsf{A}\}$ be a GFF on A with Dirichlet boundary condition (i.e., $\xi|_{\partial\mathsf{A}} = 0$). Now define mapping $g : \mathsf{A} \mapsto \mathsf{G}_n(a)$ by $g(v) = (2hv, 0)$. It is well known that (see, e.g., [18, Theorem 4.4.4 and Proposition 4.6.2])

$$R_{\mathrm{eff}}^{\mathsf{A}}(u, v) = \tfrac{1}{\pi} \log \|u - v\|_2 + O(1).$$

Combined with Lemma 2.4, it yields that for all $u, v \in \mathsf{A}$

$$R_{\mathrm{eff}}^{\mathsf{G}_n(a)}(g(u), g(v)) \ge (2ar_3\pi + 1 + o(1))h^{-1} R_{\mathrm{eff}}^{\mathsf{A}}(u, v),$$

where we have used the fact that $R_{\mathrm{eff}}^{\mathsf{A}}(u, v) \le \frac{1+o(1)}{\pi} \log n = \frac{(1+o(1))h}{a\pi}$. Applying (2.2) and Lemma 2.5, we obtain that

$$\mathbf{E}\left[\max_{v\in V_n} \eta_v\right] \ge \sqrt{2ar_3\pi + 1 + o(1)}h^{-1/2}\mathbf{E}\left[\max_{u\in\mathsf{A}} \xi_u\right].$$

Combined with [2, Theorem 2] which states that $\mathbf{E}[\max_{u\in\mathsf{A}} \xi_u] = (\sqrt{2/\pi}+o(1)) \log n$, this yields the desired lower bound on $\mathbf{E}[\max_{v\in V_n} \eta_v]$. □

As $|E_n| = 3an^2 \log n(1+o(1))$, upon combining Theorem 2.1 and Lemma 2.6, we immediately obtain Theorem 1.1.

3 Upper bound on mixing time: large deviations for admissible types

For the task of upper bounding $t_{\text{mix}}(\mathsf{G}_n(a)^\diamond, \delta)$ it suffices to compare the stationary law with a worst case initial one, for which purpose any non-random initial configuration will do. Further, since $t_{\text{mix}}(\mathsf{G}_n(a), \delta)$ is only $O(n^2)$ (see [19, Theorem 5.5]), we can and shall instead start for convenience at $X_0^\diamond$ having all lamps off and initial position uniformly chosen in V_n. Fixing $s' > s > \Psi(\phi)$ and using s in the sequel for setting the various excursion types, our goal is to show that the total-variation distance between the law of $X^\diamond_{s' t^\square_{\text{cov}}}$ and the uniform law goes to zero as $n \to \infty$. To this end, let $\widehat{\mathcal{U}}_{s'} := \mathcal{U}(s' t^\square_{\text{cov}})$ denote the subset of the vertices V_n of $\mathsf{G}_n(a)$ not visited by X up to time $s' t^\square_{\text{cov}}$, with $\widehat{\mathcal{U}}'_{s'}$ corresponding to a second, independent copy X' of the SRW on $\mathsf{G}_n(a)$. Then, with X_0 uniformly distributed, the L^2-norm of the density of the law of $X^\diamond_{s' t^\square_{\text{cov}}}$ with respect to the uniform law, is $\mathbf{E}\big[2^{|\widehat{\mathcal{U}}_{s'} \cap \widehat{\mathcal{U}}'_{s'}|}\big]$ (see [21, Proposition 3.2]). Adapting the argument of [21, Lemma 3.1], it thus suffices to find an event $\widehat{\mathcal{G}}$ measurable on the path of the SRW X on $\mathsf{G}_n(a)$ up to time $s' t^\square_{\text{cov}}$, such that as $n \to \infty$

$$\mathbf{P}[\widehat{\mathcal{G}}] \to 1, \quad \text{and} \quad \mathbf{E}\big[2^{|\widehat{\mathcal{U}}_{s'} \cap \widehat{\mathcal{U}}'_{s'}|} \mathbf{1}_{\widehat{\mathcal{G}}} \mathbf{1}_{\widehat{\mathcal{G}}'}\big] \to 1, \tag{3.1}$$

where $\widehat{\mathcal{G}}'$ corresponds to the independent copy X' of the SRW on $\mathsf{G}_n(a)$. Without $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}'$, the right side of (3.1) amounts to the L^2-convergence to 1 of the relevant density. Only L^1-convergence is needed for the total-variation mixing and using $\widehat{\mathcal{G}}$ helps eliminate some rare events that may dominate the second moment (see also the discussion immediately following [21, Proposition 3.2]).

To establish (3.1), fixing a large integer M we set hereafter

$$r := Mr' := M^2.$$

Note that for each $\underline{i} := (\underline{i}^{(1)}, \underline{i}^{(2)}) \in \{0, \ldots, 2r-1\}^3 \times \{0,1\}^3$ the points of

$$\mathsf{A}^\star_{3\mathrm{D}}(\underline{i}) := \big(\underline{i}^{(1)} + (2r\mathbb{N})^3\big) \cap \big([0,n)^2 \times [0,h) - 2r\,\underline{i}^{(2)}\big) \tag{3.2}$$

are at least $2r$ apart in $\mathsf{G}_n(a)$, whereas the union of the $(4r)^3$ sub-lattices $\mathsf{A}^\star_{3\mathrm{D}}(\underline{i})$ covers V_n. Indeed, $\mathsf{A}^\star_{3\mathrm{D}}(\underline{i})$ keeps minimal distance $2r$ from all faces that meet at the corner of $[0,n)^2 \times [0,h)$ indicated by $\underline{i}^{(2)}$, thereby assuring the stated $2r$-separation *on the torus* (even when $2r$ does not divide n or h).

Proceeding to produce in Definition 3.1 the "2D-well-centered" non-random subsets $\mathsf{A} = \mathsf{A}(\underline{i}, \underline{j})$ of $\mathsf{A}^\star_{3\mathrm{D}}(\underline{i})$, fix a large integer L and approximate the continuum of mesoscopic scales hn^ρ by $R''_k = h[n^{\rho_k}]$ for $\rho_k = k/L$, $k = 0, \ldots, L-1$ and $R''_L = [M^{-5} n]$. Setting thereafter

$$R_k := MR'_k := M^2 R''_k,$$

note that for any $L, M \geq 2$ and all n large enough,

$$2r < R''_0 < R'_0 < R_0 < 2R_0 < R''_1 < R'_1 < R_1 < 2R_1 < \cdots < R_L < n. \quad (3.3)$$

Assuming hereafter that (3.3) holds, for each $\underline{j}_k \in \{0, \ldots, (2R_k/R''_k) - 1\}^2 \times \{0, 1\}^2$ the points of

$$\mathsf{A}^\star_{2\mathrm{D},k}(\underline{j}_k) := \left(R''_k\, \underline{j}_k^{(1)} + (2R_k\mathbb{N})^2\right) \cap \left([0, n)^2 - 2R_k\, \underline{j}_k^{(2)}\right) \quad (3.4)$$

are $2R_k$ apart in the 2D torus $\mathbb{Z}_n^2$ (thanks to the guard bands associated with $\underline{j}_k^{(2)}$), whereas for each $0 \leq k \leq L$ the union of $\mathsf{A}^\star_{2\mathrm{D},k}(\underline{j}_k)$ over the $(4R_k/R''_k)^2$ possible values of $\underline{j}_k$ covers $\mathbb{Z}_n^2$.

Definition 3.1 For any $\underline{i}$ and $\underline{j} := (\underline{j}_0, \underline{j}_1, \ldots, \underline{j}_L)$, let $\mathsf{A} := \mathsf{A}(\underline{i}, \underline{j})$ denote the subset of those $x = (x_1, x_2, x_3) \in \mathsf{A}^\star_{3\mathrm{D}}(\underline{i})$ whose 2D-projection (x_1, x_2) lies for each $k = 0, 1, \ldots, L$ within the R''_k-sized square centered at some $y_k(x) \in \mathsf{A}^\star_{2\mathrm{D},k}(\underline{j}_k)$.

Note that V_n is covered by the union of the

$$\kappa' := (4r)^3(4M^2)^{2(L+1)} \quad (3.5)$$

sets $\mathsf{A}(\underline{i}, \underline{j})$, with $\kappa' = \kappa'(M, L)$ independent of n. We shall consider (3.1) for an event $\widehat{\mathcal{G}}$ of the form

$$\widehat{\mathcal{G}} = \bigcap_{\underline{i},\underline{j}} \widetilde{\mathcal{G}}_{\underline{i},\underline{j}}, \quad (3.6)$$

where each event $\widetilde{\mathcal{G}}_{\underline{i},\underline{j}}$ on the path of the SRW X on $\mathsf{G}_n(a)$ up to time $s' t^{\square}_{\mathrm{cov}}$ is defined via excursion counts associated with the points of $\mathsf{A} = \mathsf{A}(\underline{i}, \underline{j})$. Specifically, see (3.11) in the sequel (and Definition 3.3), for the precise choice of $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_{\underline{i},\underline{j}}$. Then, by the union bound

$$\mathbf{P}\left[\widehat{\mathcal{G}}^c\right] \leq \kappa' \max_{\underline{i},\underline{j}} \mathbf{P}\left[\widetilde{\mathcal{G}}^c_{\underline{i},\underline{j}}\right].$$

So, decomposing the set $\widehat{\mathcal{U}}_{s'} \cap \widehat{\mathcal{U}}'_{s'}$ in the RHS of (3.1) according to its intersections with the various $\mathsf{A}(\underline{i}, \underline{j})$, by Hölder's inequality we get (3.1) upon showing that for any $\underline{i}, \underline{j}$, as $n \to \infty$

$$\kappa' \mathbf{P}\left[\widetilde{\mathcal{G}}^c_{\underline{i},\underline{j}}\right] \to 0 \quad \text{and} \quad \mathbf{E}\left[2^{\kappa' |\mathsf{A}(\underline{i},\underline{j}) \cap \widehat{\mathcal{U}}_{s'} \cap \widehat{\mathcal{U}}'_{s'}|}\, \mathbf{1}_{\widetilde{\mathcal{G}}_{\underline{i},\underline{j}}} \mathbf{1}_{\widetilde{\mathcal{G}}'_{\underline{i},\underline{j}}}\right] \to 1. \quad (3.7)$$

Proceeding to prove (3.7) for some fixed $(\underline{i}, \underline{j})$ we avoid crowded notations by omitting hereafter the specific $(\underline{i}, \underline{j})$ from all expressions. In particular, given $(\underline{i}, \underline{j})$, to each

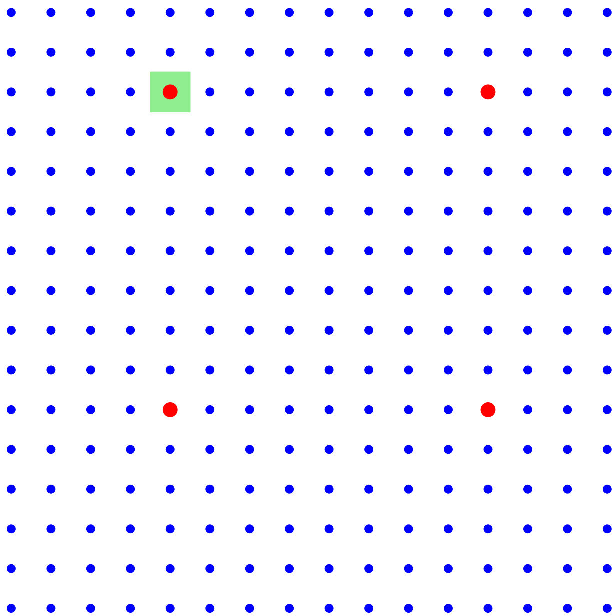

Fig. 3 Illustration of a set $\mathsf{A}^{\star}_{2\mathrm{D},k}(\underline{j}_k)$ as red dots of spacing $2R_k$ within a 2D sub-lattice of blue dots at spacing R''_k. If (x_1, x_2) is in the green square (of side length R''_k), then its center red point be $y_k(x)$. Here $R_k = 4R''_k$ (that is, $M = 2$) (color figure online)

$x \in \mathsf{A} = \mathsf{A}(\underline{i}, \underline{j})$ corresponds a unique vector $\underline{y} = (y_0, \dots, y_L)$ of *base points* $y_k = y_k(x) \in \mathsf{A}^{\star}_{2\mathrm{D},k}$ (with $y_k(x)$ the closest point to (x_1, x_2) in $\mathsf{A}^{\star}_{2\mathrm{D},k}$; See Fig. 3 for an illustration of $\mathsf{A}^{\star}_{2\mathrm{D},k}$ and $x \mapsto y_k(x)$). We further let

$$\mathsf{A}_{2\mathrm{D},k} := \{y \in \mathsf{A}^{\star}_{2\mathrm{D},k} : y = y_k(x) \text{ for some } x \in \mathsf{A}\}, \tag{3.8}$$

denote the collection of all possible k-th level base points, using the short notation $\mathsf{A}_{2\mathrm{D}}$, R, R', R'' and $y(x)$ for $\mathsf{A}_{2\mathrm{D},0}$, R_0, R'_0, R''_0 and $y_0(x)$, respectively.

Next, enumerating over $x \in \mathsf{A}$ yields the disjoint 3D-annuli of outer radius r and inner radius r', between the Euclidean balls $\mathsf{B}(x, r)$ and $\mathsf{B}(x, r')$ in $\mathsf{G}_n(a)$. For each $0 \le k \le L$, consider also the disjoint annuli of outer and inner radii R_k and R'_k, respectively, between the cylinders $\mathsf{C}(y_k, R_k)$ and $\mathsf{C}(y_k, R'_k)$ of height h in $\mathsf{G}_n(a)$, based on the 2D Euclidean disks centered at $y_k \in \mathsf{A}_{2\mathrm{D},k}$. As illustrated in Fig. 4,

for any k, each cylindrical annulus decomposes the path of the SRW on $\mathsf{G}_n(a)$ into R_k-excursions. Each such excursion starts at the outer cylinder boundary and run until hitting the inner cylinder boundary (which we call the excursion's *external part*), then goes back till exiting the outer cylinder (called the excursion's *internal part*). Note that for each k, conditional on their starting and ending points, the internal parts of various R_k-excursions of our collection of cylindrical annuli are mutually independent of each other. For n large enough so (3.3) holds, by the hierarchical structure of the sub-lattices $\mathsf{A}^{\star}_{2\mathrm{D},k}$, the vector $\underline{y}$ associated with $x \in \mathsf{A}$ is uniquely determined by $y(x)$. More generally, each R_{k-1}-sized cylindrical annulus centered at $y \in \mathsf{A}_{2\mathrm{D},k-1}$, $k \ge 1$,

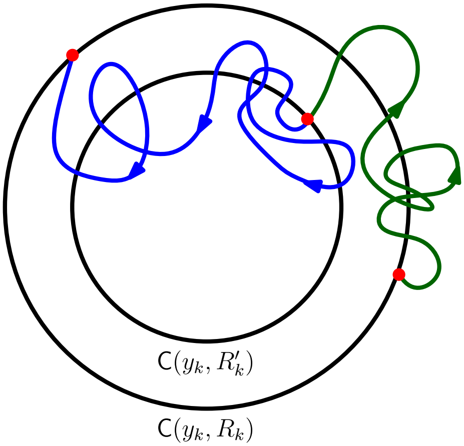

Fig. 4 The $2D$ projection of an R_k-excursion of the random walk, from the boundary of a cylinder of radius R_k back to itself via the boundary of a concentric cylinder of radius R'_k. Indicated in dark green (resp. blue) is the external (resp. internal) part of the excursion (color figure online)

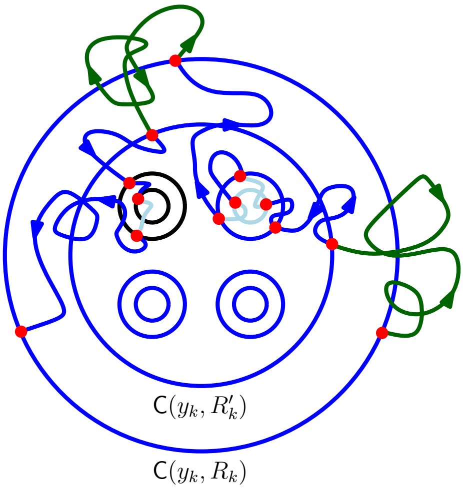

Fig. 5 The R_k-excursions across disjoint cylindrical annuli at different scales decompose into a tree structure, with the internal part of any R_{k-1}-excursions (light blue), within the internal part of some R_k-excursion (blue). For well-separated annuli, the entrance and exit points of an R_{k-1}-excursion are approximately independent of the entrance and exit points of the parent R_k-excursion (color figure online)

must be strictly inside $\mathsf{C}(y_k, R'_k)$ for *some uniquely specified* $y_k \in \mathsf{A}_{2\mathrm{D},k}$. Hence, as illustrated in Fig. 5, the R_{k-1}-excursions of the y-centered annulus decompose the internal parts of each of the R_k-excursions for the annulus centered at y_k. Similarly, for n large enough and $x \in \mathsf{A}$, each $\mathsf{B}(x, r)$ is strictly inside $\mathsf{C}(y(x), R')$, decomposing the internal parts of each of the R-excursions of the cylindrical annulus around $y(x)$, into what we call r-excursions (i.e., whose external part starts at $\partial\mathsf{B}(x, r)$ and run till hitting $\mathsf{B}(x, r')$, followed by the internal part up to the exit from $\mathsf{B}(x, r)$). Here again, conditional on their starting and ending points the internal parts of the various r-excursions associated with the collection A are independent of each other.

As shown in Sect. 4.1,

$$\overline{\mathsf{NC}}^{\star}(s) := 2s\frac{(\log n)^2}{\log(R/R')} \quad \text{and} \quad \overline{\mathsf{NB}}^{\star}(s) := \frac{4sr'}{a}\log n, \tag{3.9}$$

are the typical counts of R_k-excursions and r-excursions, respectively, by time $st_{\text{cov}}^{\square}$. Utilizing these, we next summarize which large deviations of the counts of cylindrical and ball excursions around $x \in \mathsf{A}$, are of concern in our proof of (3.7). We will show that WHP, at least $\overline{\mathsf{NC}}^{\star}(s)$ of the R_L-excursions around any $y_L \in \mathsf{A}_{2\text{D},L}$ are completed by time $s't_{\text{cov}}^{\square}$. Hence, our concepts of a z-type point $x \in \mathsf{A}$ and a $\underline{z}$-type $y(x) \in \mathsf{A}_{2\text{D}}$, amount to having about $z^2\,\overline{\mathsf{NB}}^{\star}(s)$ of the corresponding r-excursions around x, or respectively, having about $z_k^2\,\overline{\mathsf{NC}}^{\star}(s)$ of the corresponding R_k-excursions around $y_k(x)$, $k = 0, \ldots, L-1$, during the first R_L-excursions around $y_L(x)$.

Definition 3.2 Fix $s \in (\Psi(\phi), s')$ and small $\eta > 0$ such that $1/\eta$ is integer.

(a) For $\underline{z} = (z_0, \ldots, z_L)$ with $z_k \le z_L = 1$ and $z_k \in \eta\mathbb{N}$, $k = 0, \ldots, L-1$, we say that $\underline{y} = (y_0, \ldots, y_L)$, or equivalently, that $y_0 \in \mathsf{A}_{2\text{D},0}$, is of $\underline{z}$-type if the first $(z_k - 2\eta)^2\overline{\mathsf{NC}}^{\star}(s)$ of the R_k-excursions for the cylindrical annulus centered at y_k, are completed within the first $\overline{\mathsf{NC}}^{\star}(s)$ R_L-excursions for cylindrical annulus centered at y_L. In case $z_k < 1$ we further require that the first $(z_k - \eta)^2\overline{\mathsf{NC}}^{\star}(s)$ are not completed during these R_L-excursions.

(b) Similarly, $x \in \mathsf{A}$ is called *of z-type* (for $z \in \eta\mathbb{N}$), if the first $(z - 3\eta)^2\overline{\mathsf{NB}}^{\star}(s)$ of the r-excursions around x, are completed within the first $\overline{\mathsf{NC}}^{\star}(s)$ R_L-excursions for cylindrical annulus centered at $y_L(x)$, where for $z < 1$ we also require that the first $(z - 2\eta)^2\overline{\mathsf{NB}}^{\star}(s)$ of those r-excursions are not completed during said R_L-excursions.

Next, note that $\mathsf{A} \cap \widehat{\mathcal{U}}_{s'}$ is the disjoint union of

$$\widetilde{\mathcal{U}}_{s',\underline{z}} := \{x \in \mathsf{A} \cap \mathcal{U}_{s'} : y(x) \text{ of } \underline{z}\text{-type}\}, \tag{3.10}$$

over the at most $\kappa_o = \eta^{-L}$ possible $\underline{z}$-types induced on $\mathsf{A}_{2\text{D}}$ by the SRW X on $\mathsf{G}_n(a)$. Likewise, $\mathsf{A} \cap \widehat{\mathcal{U}}'_{s'}$ is the disjoint union of the sets $\widetilde{\mathcal{U}}'_{s',\underline{z}'}$ defined in terms of the types $\underline{z}'$ induced on $\mathsf{A}_{2\text{D}}$ by the independent SRW X' on $\mathsf{G}_n(a)$. We set

$$\widetilde{\mathcal{G}} := \bigcap_{\underline{z}} \mathcal{G}_{\underline{z}}, \tag{3.11}$$

where each event $\mathcal{G}_{\underline{z}}$ on the path of the SRW X on $\mathsf{G}_n(a)$ up to time $s't_{\text{cov}}^{\square}$ is now associated with a specific choice of both $\mathsf{A} = \mathsf{A}(\underline{i}, \underline{j})$ and $\underline{z}$ (see Definition 3.3 below). For κ' from (3.5) and the constants η, L from Definition 3.2, we set

$$\kappa := \kappa'\kappa_o^2 = (4r)^3(4M^2)^{2(L+1)}\eta^{-2L}, \tag{3.12}$$

which is also independent of n. Similarly to our move from (3.1) to (3.7), get by the union bound and Hölder's inequality that (3.7) holds provided that as $n \to \infty$, for any choice of $(\underline{i}, \underline{j})$ and any two types $\underline{z}, \underline{z}'$,

$$\kappa \mathbf{P}\left[\mathcal{G}^c_{\underline{z}}\right] \to 0, \tag{3.13}$$

$$\mathbf{E}\left[2^{\kappa|\widetilde{\mathcal{U}}_{s',\underline{z}}\cap\widetilde{\mathcal{U}}'_{s',\underline{z}'}|} \mathbf{1}_{\mathcal{G}_{\underline{z}}}\mathbf{1}_{\mathcal{G}'_{\underline{z}'}}\right] \to 1 \tag{3.14}$$

(with $\mathcal{G}'_{\underline{z}'}$ corresponding to the second, independent copy X' of the SRW on $\mathsf{G}_n(a)$). We proceed to define the truncation events $\mathcal{G}_{\underline{z}}$ for (3.13)–(3.14).

Definition 3.3 For each $s < s'$, $\eta > 0$ and type $\underline{z}$, let $\mathcal{G}_{\underline{z}} = \mathcal{G}_{\underline{z}}(s, \eta)$ be the event consisting of:

(a) By time $s' t^{\square}_{\text{cov}}$ the SRW on $\mathsf{G}_n(a)$ completes for each R_L-sized cylindrical annulus centered at $y_L \in \mathsf{A}_{2\text{D},L}$ the corresponding first $\overline{\mathsf{NC}}^\star(s)$ excursions.
(b) For $\rho_k = k/L$, $k = 0, \ldots, L-1$, there are at most $n^{2b_{\rho_k}(z_k)}$ points $y_k \in \mathsf{A}_{2\text{D},k}$ to which corresponds some $y_0 \in \mathsf{A}_{2\text{D},0}$ of $\underline{z}$-type.
(c) If $x \in \mathsf{A}$ is such that $y_0(x)$ is of $\underline{z}$-type (cylindrical annuli), then for some $z \geq z_0$ the point x is also of z-type (in terms of r-excursions).

From Definition 3.3(b), we see that under the event $\mathcal{G}_{\underline{z}}$ there is no $y(x)$ of $\underline{z}$-type, unless $b_{\rho_k}(z_k) \geq 0$ for all $0 \leq k < L$. This is precisely the following requirement (3.15) that $\underline{z}$ be admissible (so it suffices to establish (3.14) only for admissible types $\underline{z}$, $\underline{z}'$).

Definition 3.4 Fixing $s \geq 1$, we say that a $\underline{z}$-type is admissible, if and only if

$$\sqrt{s} \leq \min_{k=0,\ldots,L-1} \left\{\frac{1-\rho_k}{1-z_k}\right\} \tag{3.15}$$

for $\rho_k = k/L$, as in Definition 3.3.

Denoting by $H_{x,z}$ the event of not hitting x during the first $z^2\overline{\mathsf{NB}}^\star(s)$ of the r-excursions of X around x, requirements (a) and (c) of Definition 3.3 imply that under the event $\mathcal{G}_{\underline{z}}$ the set $\widetilde{\mathcal{U}}_{s',\underline{z}}$ of (3.10) is a subset of

$$\mathcal{U}_{s,\underline{z}} := \{x \in \mathsf{A} : y(x) \text{ of } \underline{z}\text{-type}, \quad H_{x,z_0-3\eta} \text{ occurs}\} \tag{3.16}$$

(see also Definition 3.2 of z-type). Similarly, $\widetilde{\mathcal{U}}'_{s',\underline{z}'} \subseteq \mathcal{U}'_{s,\underline{z}'}$ under the event $\mathcal{G}'_{\underline{z}'}$. Hence, upon proving (3.13) for $\mathcal{G}_{\underline{z}}$ of Definition 3.3, it suffices to show that for any admissible $\underline{z}$-type and $\underline{z}'$-type, as $n \to \infty$,

$$\mathbf{E}\left[2^{\kappa|\mathcal{U}_{s,\underline{z}}\cap\mathcal{U}'_{s,\underline{z}'}|} \mathbf{1}_{\mathcal{G}_{\underline{z}}}\mathbf{1}_{\mathcal{G}'_{\underline{z}'}}\right] \to 1. \tag{3.17}$$

3.1 Variational formulas and admissible annuli profiles

We first establish the variational representations of Lemma 1.7 for $\Psi(\phi)$ of (1.9) whose relevance to the asymptotic structure of $\mathcal{U}(st^{\square}_{\text{cov}})$ has already been discussed in Sect. 1.4.

Proof of Lemma 1.7 First, set $\mathsf{h}(\rho) := \sqrt{\rho(\phi + \rho/2)}$, $t := \sqrt{s}$ and

$$t_\star = \sup_{\rho \in [0,1]} \{\mathsf{h}(\rho) + 1 - \rho\}. \tag{3.18}$$

The conditions $b_\rho(z) \geq 0$ and $\alpha_\rho(z) \geq \rho$ are then re-expressed as $tz \geq t - (1-\rho)$ and $tz \geq \mathsf{h}(\rho)$, respectively. So, with the optimal choice being $z = z_\star := 1 - (1-\rho)/t$, it follows that (1.11) holds if and only if $t \geq t_\star$. That is, $\Psi(\phi) = t_\star^2$. Further, considering at $t = t_\star$ the optimal $z_\star = \mathsf{h}(\rho)/(\mathsf{h}(\rho) + 1 - \rho)$, yields the identity (1.12). Finally, in (3.18) the optimal choice is $\rho = \rho_\star = (\sqrt{2} - 1)\phi$, but in case $\phi \geq 1/(\sqrt{2} - 1)$ it is out of range and one needs to settle instead for $\rho = 1$. One easily checks that $\mathsf{h}(\rho_\star) = \phi/\sqrt{2}$, while $\mathsf{h}(1) = \sqrt{\phi + 1/2}$, hence with $t_\star$ monotone increasing in ϕ it is easy to confirm from the preceding that $t_\star^2 = \Psi(\phi)$ is given by the explicit formula (1.9), as claimed. □

Denoting hereafter $\alpha_0(\cdot)$ of (1.10) by $\alpha(\cdot)$, we proceed with an analysis lemma that is key to the success of our scheme for bounding the exponential moments as in (3.17) for all admissible $\underline{z}$-types and $s > \Psi(\phi)$.

Lemma 3.5 *Let $\Psi_{L,\eta}(\phi)$ denote, per given L and η, the minimal value of $s \geq 1$, such that if type $\underline{z}$ is* admissible *(see Definition 3.4), then for any $m = 0, \ldots, L$,*

$$\gamma_{m,\eta}(\underline{z}) := \alpha(z_0 - 4\eta) - m\eta - \frac{1}{L} - \sum_{k=1}^{m} \left[\frac{1}{L} - 2sL(z_k - z_{k-1} - 2\eta)_+^2\right] \geq \eta. \tag{3.19}$$

Then, with $\Psi(\cdot)$ given by the variational problem (1.11), we have that

$$\Psi(\phi) = \limsup_{L\to\infty} \lim_{\eta\to 0} \{\Psi_{L,\eta}(\phi)\}. \tag{3.20}$$

Proof Recall that $z_L = 1$ and note that the limit

$$\Psi_L(\phi) := \lim_{\eta\to 0} \{\Psi_{L,\eta}(\phi)\},$$

exists and corresponds to the requirement that $\gamma_{m,0}(\underline{z}) \geq 0$ for $m = 0, \ldots, L$ and admissible $\underline{z}$. Further, setting $\Delta_k := tL(z_k - z_{k-1})$, for $k = 1, \ldots, L$ and $t := \sqrt{s}$ we have from (1.10) that

$$\phi\,\alpha(z_0) = (tz_0)^2 = \left(t - \frac{1}{L}\sum_{k=1}^{L} \Delta_k\right)^2,$$

yielding that $\sqrt{\Psi_L(\phi)}$ is merely the infimum over all $t \geq 1$ such that for $m = 0, \ldots, L$,

$$\left(t - \frac{1}{L}\sum_{k=1}^{L} \Delta_k\right)^2 \geq \phi\left(\frac{m+1}{L} - \frac{2}{L}\sum_{k=1}^{m} (\Delta_k)_+^2\right), \tag{3.21}$$

whenever $\underline{z} \in [0,1]^{L+1}$ satisfies (3.15). That is, denoting by $\mathcal{D}$ the collection of all $\underline{\Delta} := (\Delta_1, \ldots, \Delta_L) \in \mathbb{R}^L$ such that

$$\delta_r := \frac{1}{L-r} \sum_{k=r+1}^{L} \Delta_k \in [0,1] \quad \forall 0 \le r < L, \tag{3.22}$$

we have that

$$\sqrt{\Psi_L(\phi)} = \max_{m=0}^{L} \max_{\underline{\Delta} \in \mathcal{D}} \{t_m(\underline{\Delta})\},$$

with $t_m(\underline{\Delta})$ the smallest $t \ge 1$ for which (3.21) holds, per given m and $\underline{\Delta}$.

The value of $t_m(\underline{\Delta})$ depends only on δ_m and $(\Delta_1, \ldots, \Delta_m)$. Further, given δ_m and $\Delta := m^{-1} \sum_{k=1}^{m} \Delta_k$, by Cauchy-Schwarz the maximal value of $t_m(\underline{\Delta})$ is attained when $\Delta_k = \Delta$ for all $1 \le k \le m$. Thus, setting $\delta = \delta_m$, we deduce that $\sqrt{\Psi_L(\phi)}$ is bounded above by the minimal $t \ge 1$ such that

$$(t - (1-\rho)\delta - \rho\Delta)^2 \ge \phi\rho \left[1 - 2(\Delta)_+^2\right] + \frac{\phi}{L}, \tag{3.23}$$

for any $\delta \in [0,1]$, $\Delta \in \mathbb{R}$ and $\rho \in [0,1]$ for which $\rho L = m$ is integer valued. Note that (3.23) trivially holds whenever $\Delta > 1$ and $\rho > 0$ (whereas for $\rho = 0$ the value of Δ is irrelevant). Further, since $t \ge 1 \ge \rho, \delta \ge 0$, if (3.23) holds for $\Delta = 0$, it also holds for any $\Delta < 0$. Consequently, it suffices to consider (3.23) only for $\Delta, \delta \in [0,1]$. Each choice of (Δ, δ) in the latter range corresponds to $\underline{\Delta} = (\Delta, \ldots, \Delta, \delta, \ldots, \delta)$ in $\mathcal{D}$, hence we conclude that the right-side of (3.20) *equals* the minimal $s = t^2 \ge 1$ satisfying (3.23) for all $\delta \in [0,1]$, $\rho \in (0,1]$ and $\Delta \ge 0$. To match this with (1.11) we equivalently set $(1-\rho)\delta = t(1-w)$ and $\rho\Delta = t(w-z)$ with $1 \ge w \ge z$ such that $b_\rho(w) \ge 0$ for $s = t^2$ (corresponding to $\delta \le 1$). This transforms (3.23), in terms of z and w, to the inequality

$$\alpha(z) + \frac{2s(w-z)^2}{\rho} \ge \rho. \tag{3.24}$$

Now, by elementary calculus we find that

$$\alpha_\rho(w) = \inf_{z \le w} \left\{ \alpha(z) + \frac{2s(w-z)^2}{\rho} \right\} \tag{3.25}$$

(with infimum attained at $z_\star := (2/\rho)w/(2/\rho + 1/\phi)$). Comparing the preceding with (1.11) we thus conclude that (3.20) holds, as claimed. □

3.2 Tail behavior for admissible excursion counts

Our approach to proving the upper bound in Theorem 1.3 is to establish (3.13) and (3.17) for

$$s' = s + \epsilon = \Psi_{L,\eta}(\phi) + 2\epsilon, \tag{3.26}$$

when $n \to \infty$ followed by $M \to \infty$. As explained before, this would imply that $t_{\text{mix}} \leq (s' + o(1))t^{\square}_{\text{cov}}$ and consequently, by Lemma 3.5, upon taking $\eta \downarrow 0$, $L \to \infty$ and finally $\epsilon \downarrow 0$ we get that $t_{\text{mix}} \leq (\Psi(\phi) + o(1))t^{\square}_{\text{cov}}$.

To this end, we use the following notation.

Definition 3.6 Let $\mathsf{NC}_{y_k,k,j,w}$, for $k < j \leq L$ and $w \in [0, 1]$ be the number of R_k-excursions for $y_k \in \mathsf{A}_{2\text{D},k}$, completed during the first $w^2\overline{\mathsf{NC}}^\star(s)$ R_j-excursions for the corresponding $y_j \in \mathsf{A}_{2\text{D},j}$ (with $\mathsf{NC}_y := \mathsf{NC}_{y,0,L,1}$). Let $\mathsf{NC}_{y_L,L}$ be the number of R_L-excursions around $y_L \in \mathsf{A}_{2\text{D},L}$ which are completed by time $s't^{\square}_{\text{cov}}$. Next, for $x' \in \mathsf{B}(x, R'')$ and $z \geq \eta$, let $\mathsf{NB}^{x'}_{x,z}$ be the number of r-excursions around $x \in \mathsf{A}$ during the first $z^2\overline{\mathsf{NC}}^\star(s)$ excursions of the R_0-cylindrical annulus centered at x'.

As detailed in Sect. 3.3, both (3.13) and (3.17) follow from the next two lemmas, whose proofs are provided in Sects. 4 and 5.

Lemma 3.7 *Fix $s > 1 \geq z > \eta > 0$. If $M \geq M_0(\eta, z)$ and $n \geq n_0(M)$, then*

$$\mathbf{P}[H_{x,z}] \leq n^{-\alpha(z-\eta)} \quad \forall x \in V_n. \tag{3.27}$$

Further, uniformly over $x \in V_n$ and $x' \in B(x, R'')$, as $n \to \infty$,

$$n^2(\log n)\mathbf{P}\left[NB^{x'}_{x,z} < (z-\eta)^2\overline{NB}^\star(s)\right] \to 0 \tag{3.28}$$

Remark 3.8 The bound (3.27) remains in effect when conditioned on $X_0 = v$ and the start and end points of all r-excursions around x (see Proposition 4.9). Similarly, from (4.37) the convergence in (3.28) holds uniformly with respect to the position of x within $\mathsf{B}(x', R'')$ and the start/end points of the R-excursions around x'.

Lemma 3.9 *For any fixed $s', s > 1$, any positive integer L, $w, z \geq \widetilde{\eta} \geq 0$ and $L \geq j > k \geq 0$, we have for all $M \geq M_1(\widetilde{\eta}, z, w, j, k)$ large enough, as $n \to \infty$, that uniformly over $y_L \in A_{2\text{D},L}$ and $y_k \in A_{2\text{D},k}$,*

$$n^M\mathbf{P}\left[|NC_{y_L,L} - \overline{NC}^\star(s')| \geq \widetilde{\eta}\,\overline{NC}^\star(s')\right] \to 0, \tag{3.29}$$

$$\limsup_{n\to\infty}\left|\frac{\log \mathbf{P}[NC_{y_k,k,j,w}(s) \leq (z-\widetilde{\eta})^2\overline{NC}^\star(s)]}{\log n} + \frac{2s(w-z)_+^2}{\rho_j - \rho_k}\right| \leq \widetilde{\eta}. \tag{3.30}$$

Remark 3.10 See Proposition 4.1 which implies (3.29). In Sect. 5 we further show that (3.30) holds uniformly in $x \in \mathsf{A}$ with $y_k(x) = y_k$ (i.e., over the relative position of

y_k in the R''_j-sized square centered at $y_j = y_j(x)$), and uniformly with respect to the start/end points of the R_j-excursions around y_j.

3.3 The proof of (3.13) and (3.17)

First, as soon as $(1-\widetilde{\eta})s' > s$ we deduce from (3.29) upon taking the union over the at most M^6 possible values of y_L, that requirement (a) in Definition 3.3 is satisfied with probability going to one as $n \to \infty$. Next, for $k < L$ let Y_k denote the number of $y_k \in \mathsf{A}_{2\mathrm{D},k}$ to which corresponds some $y_0 \in \mathsf{A}_{2\mathrm{D},0}$ of $\underline{z}$-type.

If $z_k < 1$ it follows by Definition 3.2 that necessarily $\mathsf{NC}_{y_k,k,L,1} \le (z_k-\eta)^2\overline{\mathsf{NC}}^\star(s)$ for any such y_k. With $|\mathsf{A}^\star_{2\mathrm{D},k}| \le \lceil n/(2R_k)\rceil^2 \le n^{2-2\rho_k}$ upon considering (3.30) for $j = L$, $\widetilde{\eta} = (\eta/2)^2$, $w = 1$ and $z = z_k - \eta + \widetilde{\eta}$, we see that for n large enough, $\mathbf{E}(Y_k) \le n^{2b_{\rho_k}(z_k)-\widetilde{\eta}}$. Hence, by Markov's inequality and union over $0 \le k < L$, we deduce that Definition 3.3(b) also holds with probability going to one as $n \to \infty$ (the case $z_k = 1$ trivially holds by the preceding bound on $|\mathsf{A}^\star_{2\mathrm{D},k}|$). In particular, as soon as $s(1-4\eta)^2 > 1$, necessarily $z_0 \ge 5\eta$, whereupon if $y_0(x)$ is of $\underline{z}$-type and x is not of z-type for some $z \ge z_0$, then $\mathsf{NB}^{x'}_{x,z_0-2\eta} < (z_0-3\eta)^2\overline{\mathsf{NB}}^\star(s)$, for $x' := (y_0(x), x_3) \in \mathsf{B}(x, R'')$. Combining (3.28) at $z = z_0 - 2\eta$ with a union bound over the at most $n^2 \log n$ points of A, we conclude that Definition 3.3(c) also holds with probability going to one as $n \to \infty$. With κ independent of n, this establishes (3.13) for any $s' > s > 1$ all $\eta > 0$ small enough and every possible type $\underline{z}$.

Turning to deal with (3.17), we may and shall fix $\epsilon > 0$, s, s' as in (3.26) and two admissible types $\underline{z}$, $\underline{z}'$, where as mentioned before $z_0 \ge 5\eta$ and $z'_0 \ge 5\eta$. Next, for $0 \le k \le L$, let $J_k := |\Gamma_{\underline{z}}(k) \bigcap \Gamma'_{\underline{z}'}(k)|$, where

$$\Gamma_{\underline{z}}(k) := \{y_k \in \mathsf{A}_{2\mathrm{D},k} \text{ for some } \underline{y} \text{ of } \underline{z} - \text{type}\}$$

and $\Gamma'_{\underline{z}'}(k)$ denoting the same sets for an independent SRW X' on $\mathsf{G}_n(a)$. Recall (3.16) that the image of $\mathcal{U}_{s,\underline{z}} \cap \mathcal{U}'_{s,\underline{z}'}$ via $x \mapsto y_0(x)$ is a subset of the at most J_0 points from $\mathsf{A}_{2\mathrm{D},0}$ having the corresponding types, where to each $y \in \mathsf{A}_{2\mathrm{D},0}$ correspond

$$|\{x \in \mathsf{A} : y_0(x) = y\}| \le h^3 := m \tag{3.31}$$

points from A. Given the position of their starting and ending points, the r-excursions of SRW X around each $x \in \mathsf{A}$, are mutually independent and further independent of the random subset $\Gamma_{\underline{z}}(0) \subseteq \mathsf{A}_{2\mathrm{D},0}$. Likewise, given their starting/ending points, the r-excursions of the SRW X' around each $x \in \mathsf{A}$ are mutually independent and independent of $\Gamma'_{\underline{z}'}(0)$. Further, for $x \in \mathsf{A}$ with $y(x) \in \Gamma_{\underline{z}}(0) \bigcap \Gamma'_{\underline{z}'}(0)$ to be in $\mathcal{U}_{s,\underline{z}} \cap \mathcal{U}'_{s,\underline{z}'}$ we must have $H_{x,z_0-3\eta}$ occurring for X and $H_{x,z'_0-3\eta}$ occurring for X' (see (3.16)). By (3.27), the probability of both events independently occurring at a given x, is at most

$$\bar{p} := n^{-\alpha(z_0-4\eta)-\alpha(z'_0-4\eta)}. \tag{3.32}$$

By the uniformity of (3.27) per conditioning as in Remark 3.8, we thus deduce from the preceding discussion that

$$|\mathcal{U}_{s',\underline{z}} \cap \mathcal{U}'_{s',\underline{z}'}| \quad \text{is stochastically dominated by} \quad \sum_{\ell=1}^{J_0} \xi_\ell, \tag{3.33}$$

where ξ_ℓ are i.i.d. Binomial$(m, \bar{p})$ variables independent of J_0, and m, $\bar{p}$ are given by (3.31) and (3.32), respectively. Recall that κ of (3.12) is independent of n (and of $h = [a \log n]$), while $\bar{p}\, m \to 0$ and $h^4/m \to \infty$ as $n \to \infty$. Further, with

$$(1+u)^m \le 1 + \mathrm{e}\, u\, m \quad \text{whenever} \quad u\, m \in [0, 1], \tag{3.34}$$

we deduce that for all n large enough,

$$\mathbf{E}[2^{\kappa \xi_\ell}] = [1 + (2^\kappa - 1)\bar{p}]^m \le 1 + \mathrm{e}(2^\kappa - 1) m \bar{p} \le 1 + h^4 \bar{p}. \tag{3.35}$$

In view of (3.33) and (3.35),

$$\mathbf{E}\left[2^{\kappa|\mathcal{U}_{s,\underline{z}} \cap \mathcal{U}'_{s,\underline{z}'}|} \mathbf{1}_{\mathcal{G}_{\underline{z}}} \mathbf{1}_{\mathcal{G}'_{\underline{z}'}}\right] \le \mathbf{E}\left[\prod_{\ell=1}^{J_0} 2^{\kappa \xi_\ell}\right] \le \mathbf{E}\left[\left(1 + h^4 \bar{p}\right)^{J_0}\right],$$

with (3.17) holding as soon as

$$\mathbf{E}\left[\left(1 + h^4 \bar{p}\right)^{J_0}\right] \to 1. \tag{3.36}$$

Turning to establish (3.36), note that for any $k = 0, \ldots, L-1$, given their starting and ending points, the inner parts of the R_{k+1}-excursions for different choices of $y_{k+1} \in \mathsf{A}_{2\mathrm{D},k+1}$ are independent of each other, and of the random subset $\Gamma_{\underline{z}}(k+1)$. Thus, as in the preceding derivation, the contributions $\{\xi^\star_\ell, \ell = 1, \ldots, J_{k+1}\}$ to J_k that correspond to the possible $y_{k+1} \in \Gamma_{\underline{z}}(k+1) \bigcap \Gamma'_{\underline{z}'}(k+1)$, are stochastically dominated by mutually independent random variables $\{\xi_\ell\}$, each having maximal size m_k and mean $m_k \bar{p}_k$, which are further independent of J_{k+1}. Here, $m_k := n^{2(\rho_{k+1} - \rho_k)} = n^{2/L}$ bounds the maximal number of points $y_k \in \mathsf{A}_{2\mathrm{D},k}$ inside the R_{k+1}-cylinder centered at some $y_{k+1} \in \mathsf{A}_{2\mathrm{D},k+1}$. Further, if $z_k < 1$ then $\mathsf{NC}_{y_k,k,k+1,w}(s) \le (z_k - \eta)^2 \overline{\mathsf{NC}}^\star(s)$ for $w = z_{k+1} - 2\eta$ (compare Definitions 3.2 and 3.6). Replacing $z_k < 1$ by $z'_k < 1$ and w by $w' = z'_{k+1} - 2\eta$, the same applies for the corresponding excursion counts induced by the SRW X'. Considering the upper bound (3.30) for $j = k+1$, $\widetilde{\eta} = \eta$ and such values of (w, z_k) and (w', z'_k), recall Remark 3.10 that it holds uniformly over the relative position of y_k in the R''_j-sized square centered at y_j and with respect to the start/end points of the R_j-excursions around y_j. Having here $\rho_j - \rho_k = 1/L$, we deduce by the independence of X and X' that for all n large enough,

$$\bar{p}_k := \left(n^{\eta - 2sL(z_{k+1} - 2\eta - z_k)_+^2} \wedge 1\right)\left(n^{\eta - 2sL(z'_{k+1} - 2\eta - z'_k)_+^2} \wedge 1\right).$$

Each ξ_ℓ is no longer Binomial (there are dependencies within each R_{k+1}-cylinder). Nevertheless, setting $u_{k+1} := \mathrm{e} u_k \bar{p}_k m_k$ with $u_0 := h^4 \bar{p}$ we get inductively for $k = 0, 1, \dots, L-1$, that if $u_k m_k \le 1$ then

$$\mathbf{E}\left[(1+u_k)^{J_k}\right] \le \mathbf{E}\left[\prod_{\ell=1}^{J_{k+1}} \mathbf{E}\left[(1+u_k)^{\xi_\ell}\right]\right] \le \mathbf{E}\left[(1+u_{k+1})^{J_{k+1}}\right] \tag{3.37}$$

(utilizing stochastic domination, the mutual independence of $\{J_{k+1}, \xi_\ell\}$ and finally the inequality (3.34) at u_k and $\xi_\ell \le m_k$).

With both $\underline{z}$ and $\underline{z}'$ admissible, it follows by the definition of $\Psi_{L,\eta}(\phi)$ and $\gamma_{k,\eta}(\cdot)$ (c.f. (3.19)), that for any $s > \Psi_{L,\eta}(\phi)$,

$$u_k m_k = \mathrm{e}^k u_0 m_0 \prod_{j=0}^{k-1} \bar{p}_j m_{j+1} \le \mathrm{e}^k h^4 n^{-\gamma_{k,\eta}(\underline{z})-\gamma_{k,\eta}(\underline{z}')} \le \mathrm{e}^k h^4 n^{-2\eta} \to 0$$

when $n \to \infty$. Hence, iterating (3.37) over $0 \le k \le L-1$ yields that for $n \to \infty$,

$$\mathbf{E}\Big[\big(1+u_0\big)^{J_0}\Big] \le \mathbf{E}\Big[\big(1+u_L\big)^{J_L}\Big] \to 1.$$

Indeed, the latter convergence holds since $J_L \le |\mathsf{A}_{2\mathrm{D},L}|$ is uniformly bounded (in n), whereas by the preceding, $u_L \to 0$ as $n \to \infty$.

4 Proof of Lemma 3.7: 3D-like tail probabilities

4.1 Evaluation of typical values

Setting $R = MR'$, $R' = MR''$ and $R'' \ge h$ integer valued with both M and R'' large enough,

we show that the typical excursion counts up to time $st_{\mathrm{cov}}^{\square}$ are given as in (3.9) by:

$$\overline{\mathsf{NC}}^{\star}(s) := 2s\frac{(\log n)^2}{\log(R/R')} \quad \text{and} \quad \overline{\mathsf{NB}}^{\star}(s) := \frac{4sr'}{a}\log n.$$

To this end, we start with some basic results about the 2D excursions. In particular, (4.2) establishes (3.29) and allows us to replace the random excursion counts $\mathsf{NC}_{\underline{y},L}(s)$ by their typical value $\overline{\mathsf{NC}}^{\star}(s)$, which by (4.1) and (4.3), is also where the variables $\mathsf{NC}_{\underline{y},k}(s)$, $0 \le k < L$, concentrate.

Proposition 4.1 *Fix $\underline{y} = (y_0, \dots, y_L)$ with $y_k \in \mathsf{A}_{2\mathrm{D},k}$. For $0 \le k \le L$, let $\mathsf{NC}_{\underline{y},k}(s)$ be the number of R_k-excursions for $\underline{y}$ completed during the first $\overline{\mathsf{NC}}^{\star}(s)$ of the R_L-excursions for the corresponding $y_L \in \mathsf{A}_{2\mathrm{D},L}$ with $\mathsf{NC}_{\underline{y},L}(s)$ denoting the number of latter R_L-excursions completed by time $st_{\mathrm{cov}}^{\square}$. Let $\overline{\mathsf{NC}}_{\underline{y},k}(s)$ denote the expectation of*

$NC_{\underline{y},k}(s) := NC_{y_k,k,L,1}$ in case $k < L$. Then for each $\delta > 0$, there exists $C = C(\delta) > 0$ and $M(\delta)$ such that for all $M \geq M_0(\delta)$ there exists $n_0(\delta, M)$ such that for all $n \geq n_0(\delta, M)$ and $0 \leq k \leq L$, we have that

$$(1-\delta)\overline{NC}^{\star}(s) \leq \overline{NC}_{\underline{y},k}(s) \leq (1+\delta)\overline{NC}^{\star}(s), \tag{4.1}$$

$$\mathbf{P}\big[|NC_{\underline{y},L}(s) - \overline{NC}_{\underline{y},L}(s)| \geq \delta\overline{NC}_{\underline{y},L}(s)\big] \leq \exp(-Cs(\log n)^2) \tag{4.2}$$

$$\mathbf{P}\big[|NC_{\underline{y},k}(s) - \overline{NC}_{\underline{y},k}(s)| \geq \delta\overline{NC}_{\underline{y},k}(s)\big] \leq n^{-Cs\delta^2} \tag{4.3}$$

Proof Note that $\mathsf{NC}_{\underline{y},L}(s)$ counts the number of excursions between concentric 2D-disks of radii R'_L and R_L by the projected SRW on $\mathbb{Z}_n^2$ during its first $\frac{4s}{\pi}n^2(\log n)^2(1+o(1))$ steps [9]. (As we explained earlier, the factor $2/3$ is due to the elimination of all vertical steps of the original SRW on $\mathsf{G}_n(a)$.) Our first assertion, namely (4.1) in the case $k = L$, thus follows from [10, Lemma 3.2]. That is, $\overline{\mathsf{NC}}_{\underline{y},L}(s)$ is up to leading order given by $\overline{\mathsf{NC}}^{\star}(s)$. Since $R_L/R'_L = M$ is independent of n, the bound (4.2) likewise follows from [10, Lemma 3.2]. Fixing $0 \leq k < L$ and considering [10, Lemma 3.2] for the R_k-excursions completed during the same number of steps by the projected SRW, it further follows from (4.2) that $\overline{\mathsf{NC}}_{\underline{y},k}(s) = \overline{\mathsf{NC}}_{\underline{y},L}(s)(1+o(1))$. The same argument also gives (4.3). □

We proceed to establish the mean value of the relevant 3D excursions. Hereafter, we let σ_{W} denote the first exit time of the SRW $\{X_k\}$ from a given $\mathsf{W} \subseteq V_n$ using σ_S^x for $\sigma_{\mathsf{B}(x,S)}$ and the notation $\mathsf{B}' = \mathsf{B}(x, r')$, $\mathsf{B} = \mathsf{B}(x, r)$, $\mathsf{C}' = \mathsf{C}(x', R')$ and $\mathsf{C} = \mathsf{C}(x', R)$ for balls of radii $r = Mr' \leq h$, $r' = M$ and cylinders, respectively, of any centers $x, x' \in V_n$ with $|x - x'| \leq R''$.

Proposition 4.2 *Suppose that $x, x' \in V_n$ with $|x - x'| \leq R''$. Then for each $\eta > 0$ there exists $M_0(\eta)$ such that for each $M \geq M_0(\eta)$ there exists $n_0(\eta, M)$ such that $n \geq n_0(\eta, M)$ implies that*

$$(1-\eta)z^2\overline{NB}^{\star}(s) \leq \mathbf{E}\left[NB_{x,z}^{x'}\right] \leq (1+\eta)z^2\overline{NB}^{\star}(s).$$

Proof Recall Definition 3.6 that $\mathsf{NB}_{x,z}^{x'}$ counts the SRW excursions from $\partial\mathsf{B}'$ to $\partial\mathsf{B}$ during its first $z^2\overline{\mathsf{NC}}^{\star}(s)$ excursions from $\partial\mathsf{C}'$ to $\partial\mathsf{C}$. The latter R-excursions are conditionally independent given their starting and ending points. Hence, with $Z_\star$ counting the excursions that $X|_{[0,\sigma_{\mathsf{C}}]}$ makes from $\partial\mathsf{B}'$ to $\partial\mathsf{B}$, it suffices to show that

$$\mathbf{E}_v[Z_\star \,|\, X_{\sigma_{\mathsf{C}}} = w] = F_{\mathsf{B},\mathsf{C}}(1+o(1))$$

(as $n \to \infty$ and $M \to \infty$), uniformly in $v \in \partial\mathsf{C}'$ and $w \in \partial\mathsf{C}$, where the nominal conversion factor from R-excursions to ball excursions is

$$F_{\mathsf{B},\mathsf{C}} := \frac{\overline{\mathsf{NB}}^{\star}(s)}{\overline{\mathsf{NC}}^{\star}(s)} = \frac{2r'}{h}\log(R/R'). \tag{4.4}$$

Indeed, we show in Lemma 4.8 that

$$\mathbf{P}_v[\tau_{\mathsf{B}'} < \sigma_{\mathsf{C}} \mid X_{\sigma_{\mathsf{C}}} = w] = F_{\mathsf{B},\mathsf{C}}(1 + o(1)), \tag{4.5}$$

and from part (a) of Lemma 4.5 we deduce that for $v' \in \partial \mathsf{B}'$

$$\mathbf{E}_{v'}[Z_\star \mid X_{\sigma_{\mathsf{C}}} = w] \to 1 \quad \text{as} \quad n \to \infty \quad \text{then} \quad M \to \infty \tag{4.6}$$

uniformly in v' and w, which together complete the proof. □

Our next six lemmas culminate in Lemmas 4.5 and 4.8, thereby completing the proof of Proposition 4.2. The first of these lemmas controls the fluctuations of positive harmonic functions in $\mathsf{G}_n(a)$.

Lemma 4.3 *Fixing $M \geq 2$ and $S = MS'$, we have that for all positive harmonic functions f on the ball $B(0, S)$ in $\mathbb{Z}^3$,*

$$\max_{u,u' \in B(0,S')} \frac{f(u)}{f(u')} = 1 + O(M^{-1}). \tag{4.7}$$

Likewise, if $x \in V_n$, $S < n/2$, then for any $M \geq 2$ and every positive harmonic function f on $\mathsf{C}(x, S)$ in $\mathsf{G}_n(a)$, we have that

$$\max_{u,u' \in B(x,S')} \frac{f(u)}{f(u')} = 1 + O(M^{-1}). \tag{4.8}$$

Proof We first prove (4.7). The Harnack inequality [17, Theorem 1.7.2] implies that there exists a constant $C_0 > 0$ such that

$$\max_{u,u' \in \mathsf{B}(0,S/2)} \frac{f(u)}{f(u')} \leq C_0. \tag{4.9}$$

It thus follows from [17, Theorem 1.7.1] that there exists a constant $C_1 > 0$ such that for any $u, u' \in \mathsf{B}(0, S')$ we have

$$|f(u) - f(u')| \leq S' \frac{C_1}{S} \max_{v \in \mathsf{B}(0,S/2)} f(v). \tag{4.10}$$

Combining (4.9) with (4.10) gives (4.7). Observe that (4.8) follows from (4.7) because any function which is harmonic on $\mathsf{C}(x, S)$ may be lifted to a harmonic function on a cylinder in $\mathbb{Z}^3$ with radius S and periodic boundary conditions. □

Building on the preceding lemma, we next show that starting inside $\mathsf{B}(x, S')$ any non-negative variable measurable on $X|_{[0,\sigma_{\mathsf{B}(x,S')}]}$ is almost independent of the SRW on $\mathsf{G}_n(a)$ exit location of W containing $\mathsf{B}(x, S)$.

Lemma 4.4 *Let $S = MS'$, $M \geq 2$ and $\widetilde{B} = B(x, S')$ for $x \in V_n$ and $S' \leq h$. Suppose that $Z \geq 0$ is a random variable which depends only on $X|_{[0,\sigma_{\widetilde{B}}]}$. Fix $W \subseteq V_n$ which contains $B(x, S)$. Then we have that*

$$\max_{w,w' \in \partial W} \max_{u \in \widetilde{B}} \frac{\mathbf{E}_u[Z \mid X_{\sigma_W} = w]}{\mathbf{E}_u[Z \mid X_{\sigma_W} = w']} = 1 + O(M^{-1}).$$

In particular,

$$\max_{w \in \partial W} \max_{u \in \widetilde{B}} \frac{\mathbf{E}_u[Z \mid X_{\sigma_W} = w]}{\mathbf{E}_u[Z]} = 1 + O(M^{-1}).$$

Proof Fix $u \in \widetilde{B}$ and $w \in \partial W$. Then we have that

$$\mathbf{E}_u[Z \mid X_{\sigma_W} = w] = \sum_{v \in \partial \widetilde{B}} \mathbf{E}_u[Z \mid X_{\sigma_{\widetilde{B}}} = v] \mathbf{P}_u[X_{\sigma_{\widetilde{B}}} = v \mid X_{\sigma_W} = w]. \tag{4.11}$$

By Bayes' rule, we can write

$$\mathbf{P}_u[X_{\sigma_{\widetilde{B}}} = v \mid X_{\sigma_W} = w] = \frac{\mathbf{P}_u[X_{\sigma_W} = w \mid X_{\sigma_{\widetilde{B}}} = v]}{\mathbf{P}_u[X_{\sigma_W} = w]} \mathbf{P}_u[X_{\sigma_{\widetilde{B}}} = v]. \tag{4.12}$$

By the strong Markov property, the ratio on the RHS of (4.12) is contained in $[\kappa^{-1}, \kappa]$ where

$$\kappa := \max_{v,v' \in \partial \widetilde{B}} \frac{\mathbf{P}_v[X_{\sigma_W} = w]}{\mathbf{P}_{v'}[X_{\sigma_W} = w]}. \tag{4.13}$$

Since $v \mapsto \mathbf{P}_v[X_{\sigma_W} = w]$ is harmonic on $B(x, S)$, by Lemma 4.3 we know that $\kappa = 1 + O(M^{-1})$ uniformly in w. Combining this with (4.11) and using that $Z \geq 0$ implies the stated result. □

Using the preceding lemma, we establish (4.6) and further show that if X_0 is far from x, then $X|_{[0,\sigma_C]}$ spends a negligible time in B'. To this end, we use hereafter

$$\mathcal{L}_{s,t}(W) := \sum_{k=s}^{t-1} \mathbf{1}_{\{X_k \in W\}}, \tag{4.14}$$

for the SRW local time of W between times $s \leq t$, with $\mathcal{L}_t(W) := \mathcal{L}_{0,t}(W)$.

Lemma 4.5 *Suppose that $x, x' \in V_n$ with $|x - x'| \leq R''$ and $w \in \partial C$.*

(a) *There exists a universal finite constant c_1 such that starting at any $v' \in \partial B'$ the law of $Z_\star$ conditional on $\{X_{\sigma_C} = w\}$ is stochastically dominated by $1 + Y$ where Y is a Geometric(c_1/M) variable.*

(b) *For $h' = h/(2M)$, uniformly in $v \in \partial B(x, h')$ and w,*

$$\mathbf{E}_v[\mathcal{L}_{\sigma_{\mathsf{C}}}(B') \mid X_{\sigma_{\mathsf{C}}} = w] \to 0 \quad as \quad n \to \infty \quad then \quad M \to \infty. \tag{4.15}$$

Proof (a) We first show that for some C_1 finite, any $u' \in \partial \mathsf{B}$ and all n, M,

$$\mathbf{P}_{u'}\left[X_{\sigma^x_{h/4}} \notin \mathsf{B}\left(x, \tfrac{h}{4}\right) \backslash \mathsf{C}(x, \tfrac{h'}{M}) \mid X_{\sigma_{\mathsf{C}}} = w\right] \leq \frac{C_1}{M}. \tag{4.16}$$

Indeed, applying Lemma 4.4 for $S' = h/4 \geq r$ and $\mathsf{W} = \mathsf{C} \supset \mathsf{B}(x, MS')$, we get (4.16) upon noting that due to [17, Lemma 1.7.4],

$$\mathbf{P}_{u'}\left[X_{\sigma^x_{h/4}} \notin \mathsf{B}\left(x, \tfrac{h}{4}\right) \backslash \mathsf{C}\left(x, \tfrac{h'}{M}\right)\right] \leq \frac{C_1}{M}.$$

Similarly, upon applying Lemma 4.4 for $Z = \mathbf{1}_{\{\tau_{\mathsf{B}'} < \sigma^x_{h/4}\}}$, we can deduce from [17, Theorem 1.5.4] that for some universal C_2 finite

$$\mathbf{P}_{u'}\left[\tau_{\mathsf{B}'} < \sigma^x_{h/4} \mid X_{\sigma_{\mathsf{C}}} = w\right] \leq \frac{C_2}{M}. \tag{4.17}$$

We next claim that for some $C_3(M) < \infty$ and all $u \in \mathsf{B}(x, \frac{h}{4}) \backslash \mathsf{C}(x, \frac{h'}{M})$,

$$\mathbf{P}_u[\tau_{\mathsf{B}'} < \sigma_{\mathsf{C}} \mid X_{\sigma_{\mathsf{C}}} = w] \leq \frac{C_3(M)}{\log \log n}. \tag{4.18}$$

Indeed, by Bayes' rule we can rewrite the LHS of (4.18) as

$$\frac{\mathbf{P}_u[X_{\sigma_{\mathsf{C}}} = w \mid \tau_{\mathsf{B}'} < \sigma_{\mathsf{C}}]}{\mathbf{P}_u[X_{\sigma_{\mathsf{C}}} = w]} \mathbf{P}_u[\tau_{\mathsf{B}'} < \sigma_{\mathsf{C}}]. \tag{4.19}$$

By [17, Exercise 1.6.8], the rightmost factor in (4.19) is of order $C_3(M)/\log \log n$, so to complete the proof of (4.18) it suffices to show that the left ratio in (4.19) is uniformly bounded. Applying the strong Markov property for the first time that X hits $\partial \mathsf{B}(x, \frac{h}{4})$ after $\tau_{\mathsf{B}'}$, it in turn suffices to show that

$$\max_{u, \widetilde{u} \in \partial \mathsf{B}(x, \frac{h}{4})} \frac{\mathbf{P}_{\widetilde{u}}[X_{\sigma_{\mathsf{C}}} = w]}{\mathbf{P}_u[X_{\sigma_{\mathsf{C}}} = w]}$$

is bounded. Such boundedness follows from [17, Theorem 1.7.2] since $u \mapsto \mathbf{P}_u[X_{\sigma_{\mathsf{C}}} = w]$ is harmonic. Combining (4.16), (4.17), and (4.18) yields the claimed stochastic domination of the law of Z.

(b) The same argument as in the proof of part (a) shows that here the number of excursions between B' and $\mathsf{B}(x, h')$ during the time interval $[0, \sigma_{\mathsf{C}}]$ is stochastically dominated by a Geometric$(c(M)/\log h')$ for some finite $c(M)$. Further, within each excursion between B' and $\mathsf{B}(x, h')$ we are in the setting of SRW on $\mathbb{Z}^3$. Hence, by a similar argument, relying once more on Lemma 4.4 and the relevant results from [17, Chapter 1], the expected contribution to $\mathcal{L}(\mathsf{B}')$ during such an excursion, conditional on its start/end points, is uniformly bounded by $c'(M)$. Due to the independence of these excursions given their start/end points, we thus deduce (4.15) by an application of Wald's identity. □

Turning to the proof of (4.5), our next lemma gives a precise estimate of the Green's function for the SRW on $\mathsf{G}_n(a)$ killed upon exiting C (conditioned on its exit location). We note that for large n and M the resulting Green's function exhibits both 2D (the term $\log(R/|v-x|)$) and 3D (the factor $1/h$) behaviors.

Lemma 4.6 *Suppose $x, x' \in V_n$ with $|x-x'| \le R''$. Let $G^w(v,x)$ denote the Green's function for X stopped upon hitting $\partial\mathsf{C}$ conditioned on exiting C at a given $w \in \partial\mathsf{C}$. Then, for any $v \in \partial\mathsf{C}'$ and $\beta < 2$*

$$G^w(v,x) = \frac{3+O(M^{-1})}{\pi h}\big(\log R - \log|v-x| + o(|v-x|^{-\beta}) + O(R^{-1})\big).$$

Proof Let τ_x be the first time that X hits x, and let τ_x^+ be the time of its first return to x. By the strong Markov property of X at time τ_x^+, we have

$$\mathbf{P}_x[X_{\sigma_\mathsf{C}} = w \mid \tau_x^+ \le \sigma_\mathsf{C}] = \mathbf{P}_x[X_{\sigma_\mathsf{C}} = w],$$

i.e., the events $\{X_{\sigma_\mathsf{C}} = w\}$ and $\{\tau_x^+ \le \sigma_\mathsf{C}\}$ are independent. Thus

$$\mathbf{P}_x[\tau_x^+ > \sigma_\mathsf{C} \mid X_{\sigma_\mathsf{C}} = w] = \mathbf{P}_x[\tau_x^+ > \sigma_\mathsf{C}];$$

taking reciprocals, $G^w(x,x) = G(x,x)$, where G is the (unconditioned) Green's function for X stopped upon hitting $\partial\mathsf{C}$.

Applying the strong Markov property of X conditioned on $\{X_{\sigma_\mathsf{C}} = w\}$, at the stopping time τ_x, we have that

$$G^w(v,x) = \mathbf{P}_v[\tau_x \le \sigma_\mathsf{C} \mid X_{\sigma_\mathsf{C}} = w]G^w(x,x).$$

By Bayes' rule,

$$\begin{aligned}\mathbf{P}_v[\tau_x \le \sigma_\mathsf{C} \mid X_{\sigma_\mathsf{C}} = w] &= \frac{\mathbf{P}_v[X_{\sigma_\mathsf{C}} = w \mid \tau_x \le \sigma_\mathsf{C}]}{\mathbf{P}_v[X_{\sigma_\mathsf{C}} = w]}\mathbf{P}_v[\tau_x \le \sigma_\mathsf{C}] \\ &= \frac{\mathbf{P}_x[X_{\sigma_\mathsf{C}} = w]}{\mathbf{P}_v[X_{\sigma_\mathsf{C}} = w]}\mathbf{P}_v[\tau_x \le \sigma_\mathsf{C}].\end{aligned}$$

Since $G(v,x) = \mathbf{P}_v[\tau_x \le \sigma_\mathsf{C}]G(x,x)$, combining the above we see that

$$G^w(v,x) = \frac{\mathbf{P}_x[X_{\sigma_\mathsf{C}} = w]}{\mathbf{P}_v[X_{\sigma_\mathsf{C}} = w]}G(v,x). \tag{4.20}$$

Since $u \mapsto \mathbf{P}_u[X_{\sigma_C} = w]$ is harmonic within $\mathsf{C}(x', R)$ and $v, x \in \mathsf{C}'$, applying Lemma 4.3 we arrive at

$$G^w(v, x) = (1 + O(M^{-1}))G(v, x). \tag{4.21}$$

It thus remains only to estimate $G(v, x)$. To this end, let $G_{\mathbb{Z}_n^2}$ denote the Green's function associated with the projected (unconditioned) random walk in $\mathbb{Z}_n^2$ stopped upon exiting the disk of radius R centered at $y(x')$. Note that the projected random walk has a $1/3$ holding probability since this is the probability that the (unprojected) walk moves in the vertical direction. Let W_x denote the collection of h points in V_n whose 2D projection is equal to $y(x)$. Then

$$G_{\mathbb{Z}_n^2}(v, x) = \sum_{u \in \mathsf{W}_x} G(v, u). \tag{4.22}$$

Since $u \mapsto G(v, u)$ (for v fixed) is harmonic for $u \neq v$, hence in $\mathsf{C}(x', R')$, whereas $\mathsf{W}_x \subset \mathsf{B}(x', 2R'')$, Lemma 4.3 implies that

$$\frac{G(v, u)}{G(v, u')} = 1 + O(M^{-1}) \quad \text{for all} \quad u, u' \in \mathsf{W}_x. \tag{4.23}$$

Moreover, [17, Proposition 1.6.7] gives us that for every $\beta < 2$ we have

$$G_{\mathbb{Z}_n^2}(v, x) = \frac{3}{\pi}(\log R - \log |v - x|) + o(|v - x|^{-\beta}) + O(R^{-1})$$

(recall the $1/3$ laziness). Combining this with (4.22) and (4.23) tells us that for every $\beta < 2$ we have

$$G(v, x) = \frac{(1 + O(M^{-1}))}{h} \left(\frac{3}{\pi}(\log R - \log |v - x|) + o(|v - x|^{-\beta}) + O(R^{-1}) \right).$$

Combining this with (4.21) gives the result. □

We are now going to estimate the expected amount of time that SRW starting from $\partial \mathsf{B}'$ spends in B' before exiting C. This estimate allows us to establish (4.5) in the subsequent lemma.

Lemma 4.7 *For $x, x' \in V_n$ with $|x - x'| \leq R''$ any $v' \in \partial \mathsf{B}'$ and $w \in \partial \mathsf{C}$, let*

$$\overline{\mathcal{L}}^{v', w}(\mathsf{B}'; \mathsf{C}) = \mathbf{E}_{v'}[\mathcal{L}_{\sigma_C}(\mathsf{B}') \mid X_{\sigma_C} = w]$$

(for $\mathcal{L}_t(\cdot)$ as in (4.14)). Then,

$$\frac{\overline{\mathcal{L}}^{v', w}(\mathsf{B}'; \mathsf{C})}{2(r')^2} \to 1 \quad \textit{as} \quad n \to \infty \quad \textit{then} \quad r' = M \to \infty. \tag{4.24}$$

Proof We first reduce (4.24) to a computation which involves only the transient SRW $\widetilde{X}$ on $\mathbb{Z}^3$ starting at $\widetilde{X}_0 = v'$. To this end note that for $h' = h/(2M)$,

$$\overline{\mathcal{L}}^{v',w}(\mathsf{B}';\mathsf{C}) = \mathbf{E}_{v'}\left[\mathcal{L}_{\sigma^x_{h'}}(\mathsf{B}') \mid X_{\sigma_\mathsf{C}} = w\right] + \mathbf{E}_{v'}\left[\mathcal{L}_{\sigma^x_{h'},\sigma_\mathsf{C}}(\mathsf{B}') \mid X_{\sigma_\mathsf{C}} = w\right],$$

and from part (b) of Lemma 4.5 the right most term is $o(1)$ as $n \to \infty$ followed by $M \to \infty$. Further, the other term on the RHS involves a variable of the type considered in Lemma 4.4 for $S' = h'$. With $\mathsf{C} \subset \mathsf{B}(x, h/2)$ it is thus within a uniform $1 + O(M^{-1})$ factor of $\mathbf{E}_{v'}[\mathcal{L}_{\sigma^x_{h'}}(\mathsf{B}')]$, which is precisely the local time in B' of $\widetilde{X}$ till its exit time of $\mathsf{B}(x, h')$. Let $\widetilde{Z}$ be the *total* local time of $\widetilde{X}$ in B', noting that since $h' = \Theta(\log n)$ while $r' = M$, it follows from [17, Theorem 1.5.4] that as $n \to \infty$,

$$\mathbf{E}_{v'}\left[\mathcal{L}_{\sigma^x_{h'}}(\mathsf{B}')\right] = \mathbf{E}_{v'}[\widetilde{Z}] + O(1). \tag{4.25}$$

From [17, Theorem 1.5.4], we have moreover that

$$\frac{1}{(r')^2}\mathbf{E}_{v'}[\widetilde{Z}] \to c_3 \int_{\mathsf{B}(0,1)} \frac{du}{|u - e_3|} \quad \text{as} \quad r' \to \infty, \tag{4.26}$$

where $c_3 := 3/(2\pi)$ is given explicitly in [18, Theorem 4.3.1, top of page 82], $e_3 = (0, 0, 1)$ and $\mathsf{B}(0, 1) = \{v \in \mathbb{R}^3 : |v| < 1\}$ is the unit ball in $\mathbb{R}^3$ with Lebesgue measure denoted by du; we note that an additional factor of r' appears in the normalization from spatially re-scaling. This convergence is uniform in $v' = \widetilde{X}_0$ and the proof is completed by finding after the change of coordinates $u = (t\cos\varphi\cos\theta, t\cos\varphi\sin\theta, 1 - t\sin\varphi)$ that the integral on the RHS of (4.26) is precisely $4\pi/3$. □

Combining Lemmas 4.6 and 4.7 we now establish (4.5).

Lemma 4.8 *Uniformly in $x, x' \in V_n$ with $|x - x'| \le R''$, $v \in \partial\mathsf{C}'$ and $w \in \partial\mathsf{C}$, in the limit $n \to \infty$ followed by $M \to \infty$,*

$$\mathbf{P}_v[\tau_{B'} < \sigma_\mathsf{C} \mid X_{\sigma_\mathsf{C}} = w] = \frac{2r'}{h}\log(R/R')(1 + o(1)). \tag{4.27}$$

Proof Recall that if $Z \ge 0$ and $\mathbf{P}[Z > 0] > 0$ then $\mathbf{P}[Z > 0] = \mathbf{E}[Z]/\mathbf{E}[Z \mid Z > 0]$. Applying this identity for $Z = \mathcal{L}_{\sigma_\mathsf{C}}(\mathsf{B}')$ conditional to $X_0 = v$ and $X_{\sigma_\mathsf{C}} = w$, yields

$$\mathbf{P}_v[\tau_{\mathsf{B}'} < \sigma_\mathsf{C} \mid X_{\sigma_\mathsf{C}} = w] = \frac{\overline{\mathcal{L}}^{v,w}(\mathsf{B}';\mathsf{C})}{\widehat{\mathcal{L}}^{v,w}(\mathsf{B}';\mathsf{C})},$$

where

$$\overline{\mathcal{L}}^{v,w}(\mathsf{B}';\mathsf{C}) := \mathbf{E}_v[\mathcal{L}_{\sigma_\mathsf{C}}(\mathsf{B}') \mid X_{\sigma_\mathsf{C}} = w]$$
$$\widehat{\mathcal{L}}^{v,w}(\mathsf{B}';\mathsf{C}) := \mathbf{E}_v[\mathcal{L}_{\sigma_\mathsf{C}}(\mathsf{B}') \mid X_{\sigma_\mathsf{C}} = w, \tau_{\mathsf{B}'} < \sigma_\mathsf{C}].$$

We thus arrive at (4.27) by showing that uniformly in x, x', v, w as $n \to \infty$ followed by $r' = M \to \infty$,

$$\overline{\mathcal{L}}^{v,w}(\mathsf{B}'; \mathsf{C}) \sim \frac{4(r')^3}{h} \log(R/R') \quad \text{and} \tag{4.28}$$

$$\widehat{\mathcal{L}}^{v,w}(\mathsf{B}'; \mathsf{C}) \sim 2(r')^2. \tag{4.29}$$

Note that by definition

$$\overline{\mathcal{L}}^{v,w}(\mathsf{B}'; \mathsf{C}) = \sum_{u \in \mathsf{B}'} G^w(v, u),$$

for the Green's function $G^w(\cdot, \cdot)$ of Lemma 4.6. The estimate for $G^w(\cdot, \cdot)$ given there implies that uniformly in $u \in \mathsf{B}'$ and $v \in \partial \mathsf{C}'$,

$$G^w(v, u) = \frac{3}{\pi h} \log(R/R')(1 + o(1))$$

when $n \to \infty$ followed by $M \to \infty$ (so that $|v - u| \sim R'$). Since B' has to leading order $\frac{4\pi}{3}(r')^3$ points, this yields the stated formula (4.28) for $\overline{\mathcal{L}}^{v,w}(\mathsf{B}'; \mathsf{C})$. Further,

$$\widehat{\mathcal{L}}^{v,w}(\mathsf{B}'; \mathsf{C}) = \mathbf{E}_v \left[\overline{\mathcal{L}}^{X_{\tau_{\mathsf{B}'}}, w}(\mathsf{B}') \mid X_{\sigma_{\mathsf{C}}} = w, \tau_{\mathsf{B}'} < \tau_{\mathsf{C}} \right],$$

and with $X_{\tau_{\mathsf{B}'}} \in \partial \mathsf{B}'$ we get (4.29) by the uniform in v' asymptotics of Lemma 4.7. □

4.2 Tail probabilities for 3D type events

In this section we establish tail probabilities for 3D type events, which imply (3.27) and (3.28) in the strong sense of Remark 3.8. We start with the proof of (3.27).

Proposition 4.9 *Fix $x \in V_n$ and let $\mathcal{F}_{B'}$ be the σ-algebra generated by the entrance and exit points of all the excursions of X from $\partial B'$ to ∂B. For any $s > 1$, $1 \geq z > \eta > 0$ there exists M_0 such that for every $M \geq M_0$ there exists $n_0 = n_0(M)$ such that $n \geq n_0$ implies that a.s.*

$$n^{-\alpha(z+\eta)} \leq \mathbf{P}_v[H_{x,z} \mid \mathcal{F}_{B'}] \leq n^{-\alpha(z-\eta)}. \tag{4.30}$$

The upper bound holds for all $x, v \in V_n$, with $|v - x| > R'$ for the lower bound.

In order to prove Proposition 4.9, we first estimate the probability that a SRW starting from the boundary of a ball hits the center before exiting a larger ball, conditional on its exit point.

Lemma 4.10 *Uniformly over $x \in V_n$, $v' \in \partial B'$ and $w \in \partial B$,*

$$\mathbf{P}_{v'}[\tau_x < \sigma_B \mid X_{\sigma_B} = w] = (1 + O(M^{-1}))\Delta, \tag{4.31}$$

where for $c_3 := 3/(2\pi)$ *from* [17, Theorem 1.5.4] *(see* (4.26)*), and* q *of* (1.7),

$$\Delta = \frac{c_3 q}{r'}. \tag{4.32}$$

Proof By Bayes' rule,

$$\mathbf{P}_{v'}[\tau_x < \sigma_{\mathsf{B}} \mid X_{\sigma_{\mathsf{B}}} = w] = \frac{\mathbf{P}_{v'}[X_{\sigma_{\mathsf{B}}} = w \mid \tau_x < \sigma_{\mathsf{B}}]}{\mathbf{P}_{v'}[X_{\sigma_{\mathsf{B}}} = w]} \mathbf{P}_{v'}[\tau_x < \sigma_{\mathsf{B}}].$$

By the strong Markov property of X at τ_x the ratio on the RHS is

$$\frac{\mathbf{P}_x[X_{\sigma_{\mathsf{B}}} = w]}{\mathbf{P}_{v'}[X_{\sigma_{\mathsf{B}}} = w]} = 1 + O(M^{-1})$$

(where we used once again Lemma 4.3 for $S' = r'$ and $u \mapsto \mathbf{P}_u[X_{\sigma_{\mathsf{B}}} = w]$ harmonic on B). Let $\widetilde{X}$ denote the SRW on $\mathbb{Z}^3$ starting at v' and $\widetilde{\tau}_x$, $\widetilde{\sigma}_{\mathsf{B}}$ be the corresponding stopping times. Then,

$$\mathbf{P}_{v'}[\tau_x < \sigma_{\mathsf{B}}] = 1 - \frac{\mathbf{P}_{v'}[\widetilde{\tau}_x = \infty]}{\mathbf{P}_{v'}[\widetilde{\tau}_x = \infty \mid \widetilde{\tau}_x \geq \widetilde{\sigma}_{\mathsf{B}}]}. \tag{4.33}$$

By [18, Proposition 6.5.1] (having same constant c_3 as in [17, Theorem 1.5.4]),

$$\mathbf{P}_{v'}[\widetilde{\tau}_x = \infty] \sim 1 - \frac{c_3 q}{r'}. \tag{4.34}$$

Applying the strong Markov property at $\widetilde{\sigma}_{\mathsf{B}}$, we similarly find that

$$\mathbf{P}_{v'}[\widetilde{\tau}_x = \infty \mid \widetilde{\tau}_x \geq \widetilde{\sigma}_{\mathsf{B}}] \sim 1 - \frac{c_3 q}{r}. \tag{4.35}$$

Combining (4.33)–(4.35) yields the stated estimate $\Delta(1 + O(M^{-1})$ in (4.31). □

Proof of Proposition 4.9] If $v \in \mathsf{B}'$, we only reduce the event $H_{x,z}$ by shifting v to the induced (random) first exit of X from B'. Proceeding hereafter with $v \in V_n \backslash \mathsf{B}'$ the inner parts of the r-excursions of X around x are independent of each other given $\mathcal{F}_{\mathsf{B}'}$. Thus, the conditional probability considered in (4.30) is the product of $z^2 \overline{\mathsf{NB}}^{\star}(s)$ probabilities. Lemma 4.10 implies the existence of $\delta = \delta(M) \downarrow 0$ as $M \to \infty$ such that each of these probabilities is at most $(1 - \Delta + \delta)$, uniformly in the initial and terminal points of the excursion. In view of (3.9) and (4.32),

$$(1 - \Delta)^{z^2 \overline{\mathsf{NB}}^{\star}(s)} \leq \exp\left(-\Delta z^2 \overline{\mathsf{NB}}^{\star}(s)\right) = n^{-\alpha(z)}.$$

The stated upper bound follows since $\alpha(z - \eta) < \alpha(z)$. The complementary lower bound is similarly proved for $v \notin \mathsf{B}'$. □

We now turn to establish (3.28).

Proposition 4.11 *Fix $x' \in V_n$ and let $\mathcal{F}_{\mathsf{C}'}$ be the σ-algebra generated by the entrance and exit points of all the excursions of X from $\partial \mathsf{C}'$ to $\partial \mathsf{C}$. For any $s > 1 \geq z > \eta > 0$ there exists $\gamma > 0$ such that for all $n, r' \in \mathbb{N}$ large enough and every $x \in \mathsf{B}(x', R'')$, $v \in V_n \backslash \mathsf{C}'$, we have that a.s.*

$$\mathbf{P}_v \left[\frac{NB^{x'}_{x,z}}{\overline{NB}^{\star}(s)} \notin [(z-\eta)^2, (z+\eta)^2] \mid \mathcal{F}_{\mathsf{C}'} \right] \leq n^{-\gamma r'}. \tag{4.36}$$

Proof Fixing $s > 1 \geq z > \eta > 0$ we first show that for some $\gamma > 0$ all $n, r' \in \mathbb{N}$ large enough and every $|x - x'| \leq R''$, $v \in V_n$,

$$\mathbf{P}_v \left[\mathsf{NB}^{x'}_{x,z} < (z-\eta)^2 \overline{\mathsf{NB}}^{\star}(s) \mid \mathcal{F}_{\mathsf{C}'} \right] \leq n^{-\gamma r'}. \tag{4.37}$$

Indeed, $R'' + r < R'$ hence $\mathsf{B} \subseteq \mathsf{C}'$ for all n large enough. When $v \in \mathsf{C}'$ we thus may only reduce $\mathsf{NB}^{x'}_{x,z}$ upon using the strong Markov property at the first exit of C'. Consequently, it suffices to establish (4.37) for $v \notin \mathsf{C}'$. In the latter case, by Lemma 4.8 there exist $\delta = \delta(M) \downarrow 0$ as $M \to \infty$ and $n_0 = n_0(M)$ such that for all $n \geq n_0$ the number $Z_\star$ of excursions from $\partial \mathsf{B}'$ to $\partial \mathsf{B}$ within one excursion from $\partial \mathsf{C}'$ to $\partial \mathsf{C}$ is stochastically bounded *below* by a Bernoulli(p_n) variable J with $p_n = (1-\delta) F_{\mathsf{B},\mathsf{C}}$, uniformly in x, x' as stated and in the initial and terminal points of the excursion. Letting $N := z^2 \overline{\mathsf{NB}}^{\star}(s)$ and $N' := z^2 \overline{\mathsf{NC}}^{\star}(s)$, the probability considered in (4.37) is thus bounded above by

$$P_\star := \mathbf{P} \left(\sum_{i=1}^{N'} J_i \leq (1 - \eta/z)^2 N \right),$$

for i.i.d. $\{J_i\}$. From the definition of $F_{\mathsf{B},\mathsf{C}}$ we have that $N' = N(1-\delta)/p_n$ hence by Markov's inequality we deduce that for any $\theta > 0$,

$$\frac{1}{N} \log(P_\star) \leq \theta (1 - \eta/z)^2 + \frac{1-\delta}{p_n} \log \left(1 - p_n (1 - \mathrm{e}^{-\theta}) \right). \tag{4.38}$$

The function $f(\kappa, \theta) := \theta - \kappa(1 - \mathrm{e}^{-\theta})$ decreases in κ and is strictly negative for any $\kappa > 1$ and $\theta > 0$ small enough. Since $p_n \to 0$ as $n \to \infty$, the RHS of (4.38) converges to $\kappa^{-1} f((1-\delta)\kappa, \theta)$, where $\kappa = (1 - \eta/z)^{-2} > 1$. With $\delta(M) \to 0$, there exists $\gamma' = \gamma'(\kappa) > 0$ such that using $\theta > 0$ sufficiently small we get from (4.38) that for all $M \geq M_1$ and $n \geq n_1$

$$P_\star \leq \mathrm{e}^{-\gamma' N} = n^{-\gamma r'}.$$

Note that, in view of (3.9), the value of $\gamma = \frac{4s}{a} \gamma' z^2 > 0$ is independent of r'. A similar argument shows that, by possibly decreasing $\gamma = \gamma(s, z, \eta) > 0$, for $v \notin \mathsf{C}'$ one has

$$\mathbf{P}_v\left[\mathsf{NB}^{x'}_{x,z} > (z+\eta)^2\overline{\mathsf{NB}}^{\star}(s)\,|\,\mathcal{F}_{\mathsf{C}'}\right] \le n^{-\gamma r'}.$$

Indeed, the only difference is that now we need to replace the i.i.d. copies of Bernoulli(p) by i.i.d. copies of the product of Bernoulli $\widetilde{J}$ of mean $(1+\delta)F_{\mathsf{B},\mathsf{C}}$ and $1+Y$ for the Geometric random variable Y of success probability c_1/M as established in part (a) of Lemma 4.5. □

Further, combining Propositions 4.9 and 4.11 we obtain the following.

Proposition 4.12 *For $s > 1 \ge z \ge \eta > 0$, let $\widehat{H}^{x'}_{x,z}$ be the event of not hitting x during the first $z^2\overline{\mathsf{NC}}^{\star}(s)$ excursions from $\partial\mathsf{C}'$ to $\partial\mathsf{C}$. Then, there exist finite $n_0 = n_0(M)$, $M \ge M_0$, such that for every $n \ge n_0$, $x' \in V_n$, $x \in \mathsf{B}(x', R'')$ and $v \in V_n\backslash\mathsf{C}'$ we have a.s.*

$$n^{-\alpha(z+\eta)} \le \mathbf{P}_v\left[\widehat{H}^{x'}_{x,z}\,|\,\mathcal{F}_{\mathsf{C}'}\right] \le n^{-\alpha(z-\eta)}.$$

5 Proof of Lemma 3.9: 2D excursion counts at various radii

This section is devoted to the proof of (3.30). To this end, recall our notations of $R'' = h$, $R = M^2h$ and for any fixed $L \in \mathbb{N}$ and $k \in \{0,\ldots,L-1\}$, having $\rho_k = k/L$ and $R_k = R\,[n^{\rho_k}]$, while $R_L = [n/M^5]\,M^2$. Fixing w, z and $j \in \{k+1,\ldots,L\}$ we let $\mathsf{NC}_{y_k,k,j,w}(s)$ as in Definition 3.6 count the number of R_k-excursions for $y_k \in \mathsf{A}_{2\mathrm{D},k}$ completed during the $w^2\overline{\mathsf{NC}}^{\star}(s)$ first R_j-excursions for the corresponding $y_j \in \mathsf{A}_{2\mathrm{D},j}$, with (3.30) stating that for each $\eta \in (0, w\wedge z)$ there exists $M_0 = M_0(\eta)$ such that for all $M \ge M_0$ and $n \ge n_0(\eta, M)$

$$\left|\frac{\log \mathbf{P}[\mathsf{NC}_{y_k,k,j,w}(s) \le (z-\eta)^2\overline{\mathsf{NC}}^{\star}(s)]}{\log n} + \frac{2s(w-z)_+^2}{\rho_j-\rho_k}\right| \le \eta. \qquad (5.1)$$

In Lemma 5.1 we stochastically dominate $\mathsf{NC}_{y_k,k,j,w}(s)$ from above and below by comparable variables of a much simpler form and thereby establish (5.1) upon studying in Lemma 5.3 the tail behavior of the latter variables. Specifically, fixing $0 \le k < j \le L$, set for each $n \in \mathbb{N}$,

$$p_{k\to j}(n) := \frac{\log R_k - \log R'_k}{\log R_j - \log R'_k} \quad \text{and} \quad p_{j\to k}(n) := \frac{\log R_j - \log R'_j}{\log R_j - \log R'_k}.$$

As explained in [17, Chapter 1], the hitting probabilities for SRW X within large size cylindrical annulus, have the same asymptotic as such probabilities for the corresponding 2D Brownian motion. In particular, $p_{k\to j}(n)$ (resp. $p_{j\to k}(n)$) approximates the probability that the SRW X starting from a point in $\partial\mathsf{C}(y_k, R_k)$ (resp. $\partial\mathsf{C}(y_j, R'_j)$) hits $\partial\mathsf{C}(y_j, R_j)$ before hitting $\partial\mathsf{C}(y_k, R'_k)$ (resp. hits $\partial\mathsf{C}(y_k, R'_k)$ before hitting $\partial\mathsf{C}(y_j, R_j)$). Moreover, it is easy to check that

$$\lim_{M\to\infty}\lim_{n\to\infty}\frac{p_{k\to j}(n)\overline{\mathsf{NC}}^{\star}(s)}{\log n}=\lim_{M\to\infty}\lim_{n\to\infty}\frac{p_{j\to k}(n)\overline{\mathsf{NC}}^{\star}(s)}{\log n}=\frac{2s}{\rho_j-\rho_k}. \quad (5.2)$$

We next show that the variables $\mathsf{NC}_{y_k,k,j,w}(s)$ are stochastically related to

$$Z_{w,s}(p,p') := \sum_{i=1}^{w^2\overline{\mathsf{NC}}^{\star}(s)} J_i(1+Y_i), \quad (5.3)$$

where the i.i.d. Bernoulli(p) variables (J_i) are independent of the i.i.d. Geometric(p') variables (Y_i), provided the parameters $p\in(0,1)$ and $p'\in(0,1)$ are comparable to $p_{j\to k}(n)$ and $p_{k\to j}(n)$, respectively.

Lemma 5.1 *For every $c>1$, $w>0$ and $L\geq j>k\geq 0$, all $M\geq M_0(c,L)$ and $n\geq n_0(c,L,M)$, if $p>cp_{j\to k}(n)$ and $p'<p_{k\to j}(n)/c$, then the law of $\mathsf{NC}_{y_k,k,j,w}(s)$ is stochastically dominated from above by $Z_{w,s}(p,p')$. Likewise, if $p<p_{j\to k}(n)/c$ and $p'>cp_{k\to j}(n)$ then the law of $\mathsf{NC}_{y_k,k,j,w}(s)$ is stochastically dominated from below by $Z_{w,s}(p,p')$.*

Proof For each i, let $\widetilde{J}_i$ denote the indicator of the event that the ith excursion E_i of the SRW X from $\partial\mathsf{C}(y_j,R'_j)$ to $\partial\mathsf{C}(y_j,R_j)$ hits $\partial\mathsf{C}(y_k,R'_k)$. We also let $\widetilde{Y}_i$ denote the number of returns that the SRW X makes to $\mathsf{C}(y_k,R'_k)$ from $\partial\mathsf{C}(y_k,R_k)$ before exiting $\mathsf{C}(y_j,R_j)$ during E_i. Then,

$$\mathsf{NC}_{y_k,k,j,w}(s)=\sum_{i=1}^{w^2\overline{\mathsf{NC}}^{\star}(s)}\widetilde{J}_i(1+\widetilde{Y}_i).$$

Let $\mathcal{F}_j$ denote the σ-algebra generated by the entrance and exit points of all excursions $\{E_i\}$ and $\mathcal{F}_{j,k}$ denote the σ-algebra generated by $\mathcal{F}_j$ as well as all entrance and exit points of the excursions of X from $\partial\mathsf{C}(y_k,R'_k)$ to $\partial\mathsf{C}(y_k,R_k)$. By [10, Lemma 2.3] in the limit $M\to\infty$ the probability of the occurrence of $\widetilde{J}_i$ given $\mathcal{F}_j$ does not depend on the relevant starting and ending points. The same applies for the probability that $\widetilde{Y}_i=\ell$ given $\widetilde{Y}_i\geq\ell$ and $\mathcal{F}_{j,k}$. Thus, in view of [17, Exercise 1.6.8], we conclude that,

$$\liminf_{M\to\infty}\liminf_{n\to\infty}\inf_i\left\{\frac{\mathbf{P}[\widetilde{J}_i=1\,|\,\mathcal{F}_j]}{p_{j\to k}(n)}\right\}=\limsup_{M\to\infty}\limsup_{n\to\infty}\sup_i\left\{\frac{\mathbf{P}[\widetilde{J}_i=1\,|\,\mathcal{F}_j]}{p_{j\to k}(n)}\right\}=1, \quad (5.4)$$

$$\liminf_{M\to\infty}\liminf_{n\to\infty}\inf_{i,\ell}\left\{\frac{\mathbf{P}[\widetilde{Y}_i=\ell\,|\,\mathcal{F}_{j,k},\ \widetilde{Y}_i\geq\ell]}{p_{k\to j}(n)}\right\}$$
$$=\limsup_{M\to\infty}\limsup_{n\to\infty}\sup_{i,\ell}\left\{\frac{\mathbf{P}[\widetilde{Y}_i=\ell\,|\,\mathcal{F}_{j,k},\ \widetilde{Y}_i\geq\ell]}{p_{k\to j}(n)}\right\}=1. \quad (5.5)$$

Combining (5.4) and (5.5) yields the desired result because the excursions $\{E_i\}$ are conditionally independent given $\mathcal{F}_j$. □

By Lemma 5.1, it suffices to prove the bounds of (5.1) for $Z_{w,s}(p_n, p'_n)$ in place of $\mathsf{NC}_{y_k,k,j,w}(s)$, provided that both $p_n/p_{j\to k}(n) \to 1$ and $p'_n/p_{k\to j}(n) \to 1$. Further, in view of (5.2), when doing so we may consider w.l.o.g. $p'_n = \kappa p_n$, $\kappa \in (0,\infty)$, taking $n \to \infty$ followed by $\kappa \to 1$. To this end, set

$$\Lambda_{p,p'}(\theta) := \log \mathbf{E}[\mathrm{e}^{-\theta J_1(1+Y_1)}] = \log\left(1 - p + \frac{pp'}{\mathrm{e}^\theta - 1 + p'}\right) \quad \text{for} \quad \theta \geq 0,$$

and for each $0 \leq z \leq w \leq 1$, let

$$I_{p,p'}(z,w) := \frac{1}{p}\inf_{\theta\geq 0}\left\{z^2\theta + w^2\Lambda_{p,p'}(\theta)\right\},$$

whose asymptotic as $p' = \kappa p$, $p \to 0$ shall describe the tail behavior of $Z_{w,s}(p_n, p'_n)$ which is relevant here.

Lemma 5.2 *Fix $\kappa \in (0,\infty)$. Then, we have that for $w \geq \sqrt{\kappa}z > 0$,*

$$I_\kappa(z,w) := \lim_{p\to 0} I_{p,p\kappa}(z,w) = \inf_{v\geq 0}\left(vz^2 - \frac{vw^2}{\kappa+v}\right) = -(w-\sqrt{\kappa}z)^2. \quad (5.6)$$

Let $\theta_p \in [0,\infty)$ be the unique value so that $\Lambda'_{p,\kappa p}(\theta_p) = -(z/w)^2$. Then,

$$\lim_{p\to 0}\frac{\theta_p}{p} = \sqrt{\kappa}\frac{w}{z} - \kappa := v_\star \geq 0, \quad (5.7)$$

$$\lim_{p\to 0} p^2\Lambda''_{p,\kappa p}(\theta_p) = 0. \quad (5.8)$$

Proof We begin by making the substitution $\theta := \log(1+pv)$ for $v \geq 0$, and setting $f_p(v) := p^{-1}\log(1+pv)$ rewrite $I_{p,\kappa p}(z,w)$ as

$$I_{p,\kappa p}(z,w) = \inf_{v\geq 0}\left\{z^2 f_p(v) + w^2 f_p\left(\frac{-v}{\kappa+v}\right)\right\}. \quad (5.9)$$

Since $f_p(v) \uparrow \infty$ as $v \to \infty$, the infimum in (5.9) is attained at some finite v_p. Further, with $p \mapsto f_p(v)$ non-increasing, there exists a universal finite constant V such that v_p takes its values in $[0,V]$ as $p \to 0$ and κ fixed. This allows us to change the order of the limit in p and the infimum over v, yielding

$$I_\kappa(z,w) = \inf_{v\geq 0}\lim_{p\to 0}\left\{z^2 f_p(v) + w^2 f_p\left(\frac{-v}{\kappa+v}\right)\right\}.$$

Since $f_p(v) \to v$ for $p \to 0$, the first assertion of the lemma follows upon verifying that the infimum in (5.6) is achieved at $v_\star \geq 0$.

As for confirming (5.7) and (5.8), let $F_p(v) := f_p(F_0(v))$ for $F_0(v) = -v/(\kappa+v)$, so $\Lambda_{p,\kappa p}(\theta) = pF_p(v)$, under the substitution $\theta = \log(1+pv)$. Differentiating both sides of this identity twice and rearranging, we find that

$$p^2\Lambda''_{p,\kappa p}(\theta) = p(1+pv)\left(F''_p(v)(1+pv) + pF'_p(v)\right). \tag{5.10}$$

Since the infimum in the definition of $I_{p,\kappa p}(z,w)$ is attained at θ_p, necessarily $\theta_p = pf_p(v_p)$. Thus, as $p \to 0$ we have that $p^{-1}(\mathrm{e}^{\theta_p} - 1) = v_p \to v_\star$, from which (5.7) follows. Further, $F'_p(v_p) \to F'_0(v_\star)$ and $F''_p(v_p) \to F''_0(v_\star)$, yielding (5.8) in view of (5.10). □

As explained before, the required bounds (5.1) are established by combining Lemma 5.1 with our next lemma, then taking $\kappa \to 1$ [we have the required boundedness of $p_n \log n$ by (3.9) and (5.2)].

Lemma 5.3 *Fix $s \geq 1$, $\kappa \in (0,\infty)$ and $w \geq \sqrt{\kappa}z > 0$. If $p_n \log n$ are uniformly bounded above and uniformly bounded away from zero, then*

$$\lim_{n\to\infty} \frac{1}{p_n\overline{NC}^\star(s)} \log \mathbf{P}\left[Z_{w,s}(p_n,\kappa p_n) \leq z^2\overline{\mathsf{NC}}^\star(s)\right] = -(w-\sqrt{\kappa}z)_+^2. \tag{5.11}$$

Proof Fix $s \geq 1$, $\kappa \in (0,\infty)$ and $w \geq \sqrt{\kappa}z > 0$. Now, for any $p \in (0,1)$ we get by applying Chernoff's bound, then optimizing over $\theta \geq 0$ that

$$\frac{1}{p\overline{\mathsf{NC}}^\star(s)} \log \mathbf{P}\left[Z_{w,s}(p,\kappa p) \leq z^2\overline{\mathsf{NC}}^\star(s)\right] \leq I_{p,\kappa p}(z,w). \tag{5.12}$$

Thus, in view of (5.6), considering $p = p_n \to 0$ yields the upper bound in (5.11).

For the lower bound we use a change of measure analogous to the proof of the lower bound in Cramer's theorem (see [11, Theorem 2.2.3]). Specifically, fixing $p \in (0,1)$ and $\delta > 0$ small (we eventually send $\delta \to 0$), set $\theta = \theta_p \geq 0$ be the unique value such that $\Lambda'_{p,\kappa p}(\theta_p) = -(z-\delta)^2/w^2$ and probability measure $\mathbf{P}_\theta$ given by

$$\frac{d\mathbf{P}_\theta}{d\mathbf{P}} = \exp\Big(-\theta Z_{w,s}(p,\kappa p) - w^2\overline{\mathsf{NC}}^\star(s)\Lambda_{p,\kappa p}(\theta)\Big).$$

Considering event $A_{p,\kappa p} = \{(\overline{\mathsf{NC}}^\star(s))^{-1}Z_{w,s}(p,\kappa p) \in [(z-2\delta)^2, z^2]\}$, we clearly have then

$$\mathbf{P}[A_{p,\kappa p}] \geq \mathbf{P}_\theta[A_{p,\kappa p}]\exp\left(w^2\overline{\mathsf{NC}}^\star(s)\Lambda_{p,\kappa p}(\theta) + \theta(z-2\delta)^2\overline{\mathsf{NC}}^\star(s)\right). \tag{5.13}$$

Adding and subtracting $\theta(z-\delta)^2\overline{\mathsf{NC}}^\star(s)$ in the exponent on the RHS of (5.13), then setting there $\theta = \theta_p$, we see that $\mathbf{P}[A_{p,\kappa p}]$ is further bounded below by

$$\mathbf{P}_{\theta_p}[A_{p,p'}]\exp\left(p\overline{\mathsf{NC}}^\star(s)I_{p,\kappa p}(z-\delta,w) - \eta\overline{\mathsf{NC}}^\star(s)\theta_p\right),$$

where $\eta := (z-\delta)^2-(z-2\delta)^2$. We now complete the proof by taking $p = p_n$ (we will suppress the subscript n). Indeed, note that under $\mathbf{P}_\theta$ the variables $J_i(1+Y_i)$ are i.i.d. each having mean $(z-\delta)^2/w^2$ and variance $\Lambda''_{p,\kappa p}(\theta)$. Further, $p_n^2\overline{\mathsf{NC}}^\star(s)$ is bounded away from zero, so by (5.8) we see that $\mathrm{Var}_{\theta_p}\left(\overline{\mathsf{NC}}^\star(s)^{-1}Z_{w,s}(p,\kappa p)\right) \to 0$ as $n \to \infty$, while $\mathbf{E}_{\theta_p}[\overline{\mathsf{NC}}^\star(s)^{-1}Z_{w,s}(p,\kappa p)] = -w^2\Lambda'_{p,\kappa p}(\theta_p) = (z-\delta)^2$. Consequently,

$$\lim_{n\to\infty}\frac{1}{p\overline{\mathsf{NC}}^\star(s)}\log \mathbf{P}_{\theta_p}[A_{p,\kappa p}] = 0.$$

Hence, by (5.6) and (5.7) we have that

$$\liminf_{n\to\infty}\frac{1}{p\overline{\mathsf{NC}}^\star(s)}\log \mathbf{P}\left[Z_{w,s}(p,\kappa p) \le z^2\overline{\mathsf{NC}}^\star(s)\right] \ge -(w-\sqrt{\kappa}(z-\delta))_+^2 - 2\eta.$$

The stated lower bound follows by considering $\delta \to 0$ (so $\eta \to 0$ as well). □

6 Lower bound on mixing time: effective clustering in $\mathcal{U}(st^\square_{\text{cov}})$

Let $\mathbf{Q}_{s'}$ denote the law of the lamps configuration of $X^\diamond$ at time $s't^\square_{\text{cov}}$, starting from all lamps off (and walker at the point $0 \in \mathsf{G}_n(a)$), with $\mathbf{Q}_\infty$ the uniform law over the set of $2^{|V_n|}$ possible lamp configurations. We claim that $\|\mathbf{Q}_{s'} - \mathbf{Q}_\infty\|_{\text{TV}} \to 1$ when $n \to \infty$, for fixed $s' = (1-\epsilon)s$, any $s < \Psi(\phi)$ and $\epsilon > 0$. Obviously, then $t_{\text{mix}} \ge s't^\square_{\text{cov}}$ for such s', which in view of the upper bound on t_{mix} we proved in Sect. 3, establishes the stated cut-off and thereby proves Theorem 1.3.

To prove this claim, fix $\epsilon > 0$ and $s < \Psi(\phi)$, noting that in view of (3.25) and the variational formulation (1.12) of $\Psi(\phi)$, there exist ρ and $w > z > (1+w/\rho)\delta$, all in $(0, 1]$, such that for small enough $\delta > 0$,

$$b_\rho(w-\delta) \ge 2\delta \quad \text{and} \quad \alpha(z+3\delta) + \lambda(\rho-\delta) \le \rho - 5\delta, \tag{6.1}$$

where further, by (3.24) and the assumed range of z,

$$\lambda := 2s\frac{(w-z+\delta)^2}{(\rho-\delta)^2} < 2 \quad \text{and} \quad A := \frac{(z-\delta)\rho - w\delta}{w-z+\delta} > 0. \tag{6.2}$$

Using hereafter these parameters, we considerably shorten our proof by taking advantage of the results of [10] and [8] (which we apply here for the 2D projection of the SRW on $\mathsf{G}_n(a)$). For this purpose, we change our cylinders radii somewhat and consider throughout this section

$$R_k = R'_{k+1} = (k!)^3, \quad k = 1, \dots, m$$

with $m \in \mathbb{N}$ such that for some $\bar{\gamma} \in [b+12, b+16]$

$$n = K_m := m^{\bar{\gamma}} R_m$$

(and $b \geq 10$ a universal constant from [10, Lemma 4.2]). Next, let $\mathcal{Z}_m$ denote a maximal set of $4R_{\rho m+4}$-separated points on the 2D base of $\mathsf{G}_n(a)$ excluding those within distance R_m of the starting position 0 of the 2D projected SRW, such that $(0, 2R_m) \in \mathcal{Z}_m$ (so $\mathcal{Z}_m$ is precisely the set considered in [10, Equation (10.3)], taking there $\beta = \rho$ and $K_m = n$). Further, set

$$\mathcal{Z}'_m := \mathcal{Z}_m \bigcap \bigcup_{v_i} \mathsf{C}(v_i, R_{m-2}),$$

for a collection $\{v_i\}$ that forms a maximal $4R_m$-separated set on the 2D base of $\mathsf{G}_n(a)$. Next, for any $v \in \mathcal{Z}'_m$, let $\kappa_n(v)$ count the vertices of $\mathsf{C}(v, R_{\rho m-2}) \subset \mathsf{G}_n(a)$, and D^v denote the difference of number of "off-lamps" minus "on-lamps" among these $\kappa_n(v)$ vertices. Considering the statistics

$$U_n = \max_{v \in \mathcal{Z}'_m} \{D^v\},$$

it suffices to show that as $n \to \infty$,

$$\mathbf{Q}_\infty[U_n \geq n^{\rho+\delta}] \to 0 \quad \text{and} \quad \mathbf{Q}_{s'}[U_n < n^{\rho+\delta}] \to 0. \tag{6.3}$$

We proceed with the proof of (6.3), establishing in **Step I** the easy part, namely its LHS. Introducing $n_m(2s) := 6sm^2 \log m$ and

$$\widehat{U}^v := \Big|\{x \in \mathsf{C}(v, R_{\rho m-2}) : \quad x \text{ unvisited in first } n_m(2s) \text{ excursions by} \\ \text{the SRW from } \partial\mathsf{C}(v, R'_m) \text{ to } \partial\mathsf{C}(v, R_m)\Big|, \tag{6.4}$$

we reduce in **Step II** the RHS of (6.3) to having WHP some $v \in \mathcal{Z}'_m$ with large enough $\widehat{U}^v$ (see (6.7)). We now need the following additional notations.

Definition 6.1 For a maximal set $\mathcal{Z}_{\delta m}(v)$ of $4R_{\delta m}$-separated points in the 2D projection of $\mathsf{C}(v, R_{\rho m-2})$ on the base of $\mathsf{G}_n(a)$, let:

(a) W^v count points in $\mathcal{Z}_{\delta m}(v)$ for whose $R_{\delta m}$-sized cylindrical annulus the SRW completed at most $z^2 n_m(2s)$ excursions during its first $w^2 n_m(2s)$ excursions from $\partial\mathsf{C}(v, R'_{\rho m})$ to $\partial\mathsf{C}(v, R_{\rho m})$.

(b) $\bar{U}^v \leq W^v$ count those y from (a), for which in addition $x = (y, 0)$ is not visited during the first $z^2 n_m(2s)$ excursions from $\partial\mathsf{C}(x, R'_{\delta m})$ to $\partial\mathsf{C}(x, R_{\delta m})$.

Step III shows that WHP $\widehat{U}^{v_\star} \geq \bar{U}^{v_\star}$ for some $v_\star \in \mathcal{Z}'_m$. Indeed, it clearly suffices to have at most $w^2 n_m(2s)$ of the $R_{\rho m}$-excursions of $v_\star$ within the first $n_m(2s)$ of its R_m-excursions. This applies to pre-qualified points from [10, Section 10], so we complete this step by showing that WHP the relevant count $W_{\text{P-Q}}(m)$ of pre-qualified points (see (6.10)), is positive. **Step IV** then converts the conditional statement of bounding below $\bar{U}^{v_\star}$ (for the random $v_\star$), into such a statement for non-random v, which we verify under the condition of W^v large enough (see RHS of (6.12)). We complete the proof of the latter (see LHS of (6.12)), by applying in **Step V** the concept of pre-sluggish points from [8, Section 6].

Step I Note that under $\mathbf{Q}_\infty$ the variables $\{D^v, v \in \mathcal{Z}'_m\}$ are mutually independent, with D^v having the law of the sum of $\kappa_n(v)$ i.i.d. symmetric $\{\pm 1\}$-valued variables $\{I^v_j\}$. Further, $\sup_v \kappa_n(v) \le Chn^{2\rho}$ and $|\mathcal{Z}'_m| \le Cn^{2(1-\rho)}$ for some C finite and all n. Recall that $\mathbf{E}[\mathrm{e}^{\zeta I^v_j}] \le \mathrm{e}^{\zeta^2/2}$ for all ζ, hence by the union bound over at most Chn^2 values of $v \in \mathcal{Z}_{m'}$ and the uniform tail bound

$$\sup_{r \le Chn^{2\rho}} \mathbf{P}\left[\sum_{j=1}^{r} I^v_j \ge n^{\rho+\delta}\right] \le \mathrm{e}^{-n^\delta + Ch/2}, \tag{6.5}$$

we conclude that the LHS of (6.3) holds for any $\delta > 0$.

Step II Turning to the RHS of (6.3), let $N\mathsf{C}_{v,m}(s')$ count the R_m-excursions for cylindrical annuli centered at v on the 2D base of $\mathsf{G}_n(a)$, made by the SRW on $\mathsf{G}_n(a)$ up to time $s' t^{\square}_{\mathrm{cov}}$. Note that $\log n = (3 + o(1)) m \log m$ and for $R/R' = R_m/R'_m = m^3$ the value of $\overline{\mathsf{NC}}^{\star}(s)$ of (3.9) is within $1 + o(1)$ (as $n \to \infty$), of $n_m(2s) = 6sm^2 \log m$ (from [10]). Hence, analogous to part (a) of Definition 3.3 we have that

$$\lim_{n\to\infty} \mathbf{P}\left(\max_v \{N\mathsf{C}_{v,m}(s')\} > n_m(2s)\right) = 0, \tag{6.6}$$

where the maximum is over all n^2 vertices v on the 2D base of $\mathsf{G}_n(a)$. Indeed, combining the tail bound [10, Equation (3.18)] for the aggregate number of steps during the first $n_m(2s)$ such R_m-excursions for fixed v, with standard exponential tail bounds on the number of actual steps taken by our $\frac{2}{3}$-lazy projected 2D SRW, we thus deduce that $n^2 \mathbf{P}(N\mathsf{C}_{v,m}(s') > n_m(2s)) \to 0$ and the union bound over v results with (6.6).

We now show that the RHS of (6.3) holds as soon as

$$\lim_{n\to\infty} \mathbf{P}\left[\max_{v \in \mathcal{Z}'_m} \{\widehat{U}^v\} < 2n^{\rho+\delta}\right] = 0, \tag{6.7}$$

for $\widehat{U}^v$ of (6.4). Indeed, $\mathbf{Q}_{s'}[U_n < n^{\rho+\delta}]$ is bounded by the sum of the probabilities considered in (6.6) and in (6.7), and

$$\sum_{v \in \mathcal{Z}'_m} \mathbf{Q}_{s'}\left[\sum_{j \notin \widehat{U}^v} I^v_j \le -n^{\rho+\delta}\right]. \tag{6.8}$$

Further, conditional on the whole path of the SRW on $\mathsf{G}_n(a)$ the variables $\{I^v_j, j \notin \widehat{U}^v\}$ retain under $\mathbf{Q}_{s'}$ their symmetric i.i.d. ± 1-valued law, so the sum of probabilities considered in (6.8) is small by the uniform tail bound of (6.5).

Step III Proceeding to prove (6.7), let $N^v_{m,k}$ denote the number of SRW excursions from $\partial\mathsf{C}(v, R'_k)$ to $\partial\mathsf{C}(v, R_k)$, during the first $n_m(2s)$ excursions it made from $\partial\mathsf{C}(v, R'_m)$ to $\partial\mathsf{C}(v, R_m)$. We rely on [10, Section 10] to prove the existence WHP (as $m \to \infty$), of $v \in \mathcal{Z}'_m$ such that for w of (6.1)

$$N^v_{m,\rho m} < w^2 n_m(2s). \tag{6.9}$$

Indeed, we consider the choice of parameters $a = 2s$, $\beta = \rho$ and $\gamma = (w-\delta)/\rho$, in [10, Section 10] and call $v \in \mathcal{Z}'_m$ an (m, β)-*pre-qualified point* if $N^v_{m,k} \in [\widehat{n}_k - k, \widehat{n}_k + k]$ for all $\beta m \leq k \leq m-1$ and the value of $\widehat{n}_k$ given in [10, Equation (10.2)]. Since our choices of a, β and γ result with $\widehat{n}_{\beta m} = (w-\delta)^2 n_m(2s)(1+o(1)) \gg m$, we deduce that for some universal m_0 and all $m \geq m_0$, every (m, β)-pre-qualified point satisfies (6.9). Further, in view of (1.10) and the LHS of (6.1), the value of $a^\star$ in [10, Section 10] (given the preceding choices of a, β, and γ), is such that $(1-\beta)(2-a^\star) = 2b_\rho(w-\delta) \geq 4\delta$. Thus, letting

$$W_{\text{P- Q}}(m) := |\{v \in \mathcal{Z}'_m : v \text{ is } (m, \beta)\text{- pre-qualified}\}|, \tag{6.10}$$

it suffices to show that

$$\lim_{m\to\infty} \mathbf{P}\left(W_{\text{P- Q}}(m) \geq K_m^{(1-\beta)(2-a^\star)-\delta}\right) = 1,$$

which we get by adapting the proof of [10, Equation (10.3)], in replacing the (m, β)-qualified points in $\mathcal{Z}_m$ dealt with there, by the (m, β)-pre-qualified points in $\mathcal{Z}'_m$ considered here. To this end, recall that [10, Equation (10.3)] is derived by showing that:

(a) The mean number of such points far exceeds $K_m^{(1-\beta)(2-a^\star)-\delta}$.
(b) Its variance is negligible relative to the square of its mean.

We further note that the (m, β)-qualified points of [10, Section 10] are essentially our (m, β)-pre-qualified points for which also the event $\widehat{\mathbf{A}}^v_{N^v_{m,\beta m}}$ as in the proof of [10, Lemma 10.1], occurs. In [10] one takes $2s < 2$ for which the latter event is shown to occur WHP (see [10, Equation (10.8)]). The probability that v is (m, β)-qualified, as computed in [10, Equation (10.4)], is thus within $(1 + o(1))$ of the probability that v is (m, β)-pre-qualified, and it is further easy to check that in the pre-qualified case the same formula applies also when $2s \geq 2$.

Hence, the same argument as in [10] establishes (a) here as well. The key to (b) is the bound of [10, Equation (10.7)] which builds on the correlation upper bounds [10, Equations (10.5),(10.6)]. The latter have already been derived there for (m, β)-pre-qualified points and all $s > 0$. Thus, [10, Equation (10.7)] applies here as well, apart for a minor difficulty due to the fact that we consider only points from the subset $\mathcal{Z}'_m$ of $\mathcal{Z}_m$. However, $\inf_m\{m^{12}|\mathcal{Z}'_m|/|\mathcal{Z}_m|\}$ is positive, and we have already increased by m^{12} the value of $K_m = n$, which as seen by following the derivation of [10, Equation (10.7)], well compensates this effect.

Step IV Ordering the points of $\mathcal{Z}'_m$ in some non-random fashion, we let $v^\star$ denote the first $v \in \mathcal{Z}'_m$ satisfying (6.9) (which by **Step III** exists WHP). By definition the points in $\mathcal{Z}'_m$ are $4R_{\rho m+4}$-separated and the R_m-sized cylindrical annulus around each is of distance $R_{m-1} \geq R_{\rho m}$ from any (other) point of $\mathcal{Z}'_m$. Consequently, $v^\star$ is measurable on the σ-algebra $\mathcal{F}$ generated by the SRW path excluding the interior parts of excursions between $\partial\mathsf{C}(v, R_{\rho m-1})$ and $\partial\mathsf{C}(v, R_{\rho m})$, for all $v \in \mathcal{Z}'_m$ (namely, each such part has been replaced by its entrance and exit points). Recalling Definition 6.1 of the counts $\bar{U}^v \le W^v$ of points in $\mathcal{Z}_{\delta m}(v)$, we thus get (6.7) by showing that

$$\lim_{n\to\infty} \mathbf{P}(\bar{U}^{v_\star} \ge 2n^{\rho+\delta}|\mathcal{F}) = 1. \tag{6.11}$$

Further, applying [10, Lemma 2.4] for $r = R_{\rho m-2}$, $R = R_{\rho m-1}$, $R' = R_{\rho m}$ and the event $\{\bar{U}^v \ge 2n^{\rho+\delta}\}$ which is measurable on the σ-algebra $\mathcal{H}^v(\ell)$ of the interior parts of first $\ell = w^2 n_m(2s)$ excursions for $R_{\rho m}$-sized cylindrical annulus around v, we get the conditional result (6.11), once we show that for $\theta := (2-\lambda)(\rho-\delta)-\delta$ and any non-random $v \in \mathcal{Z}'_m$, as $n \to \infty$,

$$\mathbf{P}(W^v \ge n^\theta) \to 1 \quad \text{and} \quad \mathbf{P}\left(\bar{U}^v \ge 2n^{\rho+\delta} \,|\, W^v \ge n^\theta\right) \to 1. \tag{6.12}$$

Proceeding to establish the RHS, let q_n be the minimal value over all possible excursion end points and the choice of $x \in \mathsf{G}_n(a)\backslash\mathsf{C}(0, R_{\delta m})$, of the conditional probability that x is not visited during the first $z^2 n_m(2s)$ of the SRW excursions from $\partial\mathsf{C}(x, R'_{\delta m})$ to $\partial\mathsf{C}(x, R_{\delta m})$. Since points in $\mathcal{Z}_{\delta m}(v) \subset \mathsf{C}(v, R'_{\rho m} - R_{\delta m})$ are $4R_{\delta m}$-separated, the variable W^v is measurable on the σ-algebra $\mathcal{F}^v$ generated by the SRW path excluding the interior part of the excursions between $\partial\mathsf{C}(x, R'_{\delta m})$ and $\partial\mathsf{C}(x, R_{\delta m})$, for all $x = (y, 0)$ and $y \in \mathcal{Z}_{\delta m}(v)$ (namely, each such part has been replaced by its entrance and exit points). Thus, conditionally on $W^v \ge n^\theta$, the variable $\bar{U}^v$ stochastically dominates the Binomial(n^θ, q_n) law. From (6.1) and our choice of θ we have that

$$\theta - \alpha(z+3\delta) \ge \rho + 2\delta,$$

so by the CLT for Binomial random variables, we get the RHS of (6.12) upon proving that as $n \to \infty$,

$$n^{\alpha(z+3\delta)} q_n \to \infty. \tag{6.13}$$

In view of the LBD of Proposition 4.12, we have (6.13) upon showing that for any M large enough, the probability of having at least $(z+2\delta)^2\overline{\mathsf{NC}}^\star(s)$ excursions from $\partial\mathsf{C}(x, Mh)$ to $\partial\mathsf{C}(x, M^2 h)$ during the first $z^2 n_m(2s)$ of the corresponding $R_{\delta m}$-excursions, is bounded away from one, uniformly in x, $m \to \infty$, and the possible excursion end points. Further, the stochastic comparisons of Lemma 5.1 extend to our case where $R_0 = MR'_0 = M^2 h$ as before, but we replace $R_j = n^{j/L} R_0 = MR'_j$ with $R_{\delta m} = (\delta m)!^3 = (\delta m)^3 R'_{\delta m}$ and change $\overline{\mathsf{NC}}^\star(s)$ in (5.3) to $n_m(2s)$. Since $n_m(2s)$ is within factor $1+o(1)$ of the value of $\overline{\mathsf{NC}}^\star(s)$ from (3.9) that corresponds to $R' = R'_{\delta m}$ and $R = R_{\delta m} = (\delta m)^3 R'$, the desired uniform bound on probabilities follows from the convergence $Z_{z,s}(p, p')/\mathbf{E}[Z_{z,s}(p, p')] \to 1$ as $m \to \infty$ followed by $M \to \infty$ (while both $p = 3\log m/\log R_{\delta m}$ and $p' = \log M/\log R_{\delta m}$ decay to zero).

Step V We set $\widehat{R} := R_{\rho m} + R_{\rho m-2}$, $\widehat{\rho} := R_{\rho m-1} - R_{\rho m-2}$ and $\widehat{n}_k(\lambda) := 3\lambda(k + Am)^2 \log m$, $k = 1, 2, \ldots$ for $\lambda < 2$ and $A > 0$ of (6.2). Following the proof of [8, Lemma 6.1] we call $y \in \mathcal{Z}_{\delta m}(v)$ (m, ρ)-*pre-sluggish* if for the universal constant

$b \geq 4$ found there, and all $\delta m \leq k \leq \rho m - b$ the SRW completed within $\pm k$ of $\widehat{n}_k(\lambda)$ excursions from $\partial\mathsf{C}(y, R'_k)$ to $\partial\mathsf{C}(y, R_k)$ during its first $\widehat{n}_{\rho m}(\lambda)$ excursions from $\partial\mathsf{C}(y, \widehat{\rho})$ to $\partial\mathsf{C}(y, \widehat{R})$. It is easy to check that $\widehat{n}_{\rho m}(\lambda) = w^2 n_m(2s)$ and $\widehat{n}_{\delta m}(\lambda) = (z-\delta)^2 n_m(2s) \leq z^2 n_m(2s) - \delta m$ (these analogs of [8, (6.4) and (6.5)] are behind our choice of A and λ in (6.2)). Further, if $y \in \mathcal{Z}_{\delta m}(v)$ then

$$\mathsf{C}(y, \widehat{\rho}) \subseteq \mathsf{C}\left(v, R'_{\rho m}\right) \subset \mathsf{C}(v, R_{\rho m}) \subseteq \mathsf{C}(y, \widehat{R}).$$

Hence, W^v exceeds the number $\widehat{W}^v$ of (m, ρ)-pre-sluggish $y \in \mathcal{Z}_{\delta m}(v)$. The latter points match the definition made in [8, proof of Lemma 6.1], upon taking there the parameters $\gamma := \rho$, $\beta = w$ and $\eta := \delta$. Utilizing [8, Lemma 6.2] it is shown in the course of proving [8, Lemma 6.1] that $\widehat{W}^v$ concentrates WHP around its mean value, which for our choice of parameters turns out to be $R_m^{\theta+\delta-o_m(1)}$ (see [8, Equations (6.6) and (6.7)]). The values of λ, β and γ we have here are outside the range considered in [8, Lemmas 6.1 and 6.2], but this restriction in [8] is only relevant for the *extra requirement* made in [8, Equation (6.10)] that any (m, γ)-pre-sluggish point should be WHP also (m, γ)-sluggish. We completely abandoned this requirement, so the proof of [8] easily extends to yield the LHS of (6.12).

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