# ON COLLAPSING RING BLOW UP SOLUTIONS TO THE MASS SUPERCRITICAL NLS 

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#### Abstract

We consider the nonlinear Schrödinger equation $i \partial_{t} u+\Delta u+u|u|^{p-1}=$ 0 in dimension $N \geq 2$ and in the mass super critical and energy subcritical range $1+\frac{4}{N}<p<\min \left\{\frac{N+2}{N-2}, 5\right\}$. For initial data $u_{0} \in H^{1}$ with radial symmetry, we prove a universal upper bound on the blow up speed. We then prove that this bound is sharp and attained on a family of collapsing ring blow up solutions first formally predicted in [9].


## 1. Introduction

1.1. Setting of the problem. We consider in this paper the nonlinear Schrödinger equation

$$
(N L S)\left\{\begin{array}{l}
i \partial_{t} u+\Delta u+|u|^{p-1} u=0,  \tag{1.1}\\
u_{\mid t=0}=u_{0},
\end{array} \quad(t, x) \in \mathbb{R} \times \mathbb{R}^{N}\right.
$$

in dimension $N \geq 2$ and in the mass supercritical and energy subcritical range

$$
1+\frac{4}{N}<p<2^{*}-1, \quad 2^{*}= \begin{cases}+\infty & \text { for } N=2,  \tag{1.2}\\ \frac{2 N}{N-2} & \text { for } N \geq 3\end{cases}
$$

From Ginibre and Velo [11], given $u_{0} \in H^{1}$, there exists a unique solution $u \in$ $\mathcal{C}\left([0, T), H^{1}\right)$ to (1.1) and there holds the blow up alternative:

$$
T<+\infty \text { implies } \lim _{t \rightarrow T}\|u(t)\|_{H^{1}}=+\infty .
$$

The $H^{1}$ flow admits the conservation laws:

$$
\begin{aligned}
& \text { Mass : } \quad M(u) \int|u(t, x)|^{2}=M\left(u_{0}\right), \\
& \text { Energy : } \quad E(u)=\frac{1}{2} \int|\nabla u(t, x)|^{2} d x-\frac{1}{p+1} \int|u(t, x)|^{p+1} d x=E\left(u_{0}\right) \\
& \text { Momentum : } \quad P(u)=\Im\left(\int \nabla u(t, x) \overline{u(t, x)} d x\right)=P\left(u_{0}\right)
\end{aligned}
$$

A large group of symmetries also acts in the energy space $H^{1}$, in particular the scaling symmetry

$$
\begin{equation*}
u(t, x) \mapsto \lambda_{0}^{\frac{2}{p-1}} u\left(\lambda_{0}^{2} t, \lambda_{0} x\right), \quad \lambda_{0}>0 \tag{1.3}
\end{equation*}
$$

and the Galilean drift:

$$
u(t, x) \mapsto u\left(t, x-\beta_{0} t\right) e^{i \frac{\beta_{0}}{2} \cdot\left(x-\frac{\beta_{0}}{2} t\right)}, \quad \beta_{0} \in \mathbb{R}^{N} .
$$

The scaling invariant homogeneous Sobolev space $\dot{H}^{s_{c}}$ attached to (1.1) is the one which leaves the scaling symmetry invariant, explicitly:

$$
s_{c}=\frac{N}{2}-\frac{2}{p-1} .
$$

We say that the problem is mass subcritical if $s_{c}<0$, mass critical if $s_{c}=0$ and mass supercritical if $s_{c}>0$. From standard argument, for mass subcritical problems, the energy dominates the kinetic energy and all $H^{1}$ solutions are global and bounded, see [4]. On the other hand, for $s_{c} \geq 0$ and data

$$
u_{0} \in \Sigma=H^{1} \cap\left\{x u \in L^{2}\right\},
$$

the celebrated virial identity

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}} \int|x|^{2}|u(t, x)|^{2} d x=4 N(p-1) E\left(u_{0}\right)-\frac{16 s_{c}}{N-2 s_{c}} \int|\nabla u|^{2} \leq 16 E\left(u_{0}\right) \tag{1.4}
\end{equation*}
$$

implies that solutions emerging from non positive energy initial data $E\left(u_{0}\right)<0$ cannot exist globally and hence blow up in finite time.
This dichotomy can also be seen on the stability of ground states periodic solutions $u(t, x)=Q(x) e^{i t}$ where $Q$ is from [10], [18] the unique up to symmetries solution to

$$
\begin{equation*}
\Delta Q-Q+Q^{p}=0, \quad Q \in H^{1}, \quad Q>0 \tag{1.5}
\end{equation*}
$$

From variational arguments [5], these solutions are orbitally stable for $s_{c}<0$, and unstable by blow up and scattering for $s_{c}>0,[2],[32]$.
Note that we may reformulate the condition (1.2) as

$$
0<s_{c}<1
$$

In this setting, the Cauchy problem is also well posed in $\dot{H}^{s}$ for $s_{c} \leq s \leq 1$ and from standard argument, this implies the scaling lower bound on the blow up speed for $H^{1}$ finite time blow up solutions:

$$
\begin{equation*}
\|\nabla u(t)\|_{L^{2}} \gtrsim \frac{1}{(T-t)^{\frac{1-s c}{2}}}, \tag{1.6}
\end{equation*}
$$

see [28] for further details.
1.2. Qualitative information on blow up. There is still little understanding of the blow up scenario for general initial data. The situation is better understood in the mass critical case $s_{c}=0$ since the series of works [33], [23], [24], [25], [26], [27] where a stable blow up regime of "log-log" type is exhibited in dimension $N \leq 5$ with a complete description of the associated bubble of concentration. In particular, blow up occurs at a point and the solution concentrates exactly the ground state mass

$$
\begin{equation*}
|u(t, x)|^{2} \rightharpoonup\|Q\|_{L^{2}}^{2} \delta_{x=x^{*}}+\left|u^{*}\right|^{2} \text { as } t \rightarrow T \tag{1.7}
\end{equation*}
$$

for some $\left(x^{*}, u^{*}\right) \in \mathbb{R}^{N} \times L^{2}$. This blow up dynamic is not the only one and there exist further threshold dynamics which transition from stable blow up to stable scattering, see [3], [31]. These explicit scenario correspond to an improved description of the flow near the ground state solitary wave.

For $s_{c}>0$, the situation is more poorly understood. The only general feature known on blow up is the existence of a universal upper bound on blow up rate ${ }^{1}$

$$
\begin{equation*}
\int_{0}^{T}(T-t)\|\nabla u(t)\|_{L^{2}}^{2} d t<+\infty \tag{1.8}
\end{equation*}
$$

which is a direct consequence of the time integration of the virial identity (1.4), see [4]. In [28], Merle and Raphaël consider radial data in the range $0<s_{c}<1$, and

[^0]show that if blow up occurs, the Sobolev invariant critical norm does not concentrate as in (1.7), it actually blows up with a universal lower bound
\[

$$
\begin{equation*}
\|u(t)\|_{\dot{H}^{s_{c}}} \geq|\log (T-t)|^{C(N, p)} \tag{1.9}
\end{equation*}
$$

\]

This relates to the regularity results for the 3D Navier Stokes [8] and the regularity result [14], and shows a major dynamical difference between critical and super critical blow up. Then two explicit blow up scenario have been constructed so far. In [30], a stable self similar blow up regime

$$
\|\nabla u(t)\|_{L^{2}} \sim \frac{1}{(T-t)^{\frac{1-s_{c}}{2}}}
$$

is exhibited in the range $0<s_{c} \ll 1, N \leq 5$, which bifurcates in some sense from the $\log -\log$ analysis in [24], [26]. These solutions concentrate again at a point in space.

A completely different scenario is investigated in [36], [38] for the quintic nonlinearity $p=5$ in dimensions $N \geq 2$ where "standing ring" solutions are constructed. These solutions have radial symmetry and concentrate their mass on an asymptotic fixed sphere

$$
u(t, r) \sim \frac{1}{\lambda^{\frac{2}{p-1}}(t)} Q\left(\frac{r-r^{*}}{\lambda(t)}\right), \quad r^{*}>0
$$

where $Q$ is the one dimensional mass critical ground state $p=5$, and the speed of concentration is given by the $\log \log$ law

$$
\lambda(t) \sim \sqrt{\frac{T-t}{\log |\log (t-t)|}}
$$

Note that this includes energy critical $(N=3)$ and energy super critical regimes ( $N \geq 4$ ), and this blow up scenario is shown to be stable by smooth radially symmetric perturbation of the data. We refer to [41], [12], [13] for further extensions in cylindrical symmetry.

In the breakthrough paper [9], Fibich, Gavish and Wang propose a formal generalization of the ring scenario for $1+\frac{4}{N}<p<5$ : they formally predict and numerically observe solutions with radial symmetry which concentrate on a collapsing ring

$$
u(t, r) \sim \frac{1}{\lambda^{\frac{2}{p-1}}(t)}\left(Q e^{-\beta_{\infty} y}\right)\left(\frac{r-r(t)}{\lambda(t)}\right)
$$

were $Q$ is the mass subcritical one dimensional ground state solution to (1.5), $\beta_{\infty}$ is a universal Galilean drift

$$
\begin{equation*}
\beta_{\infty}=\sqrt{\frac{5-p}{p+3}} \tag{1.10}
\end{equation*}
$$

and concentration occurs at the speed:

$$
\lambda(t) \sim(T-t)^{\frac{1}{1+\alpha}}, \quad r(t)=(T-t)^{\frac{\alpha}{1+\alpha}}
$$

for some universal interpolation number

$$
\begin{equation*}
\alpha=\frac{5-p}{(p-1)(N-1)} \tag{1.11}
\end{equation*}
$$

Moreover, numerics suggest that this blow up is stable by radial perturbation of the data. This blow up corresponds to a new type of concentration, and like the
standing ring solution for $p=5$, it recovers in the supercritical regime the mass concentration scenario (1.7).
1.3. Statement of the result. We first claim a universal space time upper bound on blow up rate for radial data in the regime $0<s_{c}<1$ which sharpens the rough virial bound (1.8).

Theorem 1.1 (Upper bound on blow up rate for radial data). Let

$$
N \geq 2, \quad 0<s_{c}<1, \quad p<5
$$

Let $u_{0} \in H^{1}$ with radial symmetry and assume that the corresponding solution $u \in$ $\mathcal{C}\left([0, T), H^{1}\right)$ of (1.1) blows up in finite time $t=T$. Then there holds the space time upper bound:

$$
\begin{equation*}
\int_{t}^{T}(T-\tau)\|\nabla u(\tau)\|_{L^{2}}^{2} d \tau \leq C\left(u_{0}\right)(T-t)^{\frac{2 \alpha}{1+\alpha}} \tag{1.12}
\end{equation*}
$$

where $\alpha$ is given by (1.11).
The proof of (1.12) is surprisingly simple and relies on a sharp version of the localized virial identity introduced in [28]. Recall that no upper bound on the blow up rate is known in the mass critical case $s_{c}=0$, and arbitrary slow type II concentration ${ }^{2}$ should be expected for the energy critical problem $s_{c}=1$ in the continuation of [17]. Note also that the bound (1.12) implies

$$
\liminf _{t \uparrow T}(T-t)^{\frac{1}{1+\alpha}}\|\nabla u(t)\|_{L^{2}}<+\infty
$$

but the derivation of a pointwise upper bound on blow up speed for all times remains open.

We now claim that the bound (1.12) is sharp in all dimensions and attained on the collapsing ring solutions:

Theorem 1.2 (Existence of collapsing ring blow up solutions). Let

$$
N \geq 2, \quad 0<s_{c}<1, \quad p<5
$$

and $\beta_{\infty}>0,0<\alpha<1$ given by (1.10), (1.11). Let $Q$ be the one dimensional mass subcritical ground state solution to (1.5). Then there exists a time $\underline{t}<0$ and a solution $u \in \mathcal{C}\left([\underline{t}, 0), H^{1}\right)$ of (1.1) with radial symmetry which blows up at time $T=0$ according to the following dynamics. There exist geometrical parameters $(r(t), \lambda(t), \gamma(t)) \in \mathbb{R}_{+}^{*} \times \mathbb{R}_{+}^{*} \times \mathbb{R}$ such that ${ }^{3}:$

$$
\begin{equation*}
u(t, r)-\frac{1}{\lambda^{\frac{2}{p-1}}(t)}\left[Q e^{-i \beta_{\infty} y}\right]\left(\frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)} \rightarrow 0 \quad \text { in } \quad L^{2}\left(\mathbb{R}^{N}\right) \tag{1.13}
\end{equation*}
$$

The speed and the radius of concentration and the phase drift are given by the asymptotic laws:

$$
\begin{equation*}
r(t) \sim|t|^{\frac{\alpha}{1+\alpha}}, \quad \lambda(t) \sim|t|^{\frac{1}{1+\alpha}}, \quad \gamma(t) \sim|t|^{-\frac{1-\alpha}{1+\alpha}} \quad \text { as } t \uparrow 0 \tag{1.14}
\end{equation*}
$$

Moreover, the blow up speed admits the equivalent:

$$
\begin{equation*}
\|\nabla u(t)\|_{L^{2}} \sim \frac{1}{(T-t)^{\frac{1}{1+\alpha}}} \text { as } t \uparrow 0 \tag{1.15}
\end{equation*}
$$

[^1]
## Comments on the result.

1. Sharp upper bound on the blow up speed. From direct inspection using (1.11), the blow up rate (1.15) of ring solutions saturates the upper bound (1.12) which is therefore optimal in the radial setting. This shows that there is some sharpness in the nonlinear interpolation estimates underlying the proof of (1.12) and the associated localized virial identity which were already at the heart of the sharp lower bound (1.9) in [28]. We may also derive from the proof the behavior of the critical norm

$$
\|u(t)\|_{H^{\dot{s_{s}}}} \sim \frac{1}{\lambda^{s_{c}}(t)} \sim \frac{1}{(T-t)^{\frac{s_{c}}{1+\alpha}}}
$$

which shows as conjectured in [28] that the logarithmic lower bound (1.9) is not always sharp, even though it is attained for the self similar blow up solutions build in [30].
2. On the restriction $s_{c}<1$. We have restricted attention in this paper to the case $s_{c}<1$. This assumption is used to control the plain nonlinear term and ensures through the energy subcritical Cauchy theory that controlling $H^{1}$ norms is enough to control the flow. We however conjecture that the sharp threshold for the existence of collapsing ring solutions is $p<5$ in any dimension $N \geq 2$. This would require exactly as in [38] the control of higher order Sobolev norms in the bootstrap regime corresponding to collapsing ring solutions exhibited in this paper. This is an independent problem which needs to be addressed in details.
3. Non dispersive solutions. The construction of the ring solution relies on the strategy to build minimal blow up elements developed in [39]. In particular, let us stress the fact that (1.13) coupled with the laws (1.14) implies that the solution is nondispersive because

$$
\left\|u_{0}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}=\|Q\|_{L^{2}(\mathbb{R})}
$$

and the solution concentrates all its $L^{2}$ mass at blow up:

$$
|u(t)|^{2} \rightharpoonup\|Q\|_{L^{2}(\mathbb{R})}^{2} \delta_{x=0} \quad \text { as } \quad t \uparrow 0 .
$$

In fact, a three parameter family of such minimal elements -indexed on scaling and phase invariance, and an additional internal Galilean drift parameter- is constructed. This is a major difference with [36], [38] where the stationary ring solutions require a non trivial dispersion, and hence the full log-log machinery developed in [24], [26]. Such minimal elements can be constructed by reintegrating the flow backwards from the singularity using a mixed Energy/virial Lyapunov functional. The key is that as observed in [39], only energy bounds on the associated linearized operator close to $Q$ are required to close this analysis, see also [15], [21] for further illustrations. We also remark that because the problem is no longer critical, we can construct an approximate solution to all orders using the slow modulated approach in [24], [16], [39], and therefore the construction of the minimal element requires less structure $^{4}$ than in [39] and the proof is particularly robust. Let us stress the fact that obtaining dispersion using dispersive bounds for the linearized operator would be particularly delicate for this problem because the leading order blow up profile is given by the mass subcritical ground state for which the linearized spectrum displays a pair of complex eigenvalues leading to oscillatory modes, see [6]. We mention that

[^2]the existence of a minimal ring solution in the particular case $N=p=3$ has been recently announced in [34].
4. Arbitrary concentration of the mass. We may let the scaling symmetry (1.3) act on the solution constructed by Theorem 1.2 and obtain solutions with an arbitrary small or large amount of mass:
$$
|u(t)|^{2} \rightharpoonup m \delta_{x=0} \quad \text { as } \quad t \uparrow 0, \quad m>0
$$

This is a spectacular difference with the mass critical problem $s_{c}=0$ where the amount of mass focused by the nonlinearity is conjectured to be quantized, see [27].

Let us stress that Theorem 1.2 gives the first explicit description of blow up dynamics for a large set of values $(N, p)$, and the robust scheme behind the proof is likely to adapt to a large class of problems. One important open problem after this work is to understand stability properties of the collapsing ring blow up solutions. The numeral experiments in [9] clearly indicate the stability of the ring mechanism by radial perturbation of the data, but the proof would involve dealing with dispersion near the subcritical ground state which is a delicate analytical problem. We moreover expect that the ring singularity scenario persists on suitably prepared finite codimensional sets of non radial initial data.

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Notations. We introduce the differential operator;

$$
\Lambda f=\frac{2}{p-1} f+y \cdot \nabla f
$$

Let $L=\left(L_{+}, L_{-}\right)$the matrix linearized operator close to the one dimensional ground state:

$$
\begin{equation*}
L_{+}=-\partial_{y}^{2}+1-p Q^{p-1}, \quad L_{-}=-\partial_{y}^{2}+1-Q^{p-1} \tag{1.16}
\end{equation*}
$$

We recall that $L$ has a generalized nullspace characterized by the following algebraic identities generated by the symmetry group:

$$
\begin{equation*}
L_{-}(Q)=0, \quad L_{+}(\Lambda Q)=-2 Q, \quad L_{+}\left(Q^{\prime}\right)=0, \quad L_{-}(y Q)=-2 Q^{\prime} \tag{1.17}
\end{equation*}
$$

We note the one dimensional scalar product:

$$
(f, g)=\int f(y) g(y) d y
$$

1.4. Strategy of the proof. Let us give a brief insight into the proof of Theorem
1.2. The scheme follows the road map designed in [39].
step 1. A rough approximate solution. Let us renormalize the flow using the time dependent rescaling:

$$
u(t, r)=\frac{1}{\lambda(t)^{\frac{2}{p-1}}} v\left(s, \frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)}, \quad \frac{d s}{d t}=\frac{1}{\lambda^{2}}
$$

which maps the finite time blow up problem (1.1) onto the global in time renormalized equation ${ }^{5}$ (3.4):

$$
\begin{align*}
& i \partial_{s} v+v_{y y}+\frac{N-1}{1+\frac{\alpha b}{2 \beta} y} \frac{\alpha b}{2 \beta} v_{y}-\left(1+\beta^{2}\right) v+i b \Lambda v+2 i \beta v_{y}+v|v|^{p-1}  \tag{1.18}\\
= & i\left(\frac{\lambda_{s}}{\lambda}+b\right) \Lambda v+i\left(\frac{r_{s}}{\lambda}+2 \beta\right) v_{y}+\left(\tilde{\gamma}_{s}-\beta^{2}\right) v
\end{align*}
$$

where we have introduced a Galilean drift parameter $\beta$ and where we have defined

$$
b=\frac{2 \beta}{\alpha} \frac{\lambda}{r} \quad \text { and } \quad \tilde{\gamma}_{s}=\gamma_{s}-1
$$

The beautiful observation of Fibich, Gavish and Wang [9] is that an approximate solution to (1.18) can be constructed of the form:

$$
w(s, y)=Q(y) e^{-i \frac{b(s) y^{2}}{4}} e^{-i \beta_{\infty} y}
$$

where $Q$ is the mass subcritical one dimensional ground state, and this relies on the specific algebra generated by the choice (1.10) of $\beta$ and the specific choices of modulation equations (3.5). Note that this choice corresponds to cancellation

$$
E\left(Q e^{-i \beta_{\infty} y}\right)=0
$$

which is indeed required for a blow up profile candidate. The explicit integration of the modulation equations

$$
\begin{equation*}
\frac{r_{s}}{\lambda}=-2 \beta, \quad-\frac{\lambda_{s}}{\lambda}=b=\frac{2 \beta}{\alpha} \frac{\lambda}{r}, \quad \frac{d s}{d t}=\frac{1}{\lambda^{2}} \tag{1.19}
\end{equation*}
$$

with the choice $\beta=\beta_{\infty}$ leads from direct check to the regime (1.14) for $r, \lambda, \gamma$. Since $\lambda(t)$ touches zero in finite time, this corresponds to a finite time blow up regime. Furthermore, there holds the relation:

$$
\begin{equation*}
b \sim \lambda^{1-\alpha} \tag{1.20}
\end{equation*}
$$

step 2. Construction of a high order approximate solution. We now proceed to the construction of a high order approximate solution to (3.4). Following the slow modulated ansatz approach developed in [24], [16], [39], we freeze the modulation equations

$$
\frac{r_{s}}{\lambda}+2 \beta=0, \quad \tilde{\gamma}_{s}=\beta^{2}, \quad b=\frac{2 \beta}{\alpha} \frac{\lambda}{r}
$$

and look for an expansion of the form

$$
\begin{equation*}
Q_{b, \widetilde{\beta}}(y)=\left[Q+\sum_{1 \leq j+l \leq k-1} b^{j} \tilde{\beta}^{l}(s)\left(T_{j, l}(y)+i S_{j, l}(y)\right)\right] e^{-i \frac{b(s) y^{2}}{4}} e^{-i \beta y} \tag{1.21}
\end{equation*}
$$

where

$$
\beta=\beta_{\infty}+\tilde{\beta}
$$

and the laws for the remaining parameters are adjusted dynamically

$$
\frac{\lambda_{s}}{\lambda}+b=\mathcal{P}_{1}(b, \tilde{\beta}), \quad \tilde{\beta}_{s}=\mathcal{P}_{2}(b, \tilde{\beta})
$$

Expanding in powers of $b, \widetilde{\beta}$, the construction reduces to an inductive linear system

$$
\left\{\begin{array}{l}
L_{+} T_{j, l}=F_{j, l}\left(T_{p, q}, \ldots, S_{p, q}\right)_{1 \leq p+q \leq j+l-1}  \tag{1.22}\\
L_{-} S_{j, l}=G_{j, l}\left(T_{p, q}, \ldots, S_{p, q}\right)_{1 \leq p+q \leq j+l-1}
\end{array}, \quad j \geq 1\right.
$$

[^3]where $\left(L_{+}, L_{-}\right)$is the matrix linearized operator (1.16) close to $Q$. The kernel of this operator is well known, [40], and the solvability of the nonlinear system (1.22) in the class of Schwarz functions is subject to the orthogonality conditions
\[

\left\{$$
\begin{array}{l}
\left(F_{j, l}\left(T_{p, q}, \ldots, S_{p, q}\right)_{1 \leq p+q \leq j+l-1}, \partial_{y} Q\right)=0,  \tag{1.23}\\
\left(G_{j, l}\left(T_{p, q}, \ldots, S_{p, q}\right)_{1 \leq p+q \leq j+l-1}, Q\right)=0
\end{array}
$$\right.
\]

which correspond respectively to the translation and phase orbital instabilities, and is ensured inductively through the construction of the polynomials $\left(\mathcal{P}_{i}(b, \tilde{\beta})\right)_{i=1,2}$. The fundamental observation is that the problem near the sub critical ground state is no longer degenerate i.e.

$$
(\Lambda Q, Q) \neq 0,
$$

and this is major difference with [39], [15]. The outcome is the construction of an approximate solution to arbitrary high order.
step 3 The mixed Energy/Morawetz functional. We now aim at building an exact solution and use for this the Schauder type compactness argument designed in [22], [19], see also [16], [39]. We let a sequence $t_{n} \uparrow 0$ and consider $u_{n}(t)$ the solution to (1.1) with initial data given by the well prepared bubble

$$
u_{n}\left(t_{n}, x\right)=\frac{1}{\lambda\left(t_{n}\right)^{\frac{2}{p-1}}} Q_{b\left(t_{n}\right), \widetilde{\beta}\left(t_{n}\right)}\left(\frac{r-r\left(t_{n}\right)}{\lambda\left(t_{n}\right)}\right) e^{i \gamma\left(t_{n}\right)}
$$

where the parameters are chosen in their asymptotic law (1.14):

$$
r\left(t_{n}\right) \sim\left|t_{n}\right|^{\frac{\alpha}{1+\alpha}}, \quad \lambda\left(t_{n}\right) \sim\left|t_{n}\right|^{\frac{1}{1+\alpha}}, \quad \gamma\left(t_{n}\right) \sim\left|t_{n}\right|^{-\frac{1-\alpha}{1+\alpha}} .
$$

We then proceed to a modulated decomposition of the flow

$$
u(t, r)=\frac{1}{\lambda(t)^{\frac{2}{p-1}}}\left(Q_{b(t), \tilde{\beta}(t)}+\varepsilon\right)\left(t, \frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)}
$$

where $\varepsilon$ satisfies suitable orthogonality conditions through the modulation on $(r(t), \gamma(t), \lambda(t), \widetilde{\beta}(t))$, and the $b$ parameter is frozen:

$$
b(t)=\frac{2 \beta(t)}{\alpha} \frac{\lambda(t)}{r(t)} .
$$

We claim that there exists a backward time $\underline{t}$ independent of $n$ such that

$$
\begin{equation*}
\forall t \in\left[\underline{t}, t_{n}\right], \quad\|\varepsilon(t)\|_{H_{\mu}^{1}} \lesssim \lambda^{c_{k}}(t) \tag{1.24}
\end{equation*}
$$

where we introduce the renormalized Sobolev norm:

$$
\|\varepsilon\|_{H_{\mu}^{1}}^{2}=\int\left(\left|\partial_{y} \varepsilon\right|^{2}+|\varepsilon|^{2}\right) \mu, \quad \mu(y)=\left(1+\frac{\alpha b}{2 \beta} y\right)^{N-1} \mathbf{1}_{1+\frac{\alpha b}{2 \beta} y>0},
$$

and where $c_{k} \rightarrow+\infty$ as $k \rightarrow+\infty$ relates to the order of expansion of the approximate solution $Q_{b, \widetilde{\beta}}$ to (1.18). The estimate (1.24) easily allows to conclude the proof existence by passing to the limit $t_{n} \uparrow 0$, and the control of the parameters $(\lambda(t), r(t))$ leading concentration follows from the standard reintegration of the corresponding modulation equation.
Following [37], [39], [31], the proof of (1.24) relies on the derivation of a mixed Energy/Morawetz Lyapunov functional. Let the Galilean shift

$$
\tilde{\varepsilon}=\varepsilon e^{i \beta y}
$$

the corresponding monotonicity formula roughly takes the form:

$$
\begin{equation*}
\frac{d}{d t} \mathcal{I}=\mathcal{J}+O\left(\frac{b^{k}}{\lambda^{4}}\right) \tag{1.25}
\end{equation*}
$$

where $\mathcal{I}, \mathcal{J}$ are given by

$$
\begin{aligned}
\mathcal{I}(\tilde{u}) & =\frac{1}{2} \int|\nabla \tilde{u}|^{2}+\frac{1+\beta^{2}}{2} \int \frac{|\tilde{u}|^{2}}{\lambda^{2}}-\int\left[F(\tilde{Q}+\tilde{u})-F(\tilde{Q})-F^{\prime}(\tilde{Q}) \cdot \tilde{u}\right] \\
& +\frac{\beta}{\lambda} \Im\left(\int \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u} \overline{\tilde{u}}\right)
\end{aligned}
$$

with $F(u)=|u|^{p+1}, \phi$ a suitable cut off function, and

$$
\mathcal{J}=O\left(\frac{b\|\varepsilon\|_{H_{\mu}^{1}}^{2}}{\lambda^{4}}\right)
$$

The power of $b$ in the right hand side of (1.25) is related to the error in the construction of $Q_{b, \widetilde{\beta}}$, and the Morawetz term in $\mathcal{I}$ is manufactured to reproduce the non trivial Galilean drift $\beta_{\infty}$ so that $\mathcal{I}$ is on the soliton core a small deformation of the linearized energy. Our choice of orthogonality conditions then ensures the coercivity of $\mathcal{I}$ :

$$
\begin{equation*}
\mathcal{I} \gtrsim \frac{\|\varepsilon\|_{H_{\mu}^{1}}^{2}}{\lambda^{2}} \tag{1.26}
\end{equation*}
$$

Now unlike in [39], we do not need to take into account further structure in the quadratic term $\mathcal{J}$. Indeed, for a large enough ${ }^{6}$ parameter $\theta \gg 1$, we obtain from $-\lambda \lambda_{t} \sim b>0$ :

$$
\frac{d}{d t}\left(\frac{\mathcal{I}}{\lambda^{\theta}}\right) \gtrsim \frac{b}{\lambda^{4+\theta}}\left[(\theta-C)\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right]+O\left(\frac{b^{k}}{\lambda^{4+\theta}}\right) \gtrsim O\left(\frac{b^{k}}{\lambda^{4+\theta}}\right)
$$

For $k$ large enough, the last term is integrable in time in the ring regime, and integrating the ODE backwards from blow up time where $\varepsilon\left(t_{n}\right) \equiv 0$ yields (1.24). Note that the strength of this energy method is in particular to completely avoid the use of weighted spaces to control the flow as in [3], [1], and the analysis is robust enough to handle rough nonlinearities $p<2$.

This paper is organized as follows. In section 2, we prove Theorem 1.1. In section 3 , we construct the approximate solution $Q_{b, \widetilde{\beta}}$ using the slowly modulated ansatz. In section 4 , we set up the bootstrap argument and derive the modulation equations. In section 5, we derive the mixed Energy/Morawetz monotonicity formula. In section 6 , we close the bootstrap and conclude the proof of Theorem 1.2.

## 2. Universal upper bound on the blow up rate

This section is devoted to the proof of Theorem 1.1. The proof is spectacularly simple and relies on a sharp version of the localized virial identity used in [28].

Proof of Theorem 1.1. step 1 Localized virial identity. Let $N \geq 2,0<s_{c}<1$ and $u \in \mathcal{C}\left([0, T), H^{1}\right)$ be a radially symmetric finite time blow up solution $0<T<+\infty$.

[^4]Pick a time $t_{0}<T$ and a radius $0<R=R\left(t_{0}\right) \ll 1$ to be chosen. Let $\chi$ a radial function in $\mathcal{D}\left(\mathbb{R}^{N}\right)$ and recall the localized virial identity ${ }^{7}$ for radial solutions:

$$
\begin{gather*}
\frac{1}{2} \frac{d}{d \tau} \int \chi|u|^{2}=\operatorname{Im}\left(\int \nabla \chi \cdot \nabla u \bar{u}\right)  \tag{2.1}\\
\frac{1}{2} \frac{d}{d \tau} \operatorname{Im}\left(\int \nabla \chi \cdot \nabla u \bar{u}\right)=\int \chi^{\prime \prime}|\nabla u|^{2}-\frac{1}{4} \int \Delta^{2} \chi|u|^{2}-\left(\frac{1}{2}-\frac{1}{p+1}\right) \int \Delta \chi|u|^{p+1} .
\end{gather*}
$$

Applying with $\chi=\psi_{R}=R^{2} \psi\left(\frac{x}{R}\right)$ where $\psi$ is a radial function such that $\psi(x)=\frac{|x|^{2}}{2}$ for $|x| \leq 2$ and $\psi(x)=0$ for $|x| \geq 3$, we get:

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d \tau} \operatorname{Im}\left(\int \nabla \psi_{R} \cdot \nabla u \bar{u}\right) \\
= & \int \psi^{\prime \prime}\left(\frac{x}{R}\right)|\nabla u|^{2}-\frac{1}{4 R^{2}} \int \Delta^{2} \psi\left(\frac{x}{R}\right)|u|^{2}-\left(\frac{1}{2}-\frac{1}{p+1}\right) \int \Delta \psi\left(\frac{x}{R}\right)|u|^{p+1} \\
\leq & \int|\nabla u|^{2}-N\left(\frac{1}{2}-\frac{1}{p+1}\right) \int|u|^{p+1}+C\left[\frac{1}{R^{2}} \int_{2 R \leq|x| \leq 3 R}|v|^{2}+\int_{|x| \geq R}|u|^{p+1}\right] .
\end{aligned}
$$

Now from the conservation of the energy:

$$
\int|u|^{p+1}=\frac{p+1}{2} \int|\nabla u|^{2}-(p+1) E\left(u_{0}\right)
$$

from which

$$
\int|\nabla u|^{2}-N\left(\frac{1}{2}-\frac{1}{p+1}\right) \int|u|^{p+1}=\frac{N(p-1)}{2} E\left(u_{0}\right)-\frac{2 s_{c}}{N-2 s_{c}} \int|\nabla u|^{2}
$$

and thus:

$$
\begin{align*}
& \frac{2 s_{c}}{N-2 s_{c}} \int|\nabla u|^{2}+\frac{1}{2} \frac{d}{d \tau} \operatorname{Im}\left(\int \nabla \psi_{R} \cdot \nabla u \bar{u}\right) \\
\lesssim & {\left[\left|E_{0}\right|+\int_{|x| \geq R}|u|^{p+1}+\frac{1}{R^{2}} \int_{2 R \leq|x| \leq 3 R}|u|^{2}\right] }  \tag{2.2}\\
\lesssim & C\left(u_{0}\right)\left[1+\frac{1}{R^{2}}+\int_{|x| \geq R}|u|^{p+1}\right]
\end{align*}
$$

from the energy and $L^{2}$ norm conservations.
step 2 Radial Gagliardo-Nirenberg interpolation estimate. In order to control the outer nonlinear term in (2.2), we recall the radial interpolation bound:

$$
\|u\|_{L^{\infty}(r \geq R)} \leq \frac{\|\nabla u\|_{L^{2}}^{\frac{1}{2}}\|u\|_{L^{2}}^{\frac{1}{2}}}{R^{\frac{N-1}{2}}}
$$

which together with the $L^{2}$ conservation law ensures:

$$
\begin{aligned}
\int_{|x| \geq R}|u|^{p+1} & \leq\|u\|_{L^{\infty}(r \geq R)}^{p-1} \int|u|^{2} \leq \frac{C\left(u_{0}\right)}{R^{\frac{(N-1)(p-1)}{2}}}\|\nabla u\|_{L^{2}}^{\frac{p-1}{2}} \\
& \leq \delta \frac{2 s_{c}}{N-2 s_{c}} \int|\nabla u|^{2}+\frac{C}{\delta R^{\frac{2(N-1)(p-1)}{(5-p)}}} \\
& =\delta \frac{2 s_{c}}{N-2 s_{c}} \int|\nabla u|^{2}+\frac{C}{\delta R^{\frac{2}{\alpha}}}
\end{aligned}
$$

[^5]where we used Hölder for $p<5$ and the definition of $\alpha$ (1.11). Injecting this into (2.2) yields for $\delta>0$ small enough using $R \ll 1$ and $0<\alpha<1$ :
\[

$$
\begin{equation*}
\frac{s_{c}}{N-2 s_{c}} \int|\nabla u|^{2}+\frac{d}{d \tau} \operatorname{Im}\left(\int \nabla \psi_{R} \cdot \nabla u \bar{u}\right) \leq \frac{C\left(u_{0}, p\right)}{R^{\frac{2}{\alpha}}} \tag{2.3}
\end{equation*}
$$

\]

step 3 Time integration. We now integrate (2.3) twice in time on $\left[t_{0}, t_{2}\right]$ using (2.1). This yields up to constants using Fubini in time:

$$
\begin{aligned}
& \int \psi_{R}\left|u\left(t_{2}\right)\right|^{2}+\int_{t_{0}}^{t_{2}}\left(t_{2}-t\right)\|\nabla u(t)\|_{L^{2}}^{2} d t \\
\lesssim & \frac{\left(t_{2}-t_{0}\right)^{2}}{R^{\frac{2}{\alpha}}}+\left(t_{2}-t_{0}\right)\left|\operatorname{Im}\left(\int \nabla \psi_{R} \cdot \nabla u \bar{u}\right)\left(t_{0}\right)\right|+\int \psi_{R}\left|u\left(t_{0}\right)\right|^{2} \\
\leq & C\left(u_{0}\right)\left[\frac{\left(t_{2}-t_{0}\right)^{2}}{R^{\frac{2}{\alpha}}}+R\left(t_{2}-t_{0}\right)\left\|\nabla u\left(t_{0}\right)\right\|_{L^{2}}+R^{2}\left\|u_{0}\right\|_{L^{2}}^{2}\right]
\end{aligned}
$$

We now let $t \rightarrow T$. We conclude that the integral in the left hand side converges ${ }^{8}$ and

$$
\begin{equation*}
\int_{t_{0}}^{T}(T-t)|\nabla u(t)|_{L^{2}}^{2} d t \leq C\left(u_{0}\right)\left[\frac{\left(T-t_{0}\right)^{2}}{R^{\frac{2}{\alpha}}}+R\left(T-t_{0}\right)\left\|\nabla u\left(t_{0}\right)\right\|_{L^{2}}+R^{2}\right] \tag{2.4}
\end{equation*}
$$

We now optimize in $R$ by choosing:

$$
\frac{\left(T-t_{0}\right)^{2}}{R^{\frac{2}{\alpha}}}=R^{2} \text { ie } R\left(t_{0}\right)=\left(T-t_{0}\right)^{\frac{\alpha}{1+\alpha}}
$$

(2.4) now becomes:

$$
\begin{align*}
\int_{t_{0}}^{T}(T-t)\|\nabla u(t)\|_{L^{2}}^{2} d t & \leq C\left(u_{0}\right)\left[\left(T-t_{0}\right)^{\frac{2 \alpha}{1+\alpha}}+\left(T-t_{0}\right)^{\frac{\alpha}{1+\alpha}}\left(T-t_{0}\right)\left\|\nabla u\left(t_{0}\right)\right\|_{L^{2}}\right] \\
& \leq C\left(u_{0}\right)\left(T-t_{0}\right)^{\frac{2 \alpha}{1+\alpha}}+\left(T-t_{0}\right)^{2}\left\|\nabla u\left(t_{0}\right)\right\|_{L^{2}}^{2} \tag{2.5}
\end{align*}
$$

In order to integrate this differential inequality, let

$$
\begin{equation*}
g\left(t_{0}\right)=\int_{t_{0}}^{T}(T-t)\|\nabla u(t)\|_{L^{2}}^{2} d t \tag{2.6}
\end{equation*}
$$

then (2.5) means:

$$
g(t) \leq C(T-t)^{\frac{2 \alpha}{1+\alpha}}-(T-t) g^{\prime}(t)
$$

ie

$$
\left(\frac{g}{T-t}\right)^{\prime}=\frac{1}{(T-t)^{2}}\left((T-t) g^{\prime}+g\right) \leq \frac{1}{(T-t)^{2-\frac{2 \alpha}{1+\alpha}}}
$$

Integrating this in time yields

$$
\frac{g(t)}{T-t} \leq C\left(u_{0}\right)+\frac{1}{(T-t)^{1-\frac{2 \alpha}{1+\alpha}}} \text { ie } g(t) \leq C\left(u_{0}\right)(T-t)^{\frac{2 \alpha}{1+\alpha}}
$$

for $t$ close enough to $T$, which together with (2.6) yields (1.12). This concludes the proof of Theorem 1.1.

[^6]
## 3. The approximate solution

The rest of the paper is dedicated to the proof of Theorem 1.2 on the existence of ring solutions. We start in this section with the construction of an approximate solution at any order.
3.1. The slow modulated ansatz. Recall the definition of the positive numbers $\alpha$ and $\beta_{\infty}$ as:

$$
\begin{equation*}
\alpha=\frac{5-p}{(p-1)(N-1)}, \beta_{\infty}=\sqrt{\frac{5-p}{p+3}} \tag{3.1}
\end{equation*}
$$

Recall also that the restrictions on $p$ yield:

$$
\begin{equation*}
0<\alpha<1 \text { and } 0<\beta_{\infty}<1 \tag{3.2}
\end{equation*}
$$

Finally, recall that $Q$ denotes the 1-dimensional groundstate, i.e. the only positive, nonzero solution in $H^{1}$ of:

$$
\begin{equation*}
Q^{\prime \prime}-Q+Q^{p}=0, \text { explicitly } Q(x)=\left(\frac{p+1}{2 \cosh ^{2}\left(\frac{p-1}{2} x\right)}\right)^{\frac{1}{p-1}} \tag{3.3}
\end{equation*}
$$

Let us consider the general modulated ansatz:

$$
u(t, r)=\frac{1}{\lambda(t)^{\frac{2}{p-1}}} v\left(s, \frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)}, \quad \frac{d s}{d t}=\frac{1}{\lambda^{2}}
$$

which maps the finite time blow up problem (1.1) onto the global in time renormalized equation ${ }^{9}$ (3.4):

$$
\begin{align*}
& i \partial_{s} v+v_{y y}+\frac{N-1}{1+\frac{\alpha b}{2 \beta} y} \frac{\alpha b}{2 \beta} v_{y}-\left(1+\beta^{2}\right) v+i b \Lambda v+2 i \beta v_{y}+v|v|^{p-1}  \tag{3.4}\\
= & i\left(\frac{\lambda_{s}}{\lambda}+b\right) \Lambda v+i\left(\frac{r_{s}}{\lambda}+2 \beta\right) v_{y}+\left(\tilde{\gamma}_{s}-\beta^{2}\right) v
\end{align*}
$$

where we have defined

$$
\begin{equation*}
b=\frac{2 \beta}{\alpha} \frac{\lambda}{r} \text { and } \tilde{\gamma}_{s}=\gamma_{s}-1 \tag{3.5}
\end{equation*}
$$

We shift a Galilean phase and let $w$ be defined by:

$$
\begin{equation*}
w(s, y)=v(s, y) e^{i \beta y} \tag{3.6}
\end{equation*}
$$

which satisfies:

$$
\begin{align*}
& i \partial_{s} w+w_{y y}-w+w|w|^{p-1}+\frac{\alpha b}{2 \beta} \frac{N-1}{1+\frac{\alpha b y}{2 \beta}}\left(w_{y}-i \beta w\right)+b(i \Lambda w+\beta y w)  \tag{3.7}\\
= & -\widetilde{\beta}_{s} y w+\left(\frac{\lambda_{s}}{\lambda}+b\right)(i \Lambda w+\beta y w)+\left(\frac{r_{s}}{\lambda}+2 \beta\right)\left(i w_{y}+\beta w\right)+\left(\tilde{\gamma}_{s}-\beta^{2}\right) w .
\end{align*}
$$

[^7]3.2. Construction of the approximate solution $Q_{b, \widetilde{\beta}}$. We now proceed to the slow modulated ansatz construction as in [16], [39]. Let
$$
\beta=\beta_{\infty}+\widetilde{\beta}
$$

We look for an approximate solution to (3.4) of the form

$$
v(s, y)=Q_{b(s), \widetilde{\beta}(s)}(y), \quad \frac{\lambda_{s}}{\lambda}=-b+\mathcal{P}_{1}(b, \widetilde{\beta}), \quad \frac{r_{s}}{\lambda}=-2 \beta, \quad \tilde{\gamma}_{s}=\beta^{2}, \quad \beta_{s}=\mathcal{P}_{2}(b, \widetilde{\beta})
$$

where $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ are polynomial in $(b, \widetilde{\beta})$ which will be chosen later to ensure suitable solvability conditions. Note from the definition (3.5) of $b$ the relation:

$$
\begin{equation*}
b_{s}+(1-\alpha) b^{2}-\frac{b}{\beta} \mathcal{P}_{2}-b \mathcal{P}_{1}=\frac{b}{\beta}\left(\widetilde{\beta}_{s}-\mathcal{P}_{2}\right)+b\left(\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}\right)-\frac{\alpha}{2 \beta} b^{2}\left(\frac{r_{s}}{\lambda}+2 \beta\right) \tag{3.8}
\end{equation*}
$$

We then define the error term:

$$
\begin{align*}
-\Psi_{b, \widetilde{\beta}}= & i\left(-(1-\alpha) b^{2}+\frac{b}{\beta} \mathcal{P}_{2}+b \mathcal{P}_{1}\right) \partial_{b} Q_{b, \widetilde{\beta}}+i \mathcal{P}_{2} \partial_{\widetilde{\beta}} Q_{b, \widetilde{\beta}}  \tag{3.9}\\
& -\left(1+\beta^{2}\right) Q_{b, \widetilde{\beta}}+i\left(b-\mathcal{P}_{1}\right)\left(\frac{2}{p-1}+y \partial_{y}\right) Q_{b, \widetilde{\beta}} \\
& +2 i \beta \partial_{y} Q_{b, \widetilde{\beta}}+\partial_{y}^{2} Q_{b, \widetilde{\beta}}+\frac{\alpha b}{2 \beta} \frac{N-1}{1+\frac{\alpha b}{2 \beta} y} \partial_{y} Q_{b, \widetilde{\beta}}+\left|Q_{b, \widetilde{\beta}}\right|^{p-1} Q_{b, \widetilde{\beta}}
\end{align*}
$$

The algebra simplifies after a mixed Galilean/pseudo conformal drift:

$$
\begin{equation*}
Q_{b, \widetilde{\beta}}(y)=P_{b, \widetilde{\beta}}(y) e^{-i \beta y-i b \frac{y^{2}}{4}} \tag{3.10}
\end{equation*}
$$

which leads to the slowly modulated equation:

$$
\begin{align*}
& i\left(-(1-\alpha) b^{2}+\frac{b}{\beta} \mathcal{P}_{2}+b \mathcal{P}_{1}\right) \partial_{b} P_{b, \widetilde{\beta}}+i \mathcal{P}_{2} \partial_{\widetilde{\beta}} P_{b, \widetilde{\beta}}  \tag{3.11}\\
- & P_{b, \widetilde{\beta}}+\partial_{y}^{2} P_{b, \widetilde{\beta}}+\frac{\alpha b}{2 \beta} \frac{N-1}{1+\frac{\alpha b}{2 \beta} y} \partial_{y} P_{b, \widetilde{\beta}}+\left|P_{b, \widetilde{\beta}}\right|^{p-1} P_{b, \widetilde{\beta}} \\
- & i \mathcal{P}_{1}\left(\frac{2}{p-1}+y \partial_{y}\right) P_{b, \widetilde{\beta}}-\mathcal{P}_{1}\left(\beta y+\frac{b y^{2}}{2}\right) P_{b, \widetilde{\beta}}+\mathcal{P}_{2} y P_{b, \widetilde{\beta}} \\
+ & \left(b \beta y+\left(\alpha b^{2}+\frac{b}{\beta} \mathcal{P}_{2}+b \mathcal{P}_{1}\right) \frac{y^{2}}{4}-i\left[\frac{N-1}{1+\frac{b \alpha y}{2 \beta}} \frac{\alpha b^{2}}{4 \beta}(1-\alpha) y\right]\right) P_{b, \widetilde{\beta}} \\
= & -\Psi_{b, \widetilde{\beta}} e^{i \beta y+i b \frac{|y|^{2}}{4}} .
\end{align*}
$$

We now claim that we can construct a well localized high order approximate solution to (3.11).

Proposition 3.1 (Approximate solution). Let an integer $k \geq 5$, then there exist polynomials $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ of the form

$$
\begin{equation*}
\mathcal{P}_{1}(b, \widetilde{\beta})=\sum_{3 \leq j+l \leq k-1} c_{1, j, l} b^{j} \widetilde{\beta}^{l}, \quad \mathcal{P}_{2}(b, \widetilde{\beta})=-2 b \widetilde{\beta}+\sum_{3 \leq j+l \leq k-1} c_{2, j, l} b^{j} \widetilde{\beta}^{l}, \tag{3.12}
\end{equation*}
$$

and smooth well localized profiles $\left(T_{j, l}, S_{j, l}\right)_{1 \leq j+l \leq k-1}$, such that

$$
\begin{equation*}
P_{b, \widetilde{\beta}}=Q+\sum_{1 \leq j+l \leq k-1} b^{j} \widetilde{\beta}^{l}\left(T_{j, l}+i S_{j, l}\right) \tag{3.13}
\end{equation*}
$$

is a solution to (3.11) with $\Psi_{b, \widetilde{\beta}}$ smooth and well localized in $y$ satisfying:

$$
\begin{equation*}
\Psi_{b, \widetilde{\beta}}=O\left(b^{k}|y|^{c_{k}} e^{-|y|}\right) \tag{3.14}
\end{equation*}
$$

Moreoever, there holds the decay estimate:

$$
\begin{equation*}
\left|P_{b, \widetilde{\beta}}\right| \lesssim\left(1+|y|^{2 k}\right) e^{-|y|} \tag{3.15}
\end{equation*}
$$

## Proof of Proposition 3.1

The proof proceeds by injecting the expansion (3.13) in (3.11), identifying the terms with the same homogeneity in $(b, \widetilde{\beta})$, and inverting the corresponding operator. Let us recall that if $L=\left(L_{+}, L_{-}\right)$is the matrix linearized operator close to $Q$ given by (1.16), then its kernel is explicit:

$$
\begin{equation*}
\operatorname{Ker}\left\{L_{+}\right\}=\operatorname{span}\left\{Q^{\prime}\right\}, \quad \operatorname{Ker}\left\{L_{-}\right\}=\operatorname{span}\{Q\} \tag{3.16}
\end{equation*}
$$

see [40], [6].
step 1 General strategy.
Let $j+l \geq 1$. Assume that $T_{p, q}, c_{1, p, q}$ and $c_{2, p, q}$ for $p+q \leq j+l-1$ have been constructed. Then, identifying the terms homogeneous of order $(j, l)$ in (3.11) yields a linear system of the following type

$$
\left\{\begin{array}{l}
L_{+}\left(T_{j, l}\right)=h_{1, j, l}-c_{1, j, l} \beta_{\infty} y Q+c_{2, j, l} y Q  \tag{3.17}\\
L_{-}\left(S_{j, l}\right)=h_{2, j, l}-c_{1, j, l} \Lambda Q
\end{array}\right.
$$

where $h_{1, j, l}$ and $h_{2, j, l}$ may be computed explicitly and only depend on $T_{p, q}, c_{1, p, q}$ and $c_{2, p, q}$ for $p+q \leq j+l-1$. The invertibility of (3.17) requires according to (3.16) to manufacture the orthogonality conditions $\left(h_{j}^{(1)}, Q^{\prime}\right)=\left(h_{j}^{(2)}, Q\right)=0$, see [37] for related issues. We also need to track the decay in space of the associated solution in a sharp way. We claim:

Lemma 3.2. For all $1 \leq j+l \leq k-1$, let:

$$
\begin{equation*}
c_{1, j, l}=\frac{1}{(Q, \Lambda Q)}\left(h_{2, j, l}, Q\right) \text { and } c_{2, j, l}=\frac{2}{\|Q\|_{L^{2}}^{2}}\left(h_{1, j, l}, Q^{\prime}\right)+\frac{\beta_{\infty}}{(Q, \Lambda Q)}\left(h_{2, j, l}, Q\right) \tag{3.18}
\end{equation*}
$$

Then, there exist $\left(T_{j, l}, S_{j, l}\right)$ solution of (3.17) for all $1 \leq j+l \leq k-1$. Furthermore, $T_{j, l}$ and $S_{j, l}$ are smooth, and decay as

$$
\begin{equation*}
T_{j, l}=O\left(|y|^{2(j+l)} e^{-|y|}\right) \text { and } S_{j, l}=O\left(|y|^{2(j+l)} e^{-|y|}\right) \text { as } y \rightarrow \pm \infty \tag{3.19}
\end{equation*}
$$

Remark 3.3. Note that the quantity $(Q, \Lambda Q)$ appearing in (3.18) is given by

$$
(Q, \Lambda Q)=\frac{5-p}{2(p-1)}
$$

and is well-defined and not zero since $1<p<5$. This is a major difference with respect to the analysis in [39].

Proof of Lemma 3.2. In order to be able to solve for $\left(T_{j, l}, S_{j, l}\right)$, we need, in view of (3.16) and (3.17)

$$
\left(h_{1, j, l}-c_{1, j, l} \beta_{\infty} y Q+c_{2, j, l} y Q, Q^{\prime}\right)=0 \text { and }\left(h_{2, j, l}-c_{1, j, l} \Lambda Q, Q\right)=0
$$

which is equivalent to (3.18). Thus, choosing $c_{1, j, l}$ and $c_{2, j, l}$ as in (3.18), we may solve for $\left(T_{j, l}, S_{j, l}\right)$ solution of (3.17).

Next, we investigate the smoothness and decay properties of $\left(T_{j, l}, S_{j, l}\right)$. Identifying the terms homogeneous of order $j+l$ in (3.11), we have for $h_{1, j, l}$ and $h_{2, j, l}$ defined in (3.17)
where we have defined by convenience $T_{0,0}=Q$, where $a_{m, p, q}$ are real numbers which may be explicitly computed, and where $\mathrm{NL}_{j}^{(1)}$ and $\mathrm{NL}_{j}^{(2)}$ are the contributions coming from the Taylor expansion of the nonlinearity near $Q$. They take the following form

$$
\begin{align*}
\mathrm{NL}_{j}^{(1)}= & \sum_{p \geq 0, q \geq 0} \sum_{j_{m} \geq 1, l_{m} \geq 1 / j_{1}+\cdots+j_{q}+l_{1}+\cdots+l_{q}=j+l} a_{j_{1}, \cdots, j_{q}, l_{1}, \cdots, l_{q}}^{(1)} \\
& \times T_{j_{1}, l_{1}} \cdots T_{j_{p}, l_{p}} S_{j_{p+1}, l+p+1} \cdots S_{j_{q}, l_{q}} Q^{p-j-l}, \tag{3.21}
\end{align*}
$$

and

$$
\begin{align*}
\mathrm{NL}_{j}^{(2)}= & \sum_{p \geq 0, q \geq 0} \sum_{j_{m} \geq 1, l_{m} \geq 1 / j_{1}+\cdots+j_{q}+l_{1}+\cdots+l_{q}=j+l} a_{j_{1}, \cdots, j_{q}, l_{1}, \cdots, l_{q}}^{(2)} \\
& \times T_{j_{1}, l_{1}} \cdots T_{j_{p}, l_{p}} S_{j_{p+1}, l+p+1} \cdots S_{j_{q}, l_{q}} Q^{p-j-l}, \tag{3.22}
\end{align*}
$$

where the real numbers $a_{j_{1}, \cdots, j_{q}, l_{1}, \cdots, l_{q}}^{(1)}$ and $a_{j_{1}, \cdots, j_{q}, l_{1}, \cdots, l_{q}}^{(2)}$ may be computed explicitly.
We argue by induction. Assume that $T_{p, q}, p+q \leq j+l-1$, satisfy the conclusions of the lemma in terms of smoothness and decay. Then, we easily check from the formulas (3.20) (3.21) (3.22) that $h_{1, j, l}$ and $h_{2, j, l}$ are smooth. Then, from standard elliptic regularity, we deduce that $T_{j, l}$ and $S_{j, l}$ are smooth.

Finally, we consider the decay properties of $T_{j, l}$ and $S_{j, l}$. Since we assume by induction that $T_{p, q}, p+q \leq j+l-1$, satisfy the decay assumption (3.19), we easily obtain from (3.21) and (3.22)

$$
\mathrm{NL}_{j}^{(1)}=O\left(|y|^{2(j+l)} e^{-p|y|}\right) \text { and } \mathrm{NL}_{j}^{(2)}=O\left(|y|^{2(j+l)} e^{-p|y|}\right) \text { as } y \rightarrow \pm \infty
$$

Together with (3.17), the fact that $T_{p, q}, p+q \leq j+l-1$ satisfy the decay assumption (3.19), and the fact that $p>1$, we deduce

$$
\begin{equation*}
h_{1, j, l}=O\left(|y|^{2 j-1} e^{-|y|}\right) \text { and } h_{2, j, l}=O\left(|y|^{2 j-1} e^{-|y|}\right) \text { as } y \rightarrow \pm \infty . \tag{3.23}
\end{equation*}
$$

Now, let us consider the solution $\left(f_{1}, f_{2}\right)$ to the system

$$
L_{+}\left(f_{1}\right)=h_{1} \text { and } L_{-}\left(f_{2}\right)=h_{2}
$$

with $\left(h_{1}, Q^{\prime}\right)=0$ and $\left(h_{2}, Q\right)=0$. Then, for $y \geq 1$ for instance, we define

$$
g_{1}(y)=Q^{\prime}(y) \int_{1}^{y} \frac{d \sigma}{Q^{\prime}(\sigma)^{2}} \text { and } g_{2}(y)=Q(y) \int_{1}^{y} \frac{d \sigma}{Q(\sigma)^{2}}
$$

so that $\left(Q^{\prime}, g_{1}\right)$ forms a basis of solutions to the second order ordinary differential equation $L_{+}(f)=0$, while $\left(Q, g_{2}\right)$ forms a basis of solutions to the second order ordinary differential equation $L_{-}(f)=0$. Note that the decay properties of $Q$ and $Q^{\prime}$ immediately yield

$$
g_{1}(y)=O\left(e^{y}\right) \text { and } g_{2}(y)=O\left(e^{y}\right) \text { as } y \rightarrow+\infty
$$

Furthermore, using the variation of constants method, we find that a solution is given by ${ }^{10}$ :

$$
\begin{aligned}
& f_{1}(y)=-g_{1}(y)\left(\int_{y}^{+\infty} h_{1}(\sigma) Q^{\prime}(\sigma) d \sigma\right)+Q^{\prime}(y)\left(\int_{1}^{y} h_{1}(\sigma) g_{1}(\sigma) d \sigma\right) \\
& f_{2}(y)=-g_{2}(y)\left(\int_{y}^{+\infty} h_{2}(\sigma) Q(\sigma) d \sigma\right)+Q(y)\left(\int_{1}^{y} h_{2}(\sigma) g_{2}(\sigma) d \sigma\right)
\end{aligned}
$$

Thus, for any integer $n$, if

$$
h_{1}=O\left(|y|^{n} e^{-|y|}\right) \text { and } h_{2}(y)=O\left(|y|^{n} e^{-|y|}\right) \text { as } y \rightarrow \pm \infty,
$$

then, we obtain

$$
f_{1}=O\left(|y|^{n+1} e^{-|y|}\right) \text { and } f_{2}(y)=O\left(|y|^{n+1} e^{-|y|}\right) \text { as } y \rightarrow \pm \infty .
$$

Applying this observation to the system (3.17) with the choice $n=2(j+l)-1$ yields, in view of (3.23), the decay (3.19). This concludes the proof of the lemma.

In view of Lemma 3.2, the proof of Proposition 3.1 will follow from the verification that

$$
c_{n, 1,0}=c_{n, 0,1}=c_{n, 2,0}=c_{n, 0,2}=0 \text { for } n=1,2, c_{1,1,1}=0 \text { and } c_{2,1,1}=-2 .
$$

step 2 Computation of $c_{1,1,0}$ and $c_{2,1,0}$.
We identify the terms homogeneous of order $(1,0)$ in (3.11) and get:

$$
\left\{\begin{array}{l}
L_{+}\left(T_{1,0}\right)=\frac{(N-1) \alpha}{2 \beta_{\infty}} Q^{\prime}+\beta_{\infty} y Q-c_{1,1,0} \beta_{\infty} y Q+c_{2,1,0} y Q  \tag{3.24}\\
L_{-}\left(S_{1,0}\right)=-c_{1,1,0} \Lambda Q
\end{array}\right.
$$

Now, note that

$$
\begin{align*}
\left(\frac{(N-1) \alpha}{2 \beta_{\infty}} Q^{\prime}+\beta_{\infty} y Q, Q^{\prime}\right) & =\frac{(N-1) \alpha}{2 \beta_{\infty}} \int\left(Q^{\prime}\right)^{2}-\frac{\beta_{\infty}}{2} \int Q^{2}  \tag{3.25}\\
& =\frac{\beta_{\infty}}{2}\left(\frac{p+3}{p-1} \int\left(Q^{\prime}\right)^{2}-\int Q^{2}\right),
\end{align*}
$$

where we used in the last inequality the definition of $\alpha$ and $\beta_{\infty}$ given by (1.10), (1.11). Now, taking the scalar product of the equation (3.3) with $Q+(p+1) y Q^{\prime}$, and integrating by parts, yields

$$
\begin{equation*}
\int Q^{2}=\frac{p+3}{p-1} \int\left(Q^{\prime}\right)^{2} \tag{3.26}
\end{equation*}
$$

which together with (3.25) implies

$$
\left(\frac{(N-1) \alpha}{2 \beta_{\infty}} Q^{\prime}+\beta_{\infty} y Q, Q^{\prime}\right)=0
$$

Together with (3.24), we obtain

$$
\left(h_{1,1,0}, Q^{\prime}\right)=\left(h_{2,1,0}, Q\right)=0
$$

which together with (3.18) yields

$$
c_{1,1,0}=c_{2,1,0}=0
$$

[^8]as desired.
step 3 Computation of $c_{1,0,1}$ and $c_{2,0,1}$.
We identify the terms homogeneous of order $(1,0)$ in $(3.11)$ and get:
\[

\left\{$$
\begin{array}{l}
L_{+}\left(T_{0,1}\right)=-c_{1,0,1} \beta_{\infty} y Q+c_{2,0,1} y Q  \tag{3.27}\\
L_{-}\left(S_{0,1}\right)=-c_{1,0,1} \Lambda Q
\end{array}
$$\right.
\]

which together with (3.18) yields

$$
c_{1,0,1}=c_{2,0,1}=0
$$

as desired.
step 4 Computation of $c_{1,2,0}$ and $c_{2,2,0}$.
We identify the terms homogeneous of order $(2,0)$ in $(3.11)$ and get:

$$
\left\{\begin{align*}
L_{+}\left(T_{2,0}\right)= & (1-\alpha) S_{1,0}+\frac{(N-1) \alpha}{2 \beta_{\infty}} T_{1,0}^{\prime}-(N-1) \frac{\alpha^{2}}{4 \beta_{\infty}^{2}} y Q^{\prime}  \tag{3.28}\\
& +\frac{p(p-1)}{2} Q^{p-2} T_{1,0}^{2}+\frac{p-1}{2} Q^{p-2} S_{1,0}^{2}+\beta_{\infty} y T_{1,0} \\
& +\frac{\alpha}{4} y^{2} Q-c_{1,2,0} \beta_{\infty} y Q+c_{2,2,0} y Q \\
L_{-}\left(S_{2,0}\right)= & -(1-\alpha) T_{1}+\frac{(N-1) \alpha}{2 \beta_{\infty}} S_{1,0}^{\prime}+(p-1) Q^{p-2} T_{1,0} S_{1,0} \\
& +\beta_{\infty} y S_{1,0}-(N-1) \frac{\alpha}{4 \beta_{\infty}} y Q(1-\alpha)-c_{1,2,0} \Lambda Q .
\end{align*}\right.
$$

Note from (3.24) that $T_{1,0}$ is an odd function, while $S_{1,0}$ is an even function. In view of (3.28), this implies that

$$
h_{1,2,0} \text { is even and } h_{2,2,0} \text { is odd. }
$$

In particular, since $Q$ is even and $Q^{\prime}$ is odd, we obtain

$$
\left(h_{1,2,0}, Q^{\prime}\right)=0 \text { and }\left(h_{2,2,0}, Q\right)=0
$$

which together with (3.18) yields

$$
c_{1,2,0}=c_{2,2,0}=0
$$

as desired.
step 5 Computation of $c_{1,1,1}$ and $c_{2,1,1}$.
We identify the terms homogeneous of order $(1,1)$ in $(3.11)$ and get:

$$
\left\{\begin{array}{l}
L_{+}\left(T_{1,1}\right)=-\frac{(N-1) \alpha}{2 \beta_{\infty}^{2}} Q^{\prime}+y Q-c_{1,1,1} \beta_{\infty} y Q+c_{2,1,1} y Q  \tag{3.29}\\
L_{-}\left(S_{1,1}\right)=-c_{1,1,1} \Lambda Q
\end{array}\right.
$$

In view of (3.29), we have

$$
\left(h_{1,1,1}, Q^{\prime}\right)=-\frac{(N-1) \alpha}{2 \beta_{\infty}^{2}} \int\left(Q^{\prime}\right)^{2}-\frac{1}{2} \int Q^{2}
$$

Using the computation

$$
\int\left(Q^{\prime}\right)^{2}=\frac{p-1}{p+3} \int Q^{2}
$$

and the definition of $\alpha$ and $\beta_{\infty}$ given by (1.10), (1.11), we deduce

$$
\left(h_{1,1,1}, Q^{\prime}\right)=-\int Q^{2},
$$

which together with (3.18) and the fact that $h_{2,1,1}=0$ yields

$$
c_{1,1,1}=0 \text { and } c_{2,1,1}=-2
$$

as desired.
step 6 Computation of $c_{1,0,1}$ and $c_{2,0,1}$.
We identify the terms homogeneous of order $(0,2)$ in $(3.11)$ and get:

$$
\left\{\begin{array}{l}
L_{+}\left(T_{0,2}\right)=-c_{1,0,2} \beta_{\infty} y Q+c_{2,0,2} y Q,  \tag{3.30}\\
L_{-}\left(S_{0,2}\right)=-c_{1,0,2} \Lambda Q
\end{array}\right.
$$

which together with (3.18) yields

$$
c_{1,0,2}=c_{2,0,2}=0
$$

as desired.
step 7 Conclusion.
We therefore have constructed an approximate solution $P_{b, \widetilde{\beta}}$ of (3.11) of the form (3.13). The decay estimate (3.15) on $P_{b, \tilde{\beta}}$ follows from (3.19). The error $\Psi_{b, \tilde{\beta}}$ consists of a polynomial in $\left(T_{j, l}, S_{j, l}\right)_{j+l \leq k-1}$ with lower order $k$, the error between the Taylor expansion of the potential terms $\frac{\alpha b}{2 \beta} \frac{N-1}{1+\frac{\alpha}{2 \beta} y}$ and $\frac{N-1}{1+\frac{b \alpha y}{2 \beta}}$ in (3.11), and the error between the nonlinear term and its Taylor expansion. The first and second type of terms are easily treated using the uniform exponential decay of $P_{b, \tilde{\beta}}$. We need to be slightly more careful for the nonlinear term. Here we recall that given $z \in \mathbb{C}$, let $P_{k-1}(z)$ be the order $k-1$ Taylor polynomial of $z \mapsto(1+z)|1+z|^{p-1}$ at $z=0$, then from ${ }^{11} p<5 \leq k$ :

$$
\forall z \in \mathbb{C}, \quad|(1+z)| 1+\left.z\right|^{p-1}-\left.P_{k-1}(z)\left|\lesssim C_{k}\right| z\right|^{k} .
$$

Let then

$$
\varepsilon_{b, \tilde{\beta}}=P_{b, \tilde{\beta}}-Q
$$

we obtain the bound by homogeneity:

$$
\begin{aligned}
& \left.\left|\left(Q+\varepsilon_{b, \widetilde{\beta}}\right)\right| Q+\left.\varepsilon_{b, \tilde{\beta}}\right|^{p-1}-Q^{p} P_{k-1}\left(\frac{\varepsilon_{b, \widetilde{\beta}}}{Q}\right) \right\rvert\, \lesssim C_{k} Q^{p} \frac{\left|\varepsilon_{b, \tilde{\beta}}\right|^{k}}{Q^{k}} \\
\lesssim & Q^{p} \sum_{1 \leq j+l \leq k-1}\left(\frac{|b|^{j}|\widetilde{\beta}|^{l}\left(\left|T_{j, l}\right|+\left|S_{j, l}\right|\right)}{Q}\right)^{k} .
\end{aligned}
$$

On the other hand, (3.19) ensures the uniform bound

$$
\frac{\left|T_{j, l}\right|+\left|S_{j, l}\right|}{Q} \lesssim 1+|y|^{c_{k}}, \quad j+l \leq k-1,
$$

and hence the bound:

$$
\left|\left(Q+\varepsilon_{b}\right)\right| Q+\left.\varepsilon_{b}\right|^{p-1}-\left.Q^{p} P_{k-1}\left(\frac{\varepsilon_{b}}{Q}\right)\left|\lesssim(|b|+|\widetilde{\beta}|)^{k}\right| y\right|^{c_{k}} e^{-|y|},
$$

[^9]and the control (3.14) of $\Psi_{b, \widetilde{\beta}}$ follows.
This concludes the proof of proposition 3.1.
3.3. Further properties of $Q_{b, \widetilde{\beta}}$. In order to avoid artificial troubles near the origin after renormalization, we introduce a smooth cut off function
\[

\zeta(y)=\left\{$$
\begin{array}{l}
0 \text { for } y \leq-2  \tag{3.31}\\
1 \text { for } y \geq-1
\end{array}
$$ \quad, \quad \zeta_{b}(y)=\zeta(\sqrt{b} y)\right.
\]

and define once and for all for the rest of this paper:

$$
\begin{equation*}
Q_{b, \widetilde{\beta}}(y)=\zeta_{b}(y) P_{b, \widetilde{\beta}}(y) e^{-i \beta y-i b \frac{y^{2}}{4}} \tag{3.32}
\end{equation*}
$$

Let us rewrite the $Q_{b, \widetilde{\beta}}$ equation using (3.4), (3.9) in the form which we will use in the forthcoming bootstrap argument.

Corollary $3.4\left(Q_{b, \widetilde{\beta}}\right.$ equation in original variables). Given $\mathcal{C}^{1}$ modulation parameters $(\lambda(t), r(t), \gamma(t), \widetilde{\beta}(t))$ such that

$$
\begin{equation*}
0<b(t)=\frac{2 \beta}{\alpha} \frac{\lambda(t)}{r(t)} \ll 1 \tag{3.33}
\end{equation*}
$$

let $\tilde{Q}$ be given by

$$
\begin{equation*}
\tilde{Q}(t, x)=\frac{1}{\lambda^{\frac{2}{p-1}}} Q_{b(t), \widetilde{\beta}(t)}\left(\frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)} \tag{3.34}
\end{equation*}
$$

Then $\tilde{Q}$ is a smooth radially symmetric function which satisfies:

$$
\begin{equation*}
i \partial_{t} \tilde{Q}+\Delta \tilde{Q}+\tilde{Q}|\tilde{Q}|^{p-1}=\psi=\frac{1}{\lambda(t)^{\frac{2 p}{p-1}}} \Psi\left(t, \frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)} \tag{3.35}
\end{equation*}
$$

with

$$
\begin{align*}
\Psi & =-\left(\gamma_{s}-1-\beta^{2}\right) Q_{b, \widetilde{\beta}}-i\left(\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}\right)\left(\Lambda Q_{b, \widetilde{\beta}}-b \partial_{b} Q_{b, \widetilde{\beta}}\right)  \tag{3.36}\\
& -i\left(\frac{r_{s}}{\lambda}+2 \beta\right)\left(\partial_{y} Q_{b, \widetilde{\beta}}+\frac{\alpha}{2 \beta} b^{2} \partial_{b} Q_{b, \widetilde{\beta}}\right)+i\left(b_{s}+(1-\alpha) b^{2}-\frac{b}{\beta} \mathcal{P}_{2}-b \mathcal{P}_{1}\right) \partial_{b} Q_{b, \widetilde{\beta}} \\
& +i\left(\beta_{s}-\mathcal{P}_{2}\right)\left(\partial_{\widetilde{\beta}} Q_{b, \widetilde{\beta}}+\frac{b}{\beta} \partial_{b} Q_{b, \widetilde{\beta}}\right)+O\left(\frac{e^{-|y|}}{b^{c_{k}}} \mathbf{1}_{y \sim \frac{1}{\sqrt{b}}}+(|b|+|\widetilde{\beta}|)^{k} \zeta_{b}|y|^{c_{k}} e^{-|y|}\right)
\end{align*}
$$

Proof of Corollary 3.4. We simply observe from (3.32), (3.33) that $\tilde{Q}$ is identically zero near the origin and hence (3.34) defines a well localized smooth radially symmetric function. The exponential decay in space of $P_{b, \widetilde{\beta}}$ ensures that the localization procedures perturbs the error term in (3.11) by an $O\left(\frac{e^{-|y|}}{b^{c k}} \mathbf{1}_{y \sim \frac{1}{\sqrt{b}}}\right)$ and the estimate (3.36) now directly follows from (3.4), (3.8), (3.9), (3.14).

## 4. Setting up the analysis

The aim of this section is to set up the bootstrap argument.
4.1. Choice of initial data. . Let us start with soling the system of exact modulation equations formally predicted by the $Q_{b, \widetilde{\beta}}$ construction. Is it easily seen that this system formally predicts a stable blow up in the ring regime. We shall simply need the following claim which proof is elementary and postponed Appendix A.

Lemma 4.1 (Integration of the exact system of modulation equations). There exists $t_{e}<0$ small enough and a solution $\left(\lambda_{e}, b_{e}, \widetilde{\beta}_{e}, r_{e}, \gamma_{e}\right)$ to the dynamical system:

$$
\left\{\begin{array}{l}
\frac{\lambda_{s}}{\lambda_{s}}+b=\mathcal{P}_{1}(b, \tilde{\beta}),  \tag{4.1}\\
\frac{r_{s}}{\lambda}+2 \beta=0 \\
\mathcal{\beta}_{s}=\mathcal{P}_{2}(b, \tilde{\beta}), \\
\gamma_{s}=1+\beta^{2}, \\
\frac{d s}{d t}=\frac{1}{\alpha^{2}}, \\
b=\frac{2 \beta}{\alpha} \frac{\lambda}{r}, \quad \beta=\beta_{\infty}+\tilde{\beta},
\end{array}\right.
$$

which is defined on $\left[t_{e}, 0\right)$. Moreover, this solution satisfies the following bounds

$$
\begin{gather*}
b_{e}(t)=\frac{1}{1+\alpha}\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{2}{1+\alpha}}|t|^{\frac{1-\alpha}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right)  \tag{4.2}\\
\lambda_{e}(t)=\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{1}{1+\alpha}}|t|^{\frac{1}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right)  \tag{4.3}\\
r_{e}(t)=g_{\infty}\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{\alpha}{1+\alpha}}|t|^{\frac{\alpha}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right)  \tag{4.4}\\
\widetilde{\beta}_{e}(t)=O\left(|t|^{\frac{2(1-\alpha)}{1+\alpha}}\right) \tag{4.5}
\end{gather*}
$$

and

$$
\begin{equation*}
\gamma_{e}(t)=\left(1+\beta_{\infty}^{2}\right)\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1-\alpha}{1+\alpha}}\left(\frac{2(1-\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{-\frac{2}{1+\alpha}}|t|^{-\frac{1-\alpha}{1+\alpha}}+O(\log (|t|)) \tag{4.6}
\end{equation*}
$$

for some universal constant $\left|g_{\infty}-1\right| \ll 1$.
From now on, we choose the integer $k$ appearing in Proposition 3.1 such that

$$
\begin{equation*}
k>\frac{2}{1-\alpha}+1 \tag{4.7}
\end{equation*}
$$

Given $t_{e}<\bar{t}<0$ small, we let $u(t)$ be the solution to (1.1) with well prepared initial data at $t=\bar{t}$ given explicitly by:

$$
\begin{equation*}
u(\bar{t}, r)=\frac{1}{\lambda_{e}(\bar{t})^{\frac{2}{p-1}}} Q_{b_{e}(\bar{t}), \widetilde{\beta}_{e}(\bar{t})}\left(\frac{r-r_{e}(\bar{t})}{\lambda_{e}(\bar{t})}\right) e^{i \gamma_{e}(\bar{t})} \tag{4.8}
\end{equation*}
$$

Our aim is to derive bounds on $u$ backwards on a time interval independent of $\bar{t}$ as $\bar{t} \rightarrow 0$. We describe in this section the bootstrap regime in which we will control the solution, and derive preliminary estimates on the flow which prepare the monotonicity formula of section 5 .
4.2. The modulation argument. We prove in this section a standard modulation lemma which relies on the implicit function theorem and the mass subcritical non degeneracy $(Q, \Lambda Q) \neq 0$.

Lemma 4.2 (Modulation). There exists a universal constant $\delta>0$ such that the following holds. Let u be a radially symmetric function of the form

$$
u(r)=\frac{1}{\lambda_{0}^{\frac{2}{p-1}}} Q_{b_{0}, \tilde{\beta}_{0}}\left(\frac{r-r_{0}}{\lambda_{0}}\right) e^{i \gamma_{0}}+\tilde{u}_{0}(r)
$$

with

$$
\lambda_{0}, r_{0}>0, \quad \beta_{0}=\beta_{\infty}+\tilde{\beta}_{0}, \quad b_{0}=\frac{2 \beta_{0}}{\alpha} \frac{\lambda_{0}}{r_{0}}
$$

the a priori bound

$$
\begin{equation*}
\frac{r_{0}}{\lambda_{0}^{\alpha}} \gtrsim 1 \tag{4.9}
\end{equation*}
$$

and the a priori smallness:

$$
\begin{equation*}
0<\left|b_{0}\right|+\left|\tilde{\beta}_{0}\right|+\left\|\tilde{u}_{0}\right\|_{L^{2}}<\delta . \tag{4.10}
\end{equation*}
$$

Then there exists a unique decomposition

$$
u(t, r)=\frac{1}{\lambda_{1}^{\frac{2}{p-1}}} Q_{b_{1}, \tilde{\beta}_{1}}\left(\frac{r-r_{1}}{\lambda_{0}}\right) e^{i \gamma_{1}}+\tilde{u}_{1}(r)
$$

with

$$
\beta_{1}=\beta_{\infty}+\tilde{\beta}_{1}, \quad b_{1}=\frac{2 \beta_{1}}{\alpha} \frac{\lambda_{1}}{r_{1}}
$$

such that

$$
\tilde{u}_{1}(r)=\frac{1}{\lambda_{1}^{\frac{2}{p-1}}} \tilde{\varepsilon}_{1}\left(\frac{r-r_{1}}{\lambda_{1}}\right) e^{i \gamma_{1}-i \beta_{1} \frac{r-r_{1}}{\lambda_{1}}}
$$

satisfies the orthogonality conditions

$$
\left(\Re\left(\tilde{\varepsilon}_{1}\right), \zeta_{b_{1}} y Q\right)=\left(\Re\left(\tilde{\varepsilon}_{1}\right), \zeta_{b_{1}} Q\right)=\left(\Im\left(\tilde{\varepsilon}_{1}\right), \zeta_{b_{1}} \Lambda Q\right)=\left(\Im\left(\tilde{\varepsilon}_{1}\right), \zeta_{b_{1}} \partial_{y} Q\right)=0
$$

Moreover, there holds the smallness:

$$
\begin{equation*}
\left|\frac{\lambda_{1}}{\lambda_{0}}-1\right|+\frac{\left|r_{0}-r_{1}\right|}{\lambda_{0}}+\left|\widetilde{\beta}_{0}-\widetilde{\beta}_{1}\right|+\left|\gamma_{0}-\gamma_{1}\right|+\left\|\tilde{u}_{1}\right\|_{L^{2}} \lesssim \delta . \tag{4.11}
\end{equation*}
$$

Proof of Lemma 4.2. This is a standard consequence of the implicit function theorem. We have by assumption:

$$
u(r)=\frac{1}{\lambda_{0}^{\frac{2}{p-1}}} Q_{b_{0}, \tilde{\beta}_{0}}\left(\frac{r-r_{0}}{\lambda_{0}}\right) e^{i \gamma_{0}}+\tilde{u}_{0}(r)
$$

and we wish to introduce a modified decomposition

$$
u(r)=\frac{1}{\lambda_{1}^{\frac{2}{p-1}}} Q_{b_{1}, \tilde{\beta}_{1}}\left(\frac{r-r_{1}}{\lambda_{1}}\right) e^{i \gamma_{1}}+\tilde{u}_{1}(r)
$$

Comparing the decompositions, we obtain the formula:

$$
\tilde{u}_{1}(r)=\frac{1}{\lambda_{0}^{\frac{2}{p-1}}} Q_{b_{0}, \tilde{\beta}_{0}}\left(\frac{r-r_{0}}{\lambda_{0}}\right) e^{i \gamma_{0}}-\frac{1}{\lambda_{1}^{\frac{2}{p-1}}} Q_{b_{1}, \tilde{\beta}_{1}}\left(\frac{r-r_{1}}{\lambda_{1}}\right) e^{i \gamma_{1}}+\tilde{u}_{0}(r)
$$

We now form the functional

$$
\begin{equation*}
F_{z, \mu, \gamma, \tilde{\beta}}(y)=\mu^{\frac{2}{p-1}} Q_{b_{0}, \tilde{\beta}_{0}}(\mu y+z) e^{-i \gamma+i\left(\beta_{0}+\tilde{\beta}\right) y}-Q_{b_{1}, \tilde{\beta}_{1}}(y) e^{i \beta_{1} y} \tag{4.12}
\end{equation*}
$$

with

$$
z=\frac{r_{1}-r_{0}}{\lambda_{0}}, \quad \mu=\frac{\lambda_{1}}{\lambda_{0}}, \quad \gamma=\gamma_{1}-\gamma_{0}, \quad \tilde{\beta}=\tilde{\beta}_{1}-\tilde{\beta}_{0}
$$

so that

$$
\tilde{\varepsilon}_{1}(y)=F_{z, \mu, \gamma, \tilde{\beta}}(y)+\lambda_{1}^{\frac{2}{p-1}} \tilde{u}_{0}\left(\lambda_{1} y+r_{1}\right) e^{-i \gamma_{1}+i \beta_{1} y}
$$

We then define the scalar products:

$$
\begin{aligned}
\rho^{(j)} & =\int_{-\infty}^{+\infty} \Re\left(\tilde{\varepsilon}_{1}\right) \zeta_{b_{1}} T^{(j)} d y \\
& =\int_{-\infty}^{+\infty} \Re\left(F_{z, \mu, \gamma, \tilde{\beta}}\right) \zeta_{b_{1}} T^{(j)} d y \\
& +\Re\left(\int_{0}^{+\infty} \tilde{u}_{0}(r) \frac{\lambda_{1}^{\frac{2}{p-1}}}{\lambda_{1}}\left(\zeta_{b_{1}} \Lambda Q\right)\left(\frac{r-r_{1}}{\lambda_{1}}\right) e^{-i \gamma_{1}+i \beta \frac{r-r_{1}}{\lambda_{1}}} d r\right) \text { for } j=1,2
\end{aligned}
$$

and

$$
\begin{aligned}
\rho^{(j)} & =\int_{-\infty}^{+\infty} \Im\left(\tilde{\varepsilon}_{1}\right) \zeta_{b_{1}} T^{(j)} d y \\
& =\int_{-\infty}^{+\infty} \Im\left(F_{z, \mu, \gamma, \tilde{\beta}}\right) \zeta_{b_{1}} T^{(j)} d y \\
& +\Im\left(\int_{0}^{+\infty} \tilde{u}_{0}(r) \frac{\lambda_{1}^{\frac{2}{p-1}}}{\lambda_{1}}\left(\zeta_{b_{1}} \Lambda Q\right)\left(\frac{r-r_{1}}{\lambda_{1}}\right) e^{-i \gamma_{1}+i \beta \frac{r-r_{1}}{\lambda_{1}}} d r\right) \text { for } j=3,4
\end{aligned}
$$

where

$$
T^{(1)}=y Q, \quad T^{(2)}=Q, \quad T^{(3)}=\partial_{y} Q, \quad T^{(4)}=\Lambda Q
$$

We now view $\rho=\left(\rho^{(j)}\right)_{1 \leq j \leq 4}$ as smooth functions of $\left(\tilde{u}_{0}, z, \mu, \tilde{\beta}, \gamma\right)$. Observe that the bound (4.9) ensures using the explicit formula (1.11) for $\alpha$ :

$$
\begin{equation*}
\left|\rho\left(\tilde{u}_{0}, 0,1,0,0\right)\right| \lesssim\left(\frac{r_{0}}{\lambda_{0}^{\frac{5-p}{(N-1)(p-1)}}}\right)^{-\frac{N-1}{2}}\left\|\tilde{u}_{0}\right\|_{L^{2}} \lesssim \delta \tag{4.13}
\end{equation*}
$$

We now compute

$$
b_{1}=\frac{2 \beta_{1}}{\alpha} \frac{\lambda_{1}}{r_{1}}=2\left(\beta_{0}+\tilde{\beta}\right) \frac{\lambda_{0}}{\alpha r_{0}} \mu \frac{r_{0}}{r_{1}}=\left(1+\frac{\tilde{\beta}}{\beta_{0}}\right) b_{0} \mu\left(1+\frac{\alpha b_{0}}{2 \beta_{0}} z\right)^{-1}
$$

We thus obtain using

$$
\left(Q_{b, \tilde{\beta}}\right)_{\mid(b, \tilde{\beta})=(0,0)}=Q e^{-i \beta_{\infty} y}
$$

the infinitesimal deformations:

$$
\begin{gathered}
\partial_{z} F_{(z=0, \mu=1, \tilde{\beta}=0, \gamma=0)}=Q^{\prime}-i \beta_{\infty} Q+O\left(\left(\left|b_{0}\right|+\left|\tilde{\beta}_{0}\right|\right) e^{-c|y|}\right) \\
\partial_{\mu} F_{\left.\right|_{(z=0, \mu=1, \tilde{\beta}=0, \gamma=0)}}=\Lambda Q-i \beta_{\infty} y Q+O\left(\left(\left|b_{0}\right|+\left|\tilde{\beta}_{0}\right|\right) e^{-c|y|}\right) \\
\partial_{\tilde{\beta}} F_{(z=0, \mu=1, \tilde{\beta}=0, \gamma=0)}=i y Q+O\left(\left(\left|b_{0}\right|+\left|\tilde{\beta}_{0}\right|\right) e^{-c|y|}\right) \\
\partial_{\gamma} F_{\left.\right|_{(z=0, \mu=1, \tilde{\beta}=0, \gamma=0)}}=-i Q+O\left(\left(\left|b_{0}\right|+\left|\tilde{\beta}_{0}\right|\right) e^{-c|y|}\right)
\end{gathered}
$$

The Jacobian matrix of $\rho$ at $\left(\tilde{u}_{0}=0, z=0, \mu=1, \tilde{\beta}=0, \gamma=0\right)$ is therefore given by:

$$
\begin{aligned}
D & =\left|\begin{array}{llll}
\left(Q^{\prime}, y Q\right) & (\Lambda Q, y Q) & 0 & 0 \\
\left(Q^{\prime}, Q\right) & (\Lambda Q, Q) & 0 & 0 \\
\left.-\beta_{\infty}\left(Q, Q^{\prime}\right)\right) & -\beta_{\infty}\left(y Q, Q^{\prime}\right) & \left(y Q, Q^{\prime}\right) & -\left(Q, Q^{\prime}\right) \\
-\beta_{\infty}(Q, \Lambda Q) & -\beta(y Q, \Lambda Q) & (y Q, \Lambda Q) & -(Q, \Lambda Q)
\end{array}\right|+O\left(\left|b_{0}\right|+\left|\tilde{\beta}_{0}\right|\right) \\
& =-\frac{1}{16}\left(\frac{5-p}{p-1}\right)^{2}\|Q\|_{L^{2}}^{8}+O\left(\left|b_{0}\right|+\left|\tilde{\beta_{0}}\right|\right) \neq 0
\end{aligned}
$$

from the smallness assumption (4.10). The existence of the desired decomposition now follows from the implicit function theorem, and the bound (4.11) follows from (4.13).
4.3. Setting up the bootstrap. Let $u(t, r)$ be the radially symmetric solution emanating from the data (4.8) at $t=\bar{t}$. From Lemma 4.1, Lemma 4.2 and a straightforward continuity argument, we can find a small time $t^{*}<\bar{t}$ such that $u(t, r)$ admits on $\left[t^{*}, \bar{t}\right]$ a unique decomposition

$$
\begin{equation*}
u(t, r)=\frac{1}{\lambda(t)^{\frac{2}{p-1}}} v\left(t, \frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)} \tag{4.14}
\end{equation*}
$$

where we froze the law:

$$
\begin{equation*}
b(t)=\frac{2 \beta}{\alpha} \frac{\lambda}{r}, \quad \frac{d s}{d t}=\frac{1}{\lambda^{2}(t)} \tag{4.15}
\end{equation*}
$$

and where there holds the decomposition

$$
\begin{equation*}
w(s, y)=v(s, y) e^{i \beta y}=Q_{b(t), \widetilde{\beta}(t)} e^{i \beta y}+\tilde{\varepsilon}(t, y), \quad \tilde{\varepsilon}=\tilde{\varepsilon}_{1}+i \tilde{\varepsilon}_{2} \tag{4.16}
\end{equation*}
$$

with the orthogonality conditions:

$$
\begin{equation*}
\left(\tilde{\varepsilon}_{1}, \zeta_{b} y Q\right)=\left(\tilde{\varepsilon}_{1}, \zeta_{b} Q\right)=\left(\tilde{\varepsilon}_{2}, \zeta_{b} \Lambda Q\right)=\left(\tilde{\varepsilon}_{2}, \zeta_{b} \partial_{y} Q\right)=0 \tag{4.17}
\end{equation*}
$$

Let us define the renormalized weight on the Lebesgue measure:

$$
\begin{equation*}
\mu=\left(1+\frac{\lambda(t)}{r(t)} y\right)^{N-1}=\left(1+\frac{\alpha b}{2 \beta} y\right)^{N-1} \tag{4.18}
\end{equation*}
$$

and the weighted Sobolev norms:

$$
\|\varepsilon\|_{L_{\mu}^{2}}^{2}=\int|\varepsilon|^{2} \mu, \quad\|\varepsilon\|_{H_{\mu}^{1}}^{2}=\int\left|\partial_{y} \varepsilon\right|^{2} \mu+\int|\varepsilon|^{2} \mu,
$$

then from Lemma 4.2, the decomposition (4.14) holds as long as

$$
\frac{r(t)}{\lambda(t)^{\alpha}} \lesssim 1
$$

and

$$
|b(t)|+|\widetilde{\beta}(t)|+\|\tilde{\varepsilon}(t)\|_{L_{\mu}^{2}}<\delta
$$

for some universal constant $\delta>0$ small enough.
We also introduce the decomposition of the flow:

$$
\begin{equation*}
u(t, r)=\tilde{Q}(t, x)+\tilde{u}(t, r), \quad \tilde{u}(t, r)=\frac{1}{\lambda(t)^{\frac{2}{p-1}}} \varepsilon\left(t, \frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)} \tag{4.19}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\tilde{\varepsilon}(s, y)=\varepsilon(s, y) e^{i \beta y} \tag{4.20}
\end{equation*}
$$

From (4.8), we have the well prepared data initialization:

$$
\varepsilon(\bar{t})=0, \quad(\lambda, b, \widetilde{\beta}, r, \gamma)(\bar{t})=\left(\lambda_{e}, b_{e}, \widetilde{\beta}_{e}, r_{e}, \gamma_{e}\right)(\bar{t})
$$

and we may thus consider a backward time $\underline{t}<\bar{t}$ such that the following bootstrap assumptions hold $\forall t \in(\underline{t}, \bar{t}]$ :

$$
\begin{gather*}
\|\varepsilon\|_{H_{\mu}^{1}}<\min (b, \lambda) \delta  \tag{4.21}\\
0<b<\delta  \tag{4.22}\\
|\widetilde{\beta}| \leq b^{\frac{3}{2}} \tag{4.23}
\end{gather*}
$$

and

$$
\begin{equation*}
\frac{g_{\infty}}{2} \leq \frac{r(t)}{\lambda(t)^{\alpha}} \leq 2 g_{\infty} \tag{4.24}
\end{equation*}
$$

In particular, the modulation decomposition of Lemma 4.2 applies. Our claim is that the above regime is trapped.

Proposition 4.3 (Improvement of the bootstrap assumptions). Let $t_{e}$ defined in Lemma 4.1. For any $\underline{t}$ such that $t_{e} \leq \underline{t}<\bar{t}$, and such that the bootstrap assumptions (4.21)-(4.24) are satisfied on $(\underline{t}, \bar{t}]$, there holds $\forall t \in(\underline{t}, \bar{t}]$ :

$$
\begin{gather*}
\|\varepsilon\|_{H_{\mu}^{1}} \lesssim \min \left(|t|^{\frac{1}{1+\alpha}}, \lambda\right)|t|^{\frac{1}{1+\alpha}}  \tag{4.25}\\
b=\frac{1}{1+\alpha}\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{2}{1+\alpha}}|t|^{\frac{1-\alpha}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right)  \tag{4.26}\\
|\widetilde{\beta}|  \tag{4.27}\\
\left.\lesssim t\right|^{\frac{2(1-\alpha)}{1+\alpha}}
\end{gather*}
$$

and

$$
\begin{equation*}
\frac{r(t)}{\lambda(t)^{\alpha}}=g_{\infty}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right) \tag{4.28}
\end{equation*}
$$

Proposition 4.3 is the heart of the proof of Theorem 1.2, and relies on a refinement of the energy method designed in [39].

We finish this section by deriving preliminary estimates on the decomposition (4.19) which are mostly a consequence of the construction of $Q_{b, \widetilde{\beta}}$ and the choice of orthogonality conditions (4.17). These estimates prepare the monotonicity formula of section 5 which is the key ingredient of the proof.
4.4. Modulation equations. We derive the modulation equations associated to the modulated parameters $(\lambda(t), r(t), \widetilde{\beta}(t), \gamma(t))$. The parameter $b$ is computed from (4.15) which yields:

$$
\begin{equation*}
b_{s}+(1-\alpha) b^{2}-\frac{b}{\beta} \mathcal{P}_{2}-b \mathcal{P}_{1}=\frac{b}{\beta}\left(\widetilde{\beta}_{s}-\mathcal{P}_{2}\right)+b\left(\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}\right)-\frac{\alpha}{2 \beta} b^{2}\left(\frac{r_{s}}{\lambda}+2 \beta\right) . \tag{4.29}
\end{equation*}
$$

The modulation equations are a consequence of the orthogonality conditions (4.17) and require the derivation of the equation for $\tilde{\varepsilon}$. Recall the equation (3.7) satisfied by $w$

$$
\begin{gathered}
i \partial_{s} w+w_{y y}-w+w|w|^{p-1}+\frac{\alpha b}{2 \beta} \frac{N-1}{1+\frac{\alpha b y}{2 \beta}}\left(w_{y}-i \beta w\right)+b(i \Lambda w+\beta y w) \\
=-\quad-\widetilde{\beta}_{s} y w+\left(\frac{\lambda_{s}}{\lambda}+b\right)(i \Lambda w+\beta y w)+\left(\frac{r_{s}}{\lambda}+2 \beta\right)\left(i w_{y}+\beta w\right)+\left(\tilde{\gamma}_{s}-\beta^{2}\right) w .
\end{gathered}
$$

We inject the decomposition (4.16) which we rewrite using (3.32):

$$
w=\zeta_{b} P_{b, \widetilde{\beta}} e^{-i b \frac{y^{2}}{4}}+\tilde{\varepsilon}
$$

We then define

$$
\operatorname{Mod}(t)=\left|\frac{r_{s}}{\lambda}+2 \beta\right|+\left|\tilde{\gamma}_{s}-\beta^{2}\right|+\left|\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}\right|+\left|\widetilde{\beta}_{s}-\mathcal{P}_{2}\right|
$$

and obtain using (4.29), the formula (3.36) and the fact that $P_{b, \widetilde{\beta}}=Q+O\left(b e^{-c|y|}\right)$ the following system of equations for $\tilde{\varepsilon}_{1}, \tilde{\varepsilon}_{2}$ :

$$
\begin{align*}
\partial_{s} \tilde{\varepsilon}_{1}-L_{-}\left(\tilde{\varepsilon}_{2}\right)= & -\frac{\alpha b}{2 \beta} \frac{(N-1)}{1+\frac{\alpha b}{2 \beta} y}\left(\left(\tilde{\varepsilon}_{2}\right)_{y}-\beta \tilde{\varepsilon}_{1}\right)-\widetilde{\beta}_{s} y \tilde{\varepsilon}_{2}+\left(\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}\right) \Lambda Q \\
& +\frac{\lambda_{s}}{\lambda}\left(\Lambda \tilde{\varepsilon}_{1}+\beta y \tilde{\varepsilon}_{2}\right)+\left(\frac{r_{s}}{\lambda}+2 \beta\right)\left(Q_{y}+\left(\tilde{\varepsilon}_{1}\right)_{y}\right)+\Gamma \tilde{\varepsilon}_{2} \\
& -\Im R(\tilde{\varepsilon})+O\left[\left(b|\tilde{\varepsilon}|+b^{k}+b \operatorname{Mod}\right) e^{-c|y|}\right] \tag{4.30}
\end{align*}
$$

and

$$
\begin{align*}
\partial_{s} \tilde{\varepsilon}_{2}+L_{+}\left(\tilde{\varepsilon}_{1}\right)= & -\frac{\alpha b}{2 \beta} \frac{(N-1)}{1+\frac{\alpha b}{2 \beta} y}\left(-\left(\tilde{\varepsilon}_{1}\right)_{y}-\beta \tilde{\varepsilon}_{2}\right)+\left(\widetilde{\beta}_{s}-\mathcal{P}_{2}\right) y Q+\widetilde{\beta}_{s} y \tilde{\varepsilon}_{1} \\
& -\beta\left(\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}\right) y Q+\frac{\lambda_{s}}{\lambda}\left(\Lambda \tilde{\varepsilon}_{2}-\beta y \tilde{\varepsilon}_{1}\right)+\left(\frac{r_{s}}{\lambda}+2 \beta\right)\left(\tilde{\varepsilon}_{2}\right)_{y} \\
& -\Gamma\left(Q+\tilde{\varepsilon}_{1}\right)+\Re R(\tilde{\varepsilon})+O\left[\left(b|\tilde{\varepsilon}|+b^{k}+b \operatorname{Mod}\right) e^{-c|y|}\right], \tag{4.31}
\end{align*}
$$

where:

$$
\begin{equation*}
\Gamma=\left(\tilde{\gamma}_{s}-\beta^{2}\right)+\beta\left(\frac{r_{s}}{\lambda}+2 \beta\right) \tag{4.32}
\end{equation*}
$$

$L_{+}$and $L_{-}$are the matrix linearized operator close to $Q$ :

$$
\begin{equation*}
L_{+}=-\partial_{y}^{2}+1-p Q^{p-1}, L_{-}=-\partial_{y}^{2}+1-Q^{p-1} \tag{4.33}
\end{equation*}
$$

and the nonlinear term is given by

$$
R(\tilde{\varepsilon})=f\left(Q_{b, \widetilde{\beta}} e^{i \beta y}+\tilde{\varepsilon}\right)-f\left(Q_{b, \widetilde{\beta}} e^{i \beta y}\right)-f^{\prime}\left(Q_{b, \widetilde{\beta}} e^{i \beta y}\right) \cdot \tilde{\varepsilon}
$$

with

$$
\begin{equation*}
f(u)=u|u|^{p-1} \tag{4.34}
\end{equation*}
$$

We are now in position to derive the modulation equations:
Lemma 4.4 (Modulation equations). There holds the bounds:

$$
\begin{gather*}
\operatorname{Mod} \lesssim b\|\tilde{\varepsilon}\|_{H_{\mu}^{1}}+b^{k}  \tag{4.35}\\
\left|b_{s}+(1-\alpha) b^{2}-\frac{b}{\beta} \mathcal{P}_{2}-b \mathcal{P}_{1}\right| \lesssim b^{2}\|\tilde{\varepsilon}\|_{H_{\mu}^{1}}+b^{k+1} . \tag{4.36}
\end{gather*}
$$

Proof of Lemma 4.4. We multiply the equation of $\tilde{\varepsilon}_{1}(4.30)$ by $\zeta_{b} y Q$ and integrate. Using the orthogonality conditions (4.17), the identity $L_{-}(y Q)=-2 Q^{\prime}$ and the non degeneracy

$$
\begin{equation*}
\left(\partial_{y} Q, \zeta_{b} y Q\right)=-\frac{1}{2}\|Q\|_{L^{2}}^{2}+O\left(e^{-\frac{c}{\sqrt{b}}}\right) \tag{4.37}
\end{equation*}
$$

we obtain:

$$
\begin{equation*}
\left|\frac{r_{s}}{\lambda}+2 \beta\right| \lesssim b\|\tilde{\varepsilon}\|_{L_{\mu}^{2}}+\operatorname{Mod}\left(b+\|\tilde{\varepsilon}\|_{L_{\mu}^{2}}\right)+b^{k}+\int|y|^{C}|R(\tilde{\varepsilon})| \zeta_{b} e^{-|y|} . \tag{4.38}
\end{equation*}
$$

Next, we multiply the equation of $\tilde{\varepsilon}_{2}$ (4.31) by $\zeta_{b} \Lambda Q$ and use the orthogonality conditions (4.17), the identity $L_{+}(\Lambda Q)=-2 Q$ and the non degeneracy

$$
\begin{equation*}
\left(\zeta_{b} \Lambda Q, Q\right)=\frac{5-p}{2(p-1)}\left(\int Q^{2}\right)+O\left(e^{-\frac{c}{\sqrt{b}}}\right) \neq 0 \tag{4.39}
\end{equation*}
$$

to compute:

$$
\begin{equation*}
|\Gamma| \lesssim b\|\tilde{\varepsilon}\|_{L_{\mu}^{2}}+\operatorname{Mod}\left(b+\|\tilde{\varepsilon}\|_{L_{\mu}^{2}}\right)+b^{k}+\int|y|^{C}|R(\tilde{\varepsilon})| \zeta_{b} e^{-|y|} . \tag{4.40}
\end{equation*}
$$

Next, we multiply the equation of $\tilde{\varepsilon}_{1}(4.30)$ by $\zeta_{b} Q$ and integrate. Using the orthogonality condition (4.17), the identity $L_{-}(Q)=0$ and the non degeneracy (4.39), we obtain:

$$
\begin{equation*}
\left|\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}\right| \lesssim b\|\tilde{\varepsilon}\|_{L_{\mu}^{2}}+\operatorname{Mod}\left(b+\|\tilde{\varepsilon}\|_{L_{\mu}^{2}}\right)+b^{k}+\int|y|^{C}|R(\tilde{\varepsilon})| \zeta_{b} e^{-|y|} . \tag{4.41}
\end{equation*}
$$

Finally, we multiply the equation of $\tilde{\varepsilon}_{2}(4.31)$ by $\zeta_{b} Q^{\prime}$ and use the orthogonality condition (4.17), the identity $L_{+}\left(Q^{\prime}\right)=0$ and the non degeneracy (4.37), we obtain

$$
\begin{equation*}
\left|\widetilde{\beta}_{s}-\mathcal{P}_{2}\right| \lesssim b\|\tilde{\varepsilon}\|_{L_{\mu}^{2}}+\operatorname{Mod}\left(b+\|\tilde{\varepsilon}\|_{L_{\mu}^{2}}\right)+b^{k}+\int|y|^{C}|R(\tilde{\varepsilon})| \zeta_{b} e^{-|y|} . \tag{4.42}
\end{equation*}
$$

In order to estimate the nonlinear term, we first use the one dimensional Sobolev ${ }^{12}$

$$
\begin{equation*}
\|\varepsilon\|_{L^{\infty}\left(y \geq-\frac{\delta}{b}\right)} \leq\left\|\varepsilon^{\prime}\right\|_{L^{2}\left(y \geq-\frac{\delta}{b}\right)}^{\frac{1}{2}}\|\varepsilon\|_{L^{2}\left(y \geq-\frac{\delta}{b}\right)}^{\frac{1}{2}} \lesssim\|\varepsilon\|_{H_{\mu}^{1}} . \tag{4.43}
\end{equation*}
$$

We then estimate by direct inspection ${ }^{13}$ :

$$
\begin{equation*}
\forall z \in \mathbb{C}, \quad\left|f(1+z)-f(1)-f^{\prime}(1) z\right| \lesssim|z|^{2}+|z|^{p} \mathbf{1}_{p>2} \tag{4.44}
\end{equation*}
$$

and hence by homogeneity:

$$
\begin{equation*}
|R(\tilde{\varepsilon})| \lesssim\left|Q_{b, \widetilde{\beta}}\right|^{p-2}|\varepsilon|^{2}+|\varepsilon|^{p} \mathbf{1}_{p>2} . \tag{4.45}
\end{equation*}
$$

We therefore conclude from the decay (3.15):

$$
\begin{aligned}
\int|y|^{C}|R(\tilde{\varepsilon})| \zeta_{b} e^{-|y|} & \lesssim \int|y|^{C_{k}} \zeta_{b}^{p-1} e^{-(p-1)|y|}|\varepsilon|^{2}+\mathbf{1}_{p>2} \int|\varepsilon|^{p} \zeta_{b} \\
& \lesssim\|\varepsilon\|_{L_{\mu}^{2}}^{2}+\mathbf{1}_{p>2}\|\varepsilon\|_{L^{\infty}}^{p-2}\|\varepsilon\|_{L_{\mu}^{2}}^{2} \lesssim\|\varepsilon\|_{L_{\mu}^{2}}^{2}
\end{aligned}
$$

where we used the Sobolev bound (4.43) and the bootstrap bound (4.21) in the last step. Injecting this estimate into (4.38), (4.40), (4.41) and (4.42) yields (4.35). (4.36) now follows from (4.35) and (4.29).

## 5. Monotonicity formula

We now turn to the core of our analysis which is the derivation of a monotonicity formula for the norm of $\varepsilon$ which relies on a mixed Energy/Morawetz functional in the continuation of [37], [39]. As in [39], the required repulsivity properties for the linearized operator are thanks to the minimal mass assumption energy bounds only which are well known for the mass subcritical ground state. The addiitional Morawetz term is designed to produce the expected non trivial Galilean drift on the soliton core after renormalization.

[^10]5.1. Algebraic identity. We recall the decomposition (4.19) which in view of (3.35) yields the equation for $\tilde{u}$ :
\[

$$
\begin{equation*}
i \partial_{t} \tilde{u}+\Delta \tilde{u}+|u|^{p-1} u-\tilde{Q}|\tilde{Q}|^{p-1}=-\psi=-\frac{1}{\lambda(t)^{\frac{2 p}{p-1}}} \Psi\left(t, \frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)} \tag{5.1}
\end{equation*}
$$

\]

with $\Psi$ given by (3.36). We let

$$
\phi:[-1,+\infty) \rightarrow \mathbb{R}
$$

be a time independent smooth compactly supported cut off function which satisfies:

$$
\begin{equation*}
\phi(z) \equiv 0 \text { for }-1 \leq z \leq-\frac{1}{2} \text { and for } z \geq \frac{1}{2} \tag{5.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\phi(0)=1, \sup _{z \geq-1}|\phi(z)|<\frac{\sqrt{1+\beta_{\infty}^{2}}}{\beta_{\infty}} \tag{5.3}
\end{equation*}
$$

Let

$$
F(u)=\frac{1}{p+1}|u|^{p+1}, \quad f(u)=u|u|^{p-1} \text { so that } F^{\prime}(u) \cdot h=\operatorname{Re}(f(u) \bar{h})
$$

We first claim a purely algebraic identity for the linearized flow (5.1) which is a mixed Energy/Morawetz functional:

Lemma 5.1 (Algebraic energy/Morawetz estimate). Let

$$
\begin{align*}
\mathcal{I}(\tilde{u})= & \frac{1}{2} \int|\nabla \tilde{u}|^{2}+\frac{1+\beta^{2}}{2} \int \frac{|\tilde{u}|^{2}}{\lambda^{2}}-\int\left[F(\tilde{Q}+\tilde{u})-F(\tilde{Q})-F^{\prime}(\tilde{Q}) \cdot \tilde{u}\right] \\
+ & \frac{\beta}{\lambda} \Im\left(\int \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u} \overline{\tilde{u}}\right)  \tag{5.4}\\
\mathcal{J}(\tilde{u})= & -\frac{1+\beta^{2}}{\lambda^{2}} \Im(f(u)-f(\tilde{Q}), \tilde{\tilde{u}})  \tag{5.5}\\
& -\frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right) \\
& -\Re\left(\partial_{t} \tilde{Q}, \overline{\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right)}\right)
\end{align*}
$$

then there holds:

$$
\begin{equation*}
\frac{d}{d t} \mathcal{I}(\tilde{u})=\mathcal{J}(\tilde{u})+O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}+\frac{b^{k}}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}\right) \tag{5.6}
\end{equation*}
$$

Proof of Lemma 5.1. step 1 Algebraic derivation of the energetic part. We compute from (5.1):

$$
\begin{align*}
& \frac{d}{d t}\left\{\frac{1}{2} \int|\nabla \tilde{u}|^{2}+\frac{1+\beta^{2}}{2} \int \frac{|\tilde{u}|^{2}}{\lambda^{2}}-\int\left[\left(F(u)-F(\tilde{Q})-F^{\prime}(\tilde{Q}) \cdot \tilde{u}\right)\right]\right\}  \tag{5.7}\\
= & -\Re\left(\partial_{t} \tilde{u}, \Delta \tilde{u}-\frac{1+\beta^{2}}{\lambda^{2}} \tilde{u}+(f(u)-f(\tilde{Q}))\right)-\frac{\left(1+\beta^{2}\right) \lambda_{t}}{\lambda^{3}} \int|\tilde{u}|^{2} \\
+ & \frac{\beta \beta_{t}}{\lambda^{2}} \int|\tilde{u}|^{2}-\Re\left(\partial_{t} \tilde{Q},\left(\overline{\left.f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right)}\right)\right. \\
= & \Im\left(\psi, \Delta \tilde{u}-\frac{1+\beta^{2}}{\lambda^{2}} \tilde{u}+(f(u)-f(\tilde{Q}))\right)-\frac{1+\beta^{2}}{\lambda^{2}} \Im(f(u)-f(\tilde{Q}), \overline{\tilde{u}}) \\
- & \frac{\left(1+\beta^{2}\right) \lambda_{t}}{\lambda^{3}} \int|\tilde{u}|^{2}+\frac{\beta \beta_{t}}{\lambda^{2}} \int|\tilde{u}|^{2}-\Re\left(\partial_{t} \tilde{Q}, \overline{\left.\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right)\right)} .\right.
\end{align*}
$$

We first estimate from (4.35):

$$
\begin{align*}
-\frac{\lambda_{t}}{\lambda^{3}} \int|\tilde{u}|^{2} & =\frac{b}{\lambda^{4}} \int|\tilde{u}|^{2}-\frac{\mathcal{P}_{1}}{\lambda^{4}} \int|\tilde{u}|^{2}-\frac{1}{\lambda^{4}}\left(\frac{\lambda_{s}}{\lambda}+b\right)\|\tilde{u}\|_{L^{2}}^{2}  \tag{5.8}\\
& =\frac{1}{\lambda^{4}} O\left(b\|\varepsilon\|_{L_{\mu}^{2}}^{2}\right)
\end{align*}
$$

where we used the bootstrap assumptions (4.21) (4.23) (4.24) in the last equality. Also, using again (4.35), we have

$$
\begin{align*}
\frac{\beta \beta_{t}}{\lambda^{2}} \int|\tilde{u}|^{2} & =\frac{\beta \mathcal{P}_{2}}{\lambda^{4}} \int|\tilde{u}|^{2}+\frac{\beta\left(\beta_{s}-\mathcal{P}_{2}\right)}{\lambda^{4}} \int|\tilde{u}|^{2}  \tag{5.9}\\
& =\frac{1}{\lambda^{4}} O\left(b\|\varepsilon\|_{L_{\mu}^{2}}^{2}\right)
\end{align*}
$$

where we used the bootstrap assumptions (4.21) (4.22) (4.23) (4.24) in the last equality.

It remains to estimate the first term in the RHS (5.7). We have

$$
\begin{aligned}
& \left|\Im\left(\psi, \overline{\Delta \tilde{u}-\frac{1+\beta^{2}}{\lambda^{2}} \tilde{u}+(f(u)-f(\tilde{Q}))}\right)-\frac{1+\beta^{2}}{\lambda^{2}} \Im(f(u)-f(\tilde{Q}), \overline{\tilde{u}})\right| \\
\lesssim & \left|\Im\left(\int\left[\Delta \psi-\left(1+\beta^{2}\right) \frac{\psi}{\lambda^{2}}+\frac{p+1}{2}|\tilde{Q}|^{p-1} \psi-\frac{p-1}{2}|\tilde{Q}|^{p-3} \tilde{Q}^{2} \bar{\psi}\right] \overline{\tilde{u}}\right)\right| \\
+ & \left|\Im\left(\psi, \overline{\left(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right)}\right)\right| .
\end{aligned}
$$

We extract from (3.36), (4.35) and (4.36) the bound:

$$
\begin{align*}
|\Psi| & \lesssim \zeta_{b}\left(b^{k}+\operatorname{Mod}\right)\left(1+|y|^{c_{k}}\right) e^{-|y|}+\frac{e^{-|y|}}{b^{c_{k}}} \mathbf{1}_{y \sim \frac{1}{\sqrt{b}}} \\
& \lesssim \zeta_{b}\left(b^{k}+b\|\varepsilon\|_{H_{\mu}^{1}}\right)\left(1+|y|^{c_{k}}\right) e^{-|y|}+\frac{e^{-|y|}}{b^{c_{k}}} \mathbf{1}_{y \sim \frac{1}{\sqrt{b}}} \tag{5.10}
\end{align*}
$$

Then, we estimate in brute force:

$$
\begin{aligned}
& \left|\Im\left(\int\left[\Delta \psi-\left(1+\beta^{2}\right) \frac{\psi}{\lambda^{2}}+\frac{p+1}{2}|\tilde{Q}|^{p-1} \psi-\frac{p-1}{2}|\tilde{Q}|^{p-3} \tilde{Q}^{2} \bar{\psi}\right] \overline{\tilde{u}}\right)\right| \\
& \lesssim \frac{\left(b^{k}+b\|\varepsilon\|_{H_{\mu}^{1}}\right)\|\varepsilon\|_{H_{\mu}^{1}}}{\lambda^{4}} .
\end{aligned}
$$

Also, we estimate using the homogeneity estimate (4.45):

$$
\begin{aligned}
& \left|\Im\left(\psi, \overline{\left(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right)}\right)\right| \\
\lesssim & \frac{1}{\lambda^{4}} \int\left[\zeta_{b}\left(b^{k}+b\|\varepsilon\|_{H_{\mu}^{1}}\right)\left(1+|y|^{c_{k}}\right) e^{-|y|}+\frac{e^{-|y|}}{b^{c_{k}}} \mathbf{1}_{y \sim \frac{1}{\sqrt{b}}}\right]\left[\left|Q_{b, \widetilde{\beta}}\right|^{p-2}|\varepsilon|^{2}+|\varepsilon|^{p} \mathbf{1}_{p>2}\right] \\
\lesssim & \frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}
\end{aligned}
$$

where we used the Sobolev bound (4.43) in the last step. We have therefore obtained the preliminary computation:

$$
\begin{align*}
& \frac{d}{d t}\left\{\frac{1}{2} \int|\nabla \tilde{u}|^{2}+\frac{1+\beta^{2}}{2} \int \frac{|\tilde{u}|^{2}}{\lambda^{2}}-\int\left[\left(F(u)-F(\tilde{Q})-F^{\prime}(\tilde{Q}) \cdot \tilde{u}\right)\right]\right\} \\
= & -\frac{1+\beta^{2}}{\lambda^{2}} \Im(f(u)-f(\tilde{Q}), \overline{\tilde{u}})-\Re\left(\partial_{t} \tilde{Q}, \overline{\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right)}\right) \\
+ & \frac{1}{\lambda^{4}} O\left(b^{k}\|\varepsilon\|_{H_{\mu}^{1}}+b\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right) . \tag{5.11}
\end{align*}
$$

step 2 Algebraic derivation of the localized virial part. We now estimate the contribution of the localized Morawetz term. We first compute using (4.15):

$$
\begin{aligned}
\frac{d}{d t}\left[\frac{r}{r(t)}\right] & =-\frac{r_{t}(t) r}{r^{2}(t)}=-\frac{r_{s}}{\lambda} \frac{r}{\lambda r^{2}(t)} \\
& =\frac{2 \beta r}{\lambda(t) r(t)^{2}}-\left(\frac{r_{s}}{\lambda}+2 \beta\right) \frac{r}{\lambda(t) r^{2}(t)} \\
& =\frac{\alpha b(t)}{\lambda^{2}(t)} \frac{r}{r(t)}-\frac{\alpha b}{2 \beta \lambda(t)^{2}} \frac{r}{r(t)}\left(\frac{r_{s}}{\lambda}+2 \beta\right) .
\end{aligned}
$$

This yields:

$$
\begin{align*}
& \frac{d}{d t}\left\{\frac{\beta}{\lambda} \Im\left(\int \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u} \overline{\tilde{u}}\right)\right\}  \tag{5.12}\\
= & \frac{\alpha \beta b}{\lambda^{3}} \Im\left(\int \frac{r}{r(t)} \phi^{\prime}\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u} \bar{u}\right) \\
- & \frac{\alpha b}{2 \lambda^{3}}\left(\frac{r_{s}}{\lambda}+2 \beta\right) \Im\left(\int \frac{r}{r(t)} \phi^{\prime}\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u} \overline{\tilde{u}}\right)+\frac{\mathcal{P}_{2}}{\lambda^{3}} \Im\left(\int \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u} \overline{\tilde{u}}\right) \\
+ & \frac{\beta_{s}-\mathcal{P}_{2}}{\lambda^{3}} \Im\left(\int \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u} \overline{\tilde{u}}\right)+\frac{\beta\left(b-\mathcal{P}_{1}\right)}{\lambda^{3}} \Im\left(\int \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u} \overline{\tilde{u}}\right) \\
- & \frac{b}{\lambda^{3}}\left(\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}\right) \Im\left(\int \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u} \bar{u}\right) \\
+ & \frac{\beta}{\lambda} \Re\left(\int i \partial _ { t } \tilde { u } \left[\overline{\left.\left.\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right) \tilde{u}+2 \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u}\right]\right)}\right.\right. \\
= & \frac{\beta}{\lambda} \Re\left(\int i \partial_{t} \tilde{u}\left[\overline{\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right) \tilde{u}+2 \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u}}\right]\right) \\
+ & O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right),
\end{align*}
$$

where we used in the last inequality (4.35), the bootstrap assumptions (4.21) (4.22) (4.23) (4.24), and the fact that

$$
\begin{equation*}
\frac{1}{r} \sim \frac{1}{r(t)} \text { on the support of } \phi\left(\frac{\cdot}{r(t)}-1\right) \tag{5.13}
\end{equation*}
$$

The first term in the right-hand side of (5.12) corresponds to the localized Morawetz multiplier, and we get from (5.1) and the classical Pohozaev integration by parts formula:

$$
\begin{aligned}
& \frac{\beta}{\lambda} \Re\left(\int i \partial_{t} \tilde{u}\left[\overline{\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right) \tilde{u}+2 \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u}}\right]\right) \\
= & \frac{\alpha b}{\lambda^{2}}\left(\int \phi^{\prime}\left(\frac{r}{r(t)}-1\right)\left|\partial_{r} \tilde{u}\right|^{2}\right) \\
- & \frac{\alpha^{2} b^{2}}{8 \beta \lambda^{3}}\left(\int \Delta\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right)|\tilde{u}|^{2}\right) \\
- & \frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right) \\
- & \frac{\beta}{\lambda} \Re\left(\int\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\tilde{u}}\right) \\
- & \frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right) \psi \overline{\partial_{r} \tilde{u}}\right)-\frac{\beta}{\lambda} \Re\left(\int\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right) \psi \overline{\tilde{u}}\right) .
\end{aligned}
$$

which together with (5.13) yields

$$
\begin{align*}
& \frac{\beta}{\lambda} \Re\left(\int i \partial_{t} \tilde{u}\left[\overline{\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right) \tilde{u}+2 \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u}}\right]\right) \\
= & -\frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right)  \tag{5.14}\\
- & \frac{\beta}{\lambda} \Re\left(\int\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right)\left(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\tilde{u}}\right) \\
- & \frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right) \psi \overline{\partial_{r} \tilde{u}}\right)-\frac{\beta}{\lambda} \Re\left(\int\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right) \psi \overline{\tilde{u}}\right) \\
+ & O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right) .
\end{align*}
$$

We estimate by direct inspection:

$$
\left|f(1+z)-f(1)-f^{\prime}(1) \cdot z\right| \lesssim|z|^{p}+|z|^{2} \mathbf{1}_{p>2}
$$

and hence the bound by homogeneity:

$$
\begin{equation*}
\left|f(\tilde{Q}+\tilde{u})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right| \lesssim|\tilde{u}|^{p}+|\tilde{Q}|^{p-2}|\tilde{u}|^{2} \mathbf{1}_{p>2} \tag{5.15}
\end{equation*}
$$

We thus obtain the bound:

$$
\begin{align*}
& \left|-\frac{\beta}{\lambda} \Re\left(\int\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right)\left(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\tilde{u}}\right)\right| \\
\lesssim & \frac{b}{\lambda^{2}}\left[\int|\tilde{u}|^{p+1}+|\tilde{u}|^{3}|\tilde{Q}|^{p-2} \mathbf{1}_{p>2}\right] \tag{5.16}
\end{align*}
$$

where we used (5.13). We claim the nonlinear bounds:

$$
\begin{align*}
\int|\tilde{u}|^{3}|\tilde{Q}|^{p-2} & \lesssim \frac{\delta\|\varepsilon\|_{L_{\mu}^{2}}^{2}}{\lambda^{2}} \text { for } p>2,  \tag{5.17}\\
\int|\tilde{u}|^{p+1} & \lesssim \frac{\delta^{p-1}\|\varepsilon\|_{H_{\mu}^{1}}^{2}}{\lambda^{2}} \tag{5.18}
\end{align*}
$$

which are proved below. The terms involving $\psi$ in (5.14) are estimateed in brute force using (5.10)

$$
\begin{align*}
& \left|\frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right) \psi \overline{\partial_{r} \tilde{u}}\right)\right|  \tag{5.19}\\
+ & \left|\frac{\beta}{\lambda} \Re\left(\int\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right) \psi \overline{\tilde{u}}\right)\right| \\
\lesssim & \frac{\left(b^{k}+b\|\varepsilon\|_{H_{\mu}^{1}}\right)\|\varepsilon\|_{H_{\mu}^{1}}}{\lambda^{4}} .
\end{align*}
$$

Injecting (5.16), (5.17), (5.18) and (5.19) into (5.14) yields:

$$
\begin{aligned}
& \frac{\beta}{\lambda} \Re\left(\int i \partial_{t} \tilde{u}\left[\overline{\left(\frac{1}{r(t)} \phi^{\prime}+\frac{N-1}{r} \phi\right)\left(\frac{r}{r(t)}-1\right) \tilde{u}+2 \phi\left(\frac{r}{r(t)}-1\right) \partial_{r} \tilde{u}}\right]\right) \\
= & -\frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right)+O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}+\frac{b^{k}}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}\right) .
\end{aligned}
$$

We now inject this into (5.12) which together with (5.11) concludes the proof of (5.6).

Proof of (5.17): Note first that $\tilde{Q}$ is localized in the region $r \geq r(t) / 2$ due to the cut-off $\zeta_{b}$ in its definition. Now, the region $r \geq r(t) / 2$ corresponds to $y \geq-\frac{r(t)}{2 \lambda(t)}$ and thus:

$$
\mu \gtrsim 1 \text { for } r \geq r(t) / 2
$$

For $p>2$, we estimate from the Sobolev bound (4.43) and the bootstrap assumption (4.21):

$$
\begin{aligned}
\int|\tilde{u}|^{3}|\tilde{Q}|^{p-2} & =\frac{1}{\lambda^{2}} \int_{y \geq-\frac{r(t)}{2 \lambda(t)}}|\varepsilon|^{3}\left|Q_{b, \widetilde{\beta}}\right|^{p-2} \mu \lesssim \frac{1}{\lambda^{2}} \int_{y \geq-\frac{r(t)}{2 \lambda(t)}}|\varepsilon|^{3} \mu \\
& \leq \frac{1}{\lambda^{2}}\|\varepsilon\|_{L^{\infty}\left(y \geq-\frac{r(t)}{2 \lambda(t)}\right.}\|\varepsilon\|_{L_{\mu}^{2}}^{2} \lesssim \frac{1}{\lambda^{2}}\|\varepsilon\|_{H_{\mu}^{1}}\|\varepsilon\|_{L_{\mu}^{2}}^{2} \\
& \lesssim \frac{\delta\|\varepsilon\|_{H_{\mu}^{1}}^{2}}{\lambda^{2}}
\end{aligned}
$$

and (5.17) is proved.
Proof of (5.18). Observe that the bootstrap bound (4.21) implies:

$$
\begin{equation*}
\|\tilde{u}\|_{H^{1}} \lesssim \frac{\|\varepsilon\|_{H_{\mu}^{1}}}{\lambda} \lesssim \delta \tag{5.20}
\end{equation*}
$$

In view of the Sobolev embeddings, this yields:

$$
\int|\tilde{u}|^{p+1} \lesssim\|\tilde{u}\|_{H^{1}}^{p+1} \lesssim \frac{\|\varepsilon\|_{H_{\mu}^{1}}^{2}\|\tilde{u}\|_{H^{1}}^{p-1} \lesssim \frac{\delta^{p-1}\|\varepsilon\|_{H_{\mu}^{1}}^{2}}{\lambda^{2}}, ~ \lambda^{2}}{\text { a }}
$$

and (5.18) is proved. This concludes the poof of Lemma 5.1.
5.2. Coercivity of $\mathcal{I}$. We now examine the various terms in Lemma 5.1 which correspond to quadratic interactions. Let us start with the boundary term in time $\mathcal{I}$ :

Lemma 5.2 (Coercivity of $\mathcal{I})$. Let $\mathcal{I}(\tilde{u})$ given by (5.4). Then:

$$
\begin{equation*}
\mathcal{I}(\tilde{u}) \geq c_{0}\left(\|\nabla \tilde{u}\|_{L^{2}}^{2}+\frac{1}{\lambda^{2}}\|\tilde{u}\|_{L^{2}}^{2}\right) \tag{5.21}
\end{equation*}
$$

for some universal constant $c_{0}>0$.
Proof of Lemma 5.2. We first renormalize:

$$
\begin{aligned}
\mathcal{I}(\tilde{u}) & =\frac{1}{2 \lambda^{2}}\left\{\int\left|\partial_{y} \varepsilon\right|^{2} \mu+2 \beta \Im\left(\int \phi(z) \partial_{y} \varepsilon \bar{\varepsilon}\right) \mu+\left(1+\beta^{2}\right) \int|\varepsilon|^{2} \mu\right. \\
& \left.-2 \int\left(F\left(Q_{b, \widetilde{\beta}}+\varepsilon\right)-F\left(Q_{b, \widetilde{\beta}}\right)-F^{\prime}\left(Q_{b, \widetilde{\beta}}\right) \cdot \varepsilon\right) \mu\right\} .
\end{aligned}
$$

where

$$
\begin{equation*}
z=\frac{r}{r(t)}-1=\frac{\alpha b}{2 \beta} y, \quad \mu=(1+z)^{N-1} . \tag{5.22}
\end{equation*}
$$

We compute:

$$
F^{\prime \prime}\left(Q_{b, \widetilde{\beta}}\right) \cdot \varepsilon \cdot \varepsilon=\frac{p-1}{4} \overline{Q_{b, \widetilde{\beta}}^{2}} \varepsilon^{2}+\frac{p+1}{2}\left|Q_{b, \widetilde{\beta}}\right|^{p-1}|\varepsilon|^{2}+\frac{p-1}{4}\left|Q_{b, \widetilde{\beta}}\right|^{p-3} Q_{b, \widetilde{\beta}^{2}}^{2}
$$

and estimate by homogeneity:

$$
\begin{align*}
& \left|F(Q+\varepsilon)-F\left(Q_{b, \widetilde{\beta}}\right)-F^{\prime}\left(Q_{b, \widetilde{\beta}}\right) \cdot \varepsilon-\frac{1}{2} F^{\prime \prime}\left(Q_{b, \widetilde{\beta}}\right) \cdot \varepsilon \cdot \varepsilon\right|  \tag{5.23}\\
\lesssim & |\varepsilon|^{p+1}+|\varepsilon|^{3}\left|Q_{b, \widetilde{\beta}}\right|^{p-2} \mathbf{1}_{p>2} .
\end{align*}
$$

We conclude using the bounds (5.17), (5.18):

$$
\begin{aligned}
& 2 \int\left[F\left(Q_{b, \widetilde{\beta}}+\varepsilon\right)-F\left(Q_{b, \widetilde{\beta}}\right)-F^{\prime}\left(Q_{b, \widetilde{\beta}}\right) \cdot \varepsilon\right] \mu \\
= & \int\left[\frac{p-1}{4} \overline{Q_{b, \widetilde{\beta}}^{2}} \varepsilon^{2}+\frac{p+1}{2}\left|Q_{b, \widetilde{\beta}}\right|^{p-1}|\varepsilon|^{2}+\frac{p-1}{4}\left|Q_{b, \widetilde{\beta}}\right|^{p-3} Q_{b, \widetilde{\beta}}^{2} \bar{\varepsilon}^{2}\right] \mu \\
& +O\left(\lambda^{2} \int|\tilde{u}|^{3}|\tilde{Q}|^{p-2} \mathbf{1}_{p>2}+|\tilde{u}|^{p+1}\right) \\
= & p \int \tilde{\varepsilon}_{1}^{2} \zeta_{b} Q^{p-1}+\int \tilde{\varepsilon}_{2}^{2} \zeta_{b} Q^{p-1}+O\left(b\|\varepsilon\|_{L_{\mu}^{2}}^{2}+\|\varepsilon\|_{H_{\mu}^{1}}^{3}\right) \\
= & p \int \tilde{\varepsilon}_{1}^{2} \zeta_{b} Q^{p-1}+\int \tilde{\varepsilon}_{2}^{2} \zeta_{b} Q^{p-1}+O\left(\delta^{C}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right)
\end{aligned}
$$

where we used the estimates (5.17) and (5.18), the bootstrap assumption (4.22), the fact that

$$
Q_{b, \widetilde{\beta}}=\zeta_{b} Q e^{-i \beta y}+O\left(b e^{-c|y|}\right) \quad \text { and } \quad \zeta_{b} Q^{p-1} \mu=\zeta_{b} Q^{p-1}+O\left(b \zeta_{b} e^{-c|y|}\right)
$$

and where we recall from (4.20) that:

$$
\tilde{\varepsilon}=\varepsilon e^{i \beta y}
$$

Together with $\beta=\beta_{\infty}+\widetilde{\beta}$ and the bootstrap assumptions (4.22) (4.23), this yields the preliminary estimate:

$$
\begin{aligned}
\mathcal{I}(\tilde{u}) & =\frac{1}{2 \lambda^{2}}\left\{\int\left|\partial_{y} \varepsilon\right|^{2} \mu+2 \beta_{\infty} \Im\left(\int \phi(z) \partial_{y} \varepsilon \bar{\varepsilon} \mu\right)+\int\left(1+\beta_{\infty}^{2}\right)|\varepsilon|^{2} \mu(5.24)\right. \\
& \left.-p \int \tilde{\varepsilon}_{1}^{2} Q^{p-1}-\int \tilde{\varepsilon}_{2}^{2} Q^{p-1}+O\left(\delta^{C}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right)\right\} .
\end{aligned}
$$

Let us now split the potential part in the zones $|y| \leq \frac{1}{\sqrt{b}},|y| \geq \frac{1}{\sqrt{b}}$. Away from the soliton, the reduced discriminant of the quadratic form

$$
\left|\partial_{y} \varepsilon\right|^{2}+2 \beta_{\infty} \Im\left(\phi(z) \partial_{y} \varepsilon \bar{\varepsilon}\right)+\left(1+\beta_{\infty}^{2}\right)|\varepsilon|^{2}
$$

is given by

$$
\Delta=\beta_{\infty}^{2} \phi^{2}(z)-\left(1+\beta_{\infty}^{2}\right)^{2}<0
$$

from (5.3) and thus:

$$
\int_{|y| \geq \frac{1}{\sqrt{b}}}\left[\left|\partial_{y} \varepsilon\right|^{2}+2 \beta_{\infty} \Im\left(\phi(z) \partial_{y} \varepsilon \bar{\varepsilon}\right)+\left(1+\beta_{\infty}^{2}\right)|\varepsilon|^{2}\right] \gtrsim \int_{|y| \geq \frac{1}{\sqrt{b}}}\left[\left|\partial_{y} \varepsilon\right|^{2}+|\varepsilon|^{2}\right] .
$$

On the singularity $|y| \lesssim \frac{1}{\sqrt{b}}$, we have from (5.3):

$$
|\phi(z)-1| \lesssim|z| \lesssim \sqrt{b}
$$

and thus:

$$
\begin{aligned}
& \int_{|y| \leq \frac{1}{\sqrt{b}}}\left[\left|\partial_{y} \varepsilon\right|^{2}+2 \beta_{\infty} \Im\left(\phi(z) \partial_{y} \varepsilon \bar{\varepsilon}\right)+\left(1+\beta_{\infty}^{2}\right)|\varepsilon|^{2}\right] \mu \\
= & \int_{|y| \leq \frac{1}{\sqrt{b}}}\left[\left|\partial_{y} \tilde{\varepsilon}\right|^{2}+|\tilde{\varepsilon}|^{2}\right]+O\left(\sqrt{b}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right)
\end{aligned}
$$

Collecting the above bounds yields:

$$
\begin{align*}
2 \mathcal{I}(\tilde{u}) & =\int_{|y| \leq \frac{1}{\sqrt{b}}}\left[\left|\partial_{y} \tilde{\varepsilon}\right|^{2}+|\tilde{\varepsilon}|^{2}\right]-p \int \tilde{\varepsilon}_{1}^{2} \zeta_{b} Q^{p-1}+\int \tilde{\varepsilon}_{2}^{2} \zeta_{b} Q^{p-1} \\
& +\int_{|y| \geq \frac{1}{\sqrt{b}}}\left[\left|\partial_{y} \varepsilon\right|^{2}+|\varepsilon|^{2}\right] \mu+O\left(\delta^{C}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right) \tag{5.25}
\end{align*}
$$

We now recall the following coercivity property of the linearized energy in the one dimensional subcritical case which is a well known consequence of the variational characterization of $Q$, see for example [6]:

Lemma 5.3 (Coercivity of the linearized energy). There holds for some universal constant $c_{0}>0: \forall \varepsilon \in H^{1}(\mathbb{R})$,

$$
\begin{align*}
\left(L_{+}\left(\varepsilon_{1}\right), \varepsilon_{1}\right)+\left(L_{-}\left(\varepsilon_{2}\right), \varepsilon_{2}\right) & \geq c_{0}\|\varepsilon\|_{H^{1}}^{2}  \tag{5.26}\\
& -\frac{1}{c_{0}}\left\{\left(\varepsilon_{1}, Q\right)^{2}+\left(\varepsilon_{1}, y Q\right)^{2}+\left(\varepsilon_{2}, \Lambda Q\right)^{2}\right\}
\end{align*}
$$

We now inject the choice of orthogonality conditions (4.17) into (5.26) and obtain using a standard localization argument ${ }^{14}$ :

$$
\begin{aligned}
& \int_{|y| \leq \frac{1}{\sqrt{b}}}\left[\left|\partial_{y} \tilde{\varepsilon}\right|^{2}+|\tilde{\varepsilon}|^{2}\right]-p \int \tilde{\varepsilon}_{1}^{2} \zeta_{b} Q^{p-1}+\int \tilde{\varepsilon}_{2}^{2} \zeta_{b} Q^{p-1} \\
\gtrsim & \int_{|y| \leq \frac{1}{\sqrt{b}}}\left[\left|\partial_{y} \tilde{\varepsilon}\right|^{2}+|\tilde{\varepsilon}|^{2}\right]+O\left(\delta^{C}\|\tilde{\varepsilon}\|_{H_{\mu}^{1}}^{2}\right) \\
\gtrsim & \int_{|y| \leq \frac{1}{\sqrt{b}}}\left[\left|\partial_{y} \varepsilon\right|^{2}+|\varepsilon|^{2}\right] \mu+O\left(\delta^{C}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right)
\end{aligned}
$$

which together with (5.25) concludes the proof of (5.21).
Remark 5.4. One can easily extract from the above proof the upper bound:

$$
\begin{equation*}
\mathcal{I} \lesssim\|\nabla \tilde{u}\|_{L^{2}}^{2}+\frac{1}{\lambda^{2}}\|\tilde{u}\|_{L^{2}}^{2} \tag{5.27}
\end{equation*}
$$

5.3. Estimate for $\mathcal{J}(\tilde{u})$. We now treat the $\mathcal{J}(\tilde{u})$ term given by (5.5). We first extract the leading order quadratic terms in $\mathcal{J}(\tilde{u})$ and claim that is a $b$ degenerate quadratic term. A suitable choice of the cut off function $\phi$ would allow us sign this term again as in [39], but we shall not need this additional structural fact here.
Lemma 5.5 (Leading order terms in $\mathcal{J}(\tilde{u}))$. We have the rough bound:

$$
\begin{equation*}
|\mathcal{J}(\tilde{u})| \lesssim \frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2} \tag{5.28}
\end{equation*}
$$

Proof of Lemma 5.5. step 1 The $\partial_{t} \tilde{Q}$ term. We compute $\partial_{t} \tilde{Q}$ from (3.34):

$$
\begin{aligned}
\tilde{Q}_{t}= & i \gamma_{t} \tilde{Q}-\frac{2}{p-1} \frac{\lambda_{t}}{\lambda} \tilde{Q}-\frac{r-r(t)}{\lambda} \frac{\lambda_{t}}{\lambda} \frac{1}{\lambda^{\frac{2}{p-1}} Q_{b, \widetilde{\beta}}^{\prime}\left(\frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma}} \begin{aligned}
& -\frac{r_{t}(t)}{\lambda} \frac{1}{\lambda^{\frac{2}{p-1}}} Q_{b, \widetilde{\beta}}^{\prime}\left(\frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma}+b_{t} \frac{1}{\lambda^{\frac{2}{p-1}}} \partial_{b} Q_{b(t), \widetilde{\beta}(t)}\left(\frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)} \\
& +\widetilde{\beta}_{t} \frac{1}{\lambda^{\frac{2}{p-1}}} \partial_{\widetilde{\beta}} Q_{b(t), \widetilde{\beta}(t)}\left(\frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)} \\
= & \left(\frac{i\left(1+\beta^{2}\right)}{\lambda^{2}}+\frac{2}{p-1} \frac{b}{\lambda^{2}}\right) \tilde{Q}+\frac{b}{\lambda} \frac{r-r(t)}{\lambda} \partial_{r} \tilde{Q}+\frac{2 \beta}{\lambda} \partial_{r} \tilde{Q} \\
+ & \frac{1}{\lambda^{2+\frac{2}{p-1}}} O\left(\left[b^{2}+\operatorname{Mod}+\left|b_{s}+(1-\alpha) b^{2}-\frac{b}{\beta} \mathcal{P}_{2}-b \mathcal{P}_{1}\right|\right] \zeta_{b}|y|^{c} e^{-|y|}\right) \\
= & \frac{i\left(1+\beta^{2}\right)}{\lambda^{2}} \tilde{Q}+\frac{2 \beta}{\lambda} \partial_{r} \tilde{Q}+\frac{1}{\lambda^{2+\frac{2}{p-1}}} O\left(b \zeta_{b}|y|^{c} e^{-|y|}\right)
\end{aligned}, l
\end{aligned}
$$

where we used (4.35) and the decay estimate (3.15) in the last step. This yields:

$$
\begin{aligned}
& -\Re\left(\partial_{t} \tilde{Q}, \overline{\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right)}\right) \\
& =-\frac{1+\beta^{2}}{\lambda^{2}} \Im\left(\int\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\tilde{Q}}\right) \\
& -\frac{2 \beta}{\lambda} \Re\left(\int\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\partial_{r} \tilde{Q}}\right) \\
& +\frac{1}{\lambda^{4}} O\left(\int b \zeta_{b}|y|^{c} e^{-|y|}\left|f\left(Q_{b, \widetilde{\beta}}+\varepsilon\right)-f\left(Q_{b, \widetilde{\beta}}\right)-f^{\prime}\left(Q_{b, \widetilde{\beta}}\right) \cdot \varepsilon\right| \mu\right)
\end{aligned}
$$

[^11]where we used the estimates (5.17) and the bootstrap assumptions (4.21) and (4.22). We estimate the nonlinear terms using (4.45), (4.21), (4.43):
\[

$$
\begin{aligned}
& \int b \zeta_{b}|y|^{c} e^{-|y|}\left|f\left(Q_{b, \widetilde{\beta}}+\varepsilon\right)-f\left(Q_{b, \widetilde{\beta}}\right)-f^{\prime}\left(Q_{b, \widetilde{\beta}}\right) \cdot \varepsilon\right| \mu \\
\lesssim & b \int \zeta_{b}|y|^{c} e^{-|y|}\left[\left|Q_{b, \widetilde{\beta}}\right|^{p-2}|\varepsilon|^{2}+|\varepsilon|^{p} \mathbf{1}_{p>2}\right] \mu \\
\lesssim & b\left[1+\|\varepsilon\|_{L^{\infty}\left(y \geq-\frac{\delta}{b}\right)}^{p-2} \mathbf{1}_{p>2}\right] \int|\varepsilon|^{2} \mu \\
\lesssim & b\|\varepsilon\|_{H_{\mu}^{1}}^{2}
\end{aligned}
$$
\]

Injecting the collection of above bounds into (5.5) yields the preliminary computation:

$$
\begin{align*}
\mathcal{J}(\tilde{u}) & =-\frac{1+\beta^{2}}{\lambda^{2}} \Im \int(f(u)-f(\tilde{Q}), \overline{\tilde{u}})  \tag{5.29}\\
& -\frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right) \\
& -\frac{1+\beta^{2}}{\lambda^{2}} \Im\left(\int\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \tilde{\tilde{Q}}\right) \\
& -\frac{2 \beta}{\lambda} \Re\left(\int\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\partial_{r} \tilde{Q}}\right)+O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right) .
\end{align*}
$$

step 2 Nonlinear cancellation on the phase term. We observe using the explicit formula for $f$ and

$$
\begin{equation*}
f^{\prime}(\tilde{Q}) \cdot \tilde{u}=\frac{p+1}{2}|\tilde{Q}|^{p-1} \tilde{u}+\frac{p-1}{2}|\tilde{Q}|^{p-3} \tilde{Q}^{2} \overline{\tilde{u}} \tag{5.30}
\end{equation*}
$$

the nonlinear cancellation:

$$
\begin{align*}
& -\frac{1+\beta^{2}}{\lambda^{2}} \Im \int(f(u)-f(\tilde{Q}), \overline{\tilde{u}})-\frac{1+\beta^{2}}{\lambda^{2}} \Im\left(\int\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\tilde{Q}}\right) \\
& =-\frac{1+\beta^{2}}{\lambda^{2}} \Im\left(\int f(\tilde{u}+\tilde{Q}), \bar{Q}+\tilde{u}\right)+\frac{1+\beta^{2}}{\lambda^{2}} \Im\left(\int f(\tilde{Q}), \overline{\tilde{Q}}\right) \\
& +\frac{1+\beta^{2}}{\lambda^{2}} \Im\left(\int f(\tilde{Q}) \overline{\tilde{u}}+f^{\prime}(\tilde{Q}) \cdot \tilde{u} \tilde{Q}\right) \\
& =\frac{1+\beta^{2}}{\lambda^{2}} \Im\left(\int f(\tilde{Q}) \overline{\tilde{u}}+f^{\prime}(\tilde{Q}) \cdot \tilde{u} \bar{Q}\right) \\
& =\frac{1+\beta^{2}}{\lambda^{2}} \Im\left(\int|\tilde{Q}|^{p-1}\left(\tilde{Q} \overline{\tilde{u}}+\frac{p+1}{2} \tilde{u} \overline{\tilde{Q}}+\frac{p-1}{2} \tilde{Q} \overline{\tilde{u}}\right)\right) \\
& =0 \tag{5.31}
\end{align*}
$$

step 3 Conclusion. Let $\varphi$ be a smooth compactly supported cut-off function which is 1 in the neighborhood of the support of $\phi$, and 0 in the neighborhood of
$z=-1$. We compute:

$$
\begin{aligned}
A_{1} & =-\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right) \\
& -\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right)\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\partial_{r} \tilde{Q}}\right) \\
& =-\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right) f(\tilde{Q}+\tilde{u}) \overline{\partial_{r} \tilde{Q}+\partial_{r} \tilde{u}}\right) \\
& +\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right) f(\tilde{Q}) \overline{\partial_{r} \tilde{Q}}\right) \\
& +\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right)\left(f(\tilde{Q}) \overline{\partial_{r} \tilde{u}}+f^{\prime}(\tilde{Q}) \cdot \tilde{u} \overline{\partial_{r} \tilde{Q}}\right)\right) \\
& =-\frac{2 \beta}{\lambda} \Re \int \varphi\left(\frac{r}{r(t)}-1\right) \partial_{r}[F(u)-F(\tilde{Q})-f(\tilde{Q}) \overline{\tilde{u}}]
\end{aligned}
$$

Integrating by parts in $r$, we obtain:

$$
A_{1}=\frac{2 \beta}{\lambda} \Re \int\left[\frac{1}{r(t)} \varphi^{\prime}+\frac{N-1}{r} \varphi\right]\left(\frac{r}{r(t)}-1\right)(F(u)-F(\tilde{Q})-f(\tilde{Q}) \overline{\tilde{u}})
$$

In view of the properties of $\varphi$, we have

$$
\begin{equation*}
\frac{1}{r} \sim \frac{1}{r(t)} \text { on the support of } \varphi\left(\frac{\cdot}{r(t)}-1\right) \tag{5.32}
\end{equation*}
$$

and thus

$$
\begin{align*}
A_{1}= & \frac{2 \beta}{\lambda} \Re \int\left[\frac{1}{r(t)} \varphi^{\prime}+\frac{N-1}{r} \varphi\right]\left(\frac{r}{r(t)}-1\right)  \tag{5.33}\\
& \times\left(F(u)-F(\tilde{Q})-f(\tilde{Q}) \overline{\tilde{u}}-\frac{1}{2} F^{\prime \prime}(\tilde{Q})(\tilde{u}, \tilde{u})\right)+O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right) .
\end{align*}
$$

Next, we estimate using (5.23), the nonlinear estimates (5.17), (5.18), (5.32) and (4.15):

$$
\begin{aligned}
& \left|\frac{2 \beta}{\lambda} \Re \int\left[\frac{1}{r(t)} \varphi^{\prime}+\frac{N-1}{r} \varphi\right]\left(\frac{r}{r(t)}-1\right)\left(F(u)-F(\tilde{Q})-f(\tilde{Q}) \overline{\tilde{u}}-\frac{1}{2} F^{\prime \prime}(\tilde{Q})(\tilde{u}, \tilde{u})\right)\right| \\
\lesssim & \frac{b}{\lambda^{4}} \int\left[|\varepsilon|^{p+1}+|\varepsilon|^{3}\left|Q_{b, \widetilde{\beta}}\right|^{p-2} \mathbf{1}_{p>2}\right] \mu \lesssim \frac{b \delta^{C}}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2},
\end{aligned}
$$

which together with (5.33) yields

$$
\begin{equation*}
A_{1}=O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right) \tag{5.34}
\end{equation*}
$$

Since $\varphi=1$ on the support of $\phi$, we have:

$$
\begin{aligned}
& \frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right) \\
= & \frac{2 \beta}{\lambda} \Re\left(\int(\varphi \phi)\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right)
\end{aligned}
$$

and thus from (4.15):

$$
\begin{align*}
A_{2} & =-\frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right)  \tag{5.35}\\
& +\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right) \\
& =-\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right)\left[\phi\left(\frac{r}{r(t)}-1\right)-1\right](f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right) .
\end{align*}
$$

We then observe the identity:

$$
\begin{aligned}
& \Re\left(\partial_{r}[F(\tilde{Q}+\tilde{u})-F(\tilde{Q})-f(\tilde{Q}) \overline{\tilde{u}}]-\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\partial_{r} \tilde{Q}}\right) \\
= & \Re\left(f(\tilde{Q}+\tilde{u})\left(\overline{\partial_{r} \tilde{Q}}+\overline{\partial_{r} \tilde{u}}\right)-f(\tilde{Q}) \overline{\partial_{r} \tilde{Q}}-f^{\prime}(\tilde{Q}) \cdot \partial_{r} \tilde{Q} \overline{\tilde{u}}-f(\tilde{Q}) \overline{\partial_{r} \tilde{u}}-f(\tilde{Q}+\tilde{u}) \overline{\partial_{r} \tilde{Q}}\right. \\
+ & \left.f(\tilde{Q}) \overline{\partial_{r} \tilde{Q}}+f^{\prime}(\tilde{Q}) \cdot \tilde{u} \overline{\partial_{r} \tilde{Q}}\right) \\
= & \Re\left((f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right)+\Re\left(-f^{\prime}(\tilde{Q}) \cdot \partial_{r} \tilde{Q} \overline{\tilde{u}}+f^{\prime}(\tilde{Q}) \cdot \tilde{u} \overline{\partial_{r} \tilde{Q}}\right) \\
= & \Re\left((f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right) \\
+ & \Re\left(-\partial_{u} f(\tilde{Q}) \partial_{r} \tilde{Q} \overline{\tilde{u}}-\partial_{\bar{u}} f(\tilde{Q}) \overline{\partial_{r} \tilde{Q}} \overline{\tilde{u}}+\partial_{u} f(\tilde{Q}) \tilde{u} \overline{\partial_{r} \tilde{Q}}+\partial_{\bar{u}} f(\tilde{Q}) \overline{\tilde{u}} \overline{\partial_{r} \tilde{Q}}\right) \\
= & \Re\left((f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right)+\Re\left(-\partial_{u} f(\tilde{Q}) \partial_{r} \tilde{Q} \overline{\tilde{u}}+\partial_{u} f(\tilde{Q}) \tilde{u} \overline{\partial_{r} \tilde{Q}}\right) \\
= & \Re\left((f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right),
\end{aligned}
$$

where we used in the last inequality the fact that

$$
\partial_{u} f(\tilde{Q})=\frac{p+1}{2}|\tilde{Q}|^{p-1} \in \mathbb{R}
$$

Injecting this into (5.35) and using (4.15), (5.22) yields:

$$
\begin{align*}
A_{2} & =-\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right)\left[\phi\left(\frac{r}{r(t)}-1\right)-1\right]\right. \\
& \left.\times \partial_{r}[F(\tilde{u}+\tilde{Q})-F(\tilde{Q})-f(\tilde{Q}) \overline{\tilde{u}}]\right) \\
& +\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right)\left[\phi\left(\frac{r}{r(t)}-1\right)-1\right]\right. \\
& \left.\times\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\partial_{r} \tilde{Q}}\right) \\
& =\frac{2 \beta}{\lambda^{2}} \Re\left(\int\left(\frac{\alpha b}{2 \beta} \partial_{z}(\varphi(z)[\phi(z)-1])+\frac{(N-1) \lambda}{r} \varphi(z)[\phi(z)-1]\right)\left(\frac{r}{r(t)}-1\right)\right. \\
& \times(F(\tilde{u}+\tilde{Q})-F(\tilde{Q})-f(\tilde{Q}) \overline{\tilde{u}})) \\
& +\frac{2 \beta}{\lambda} \Re\left(\int \varphi\left(\frac{r}{r(t)}-1\right)\left[\phi\left(\frac{r}{r(t)}-1\right)-1\right]\right. \\
& \left.\times\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\partial_{r} \tilde{Q}}\right) . \tag{5.36}
\end{align*}
$$

Since $\phi(0)=1$, we obtain from (5.22):

$$
\begin{equation*}
|\phi(z)-1| \lesssim|z| \lesssim b|y|, \quad\left|\partial_{z}(\phi(z)-1)\right| \lesssim 1 \tag{5.37}
\end{equation*}
$$

We inject this into (5.36), use the homogeneity bounds (4.44) and

$$
\left|F\left(Q_{b, \widetilde{\beta}}+\varepsilon\right)-F\left(Q_{b, \widetilde{\beta}}\right)-f\left(Q_{b, \widetilde{\beta}}\right) \varepsilon\right| \lesssim\left|Q_{b, \widetilde{\beta}}\right|^{p-1}|\varepsilon|^{2}+|\varepsilon|^{p+1},
$$

the pointwise bound:

$$
|\phi(z)|+\left|\partial_{z} \phi(z)\right| \lesssim 1, \frac{\lambda}{r}(1+|z|) \lesssim \frac{\lambda}{r}\left|\frac{r}{r(t)}\right| \lesssim b \text { on } \operatorname{Supp}(\phi)
$$

and the decay (3.15) to estimate:

$$
\begin{align*}
A_{2} & \lesssim \frac{b}{\lambda^{4}} \int\left[\left|Q_{b, \widetilde{\beta}}\right|^{p-1}|\varepsilon|^{2}+|\varepsilon|^{p+1}\right]\left[\mid \partial_{z}(\varphi(z)(\phi(z)-1)|+\varphi(z)| \phi(z)-1 \mid] \mu\right. \\
& +\frac{1}{\lambda^{4}} \int\left[\left|Q_{b, \widetilde{\beta}}\right|^{p-2}|\varepsilon|^{2}+|\varepsilon|^{p} \mathbf{1}_{p>2}\right] \varphi(z)|\phi(z)-1|\left|\partial_{y} Q_{b, \widetilde{\beta}}\right| \mu \\
& \lesssim \frac{b}{\lambda^{4}} \int\left[|\varepsilon|^{p+1}+b|y|^{C} \zeta_{b} e^{-(p-1)|y|}|\varepsilon|^{2}\right] \mu \\
& +\frac{1}{\lambda^{4}} \int b|y|\left[|\varepsilon|^{2}|y|^{C} \zeta_{b} e^{-(p-1)|y|}+\zeta_{b}|\varepsilon|^{p} \mathbf{1}_{p>2} e^{-c|y|}\right] \mu \\
& \lesssim \frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2} \tag{5.38}
\end{align*}
$$

where we used (5.18) and the Sobolev bound (4.43) in the last step. We conclude from (5.34), (5.38):

$$
\begin{equation*}
A_{1}-A_{2}=O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right) \tag{5.39}
\end{equation*}
$$

The function $1-\varphi$ is supported by construction in $y \lesssim-\frac{1}{b}$ where $\tilde{Q}$ vanishes, and hence (5.39) ensures:

$$
\begin{align*}
& -\frac{2 \beta}{\lambda} \Re\left(\int \phi\left(\frac{r}{r(t)}-1\right)(f(\tilde{Q}+\tilde{u})-f(\tilde{Q})) \overline{\partial_{r} \tilde{u}}\right)  \tag{5.40}\\
& \quad-\frac{2 \beta}{\lambda} \Re\left(\int\left(f(\tilde{u}+\tilde{Q})-f(\tilde{Q})-f^{\prime}(\tilde{Q}) \cdot \tilde{u}\right) \overline{\partial_{r} \tilde{Q}}\right)=O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right) .
\end{align*}
$$

In view of (5.29), (5.31) and (5.40), we obtain the expansion of quadratic terms in $\mathcal{J}(\tilde{u})$ :

$$
J(\tilde{u})=O\left(\frac{b}{\lambda^{4}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}\right)
$$

which is the wanted estimate (5.28). This concludes the proof of Lemma 5.5.

## 6. Existence of ring solutions

We conclude in this section the proof of Theorem 1.2. We start with closing the bootstrap Proposition 4.3 using the monotonicity tools developed in the previous section, and then prove the existence of a ring solution using a now standard Schauder type compactness argument and a backwards integration of the flow from blow up time.
6.1. Closing the bootstrap. We are now in position to close the bootstrap i.e. Proposition 4.3.

Proof of Proposition 4.3. step 1 Pointwise control of $\varepsilon$. In view of (5.6), we have:

$$
\begin{aligned}
\frac{d\left(\frac{\mathcal{I}(\tilde{u})}{\lambda^{\theta}}\right)}{d t}= & \frac{1}{\lambda^{\theta}} \frac{d \mathcal{I}(\tilde{u})}{d t}-\theta \frac{\lambda_{t}}{\lambda^{\theta+1}} \mathcal{I}(\tilde{u}) \\
= & \frac{1}{\lambda^{\theta}} \mathcal{J}(\tilde{u})+\theta \frac{b}{\lambda^{2+\theta}} \mathcal{I}(\tilde{u})-\theta \frac{\mathcal{P}_{1}}{\lambda^{2+\theta}} \mathcal{I}(\tilde{u}) \\
& -\left(\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}\right) \frac{\mathcal{I}(t)}{\lambda^{2+\theta}}+O\left(\frac{b}{\lambda^{4+\theta}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}+\frac{b^{k}}{\lambda^{4+\theta}}\|\varepsilon\|_{H_{\mu}^{1}}\right)
\end{aligned}
$$

We estimate from (4.35), the bootstrap assumptions (4.22) (4.23), and (5.27):

$$
\begin{aligned}
\left|\frac{\mathcal{P}_{1}}{\lambda^{2+\theta}} \mathcal{I}(\tilde{u})\right|+\left|\left(\frac{\lambda_{s}}{\lambda}+b\right) \frac{\mathcal{I}(t)}{\lambda^{2+\theta}}\right| & \lesssim \frac{b}{\lambda^{4+\theta}}\left[b^{2}+b\|\varepsilon\|_{H_{\mu}^{1}}+b^{k}\right]\|\varepsilon\|_{H_{\mu}^{1}}^{2} \\
& \lesssim \frac{b \delta^{C}}{\lambda^{4+\theta}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}
\end{aligned}
$$

which together with (5.21) yields the ecistence of a constant $C>0$ such that

$$
\begin{equation*}
\frac{d\left(\frac{\mathcal{I}(\tilde{u})}{\lambda^{\theta}}\right)}{d t} \geq\left(c_{0} \theta-C\right) \frac{b}{\lambda^{4+\theta}}\|\varepsilon\|_{H_{\mu}^{1}}^{2}-C \frac{b^{2 k-1}}{\lambda^{4+\theta}} \tag{6.1}
\end{equation*}
$$

We fix $\theta$ such that

$$
\theta>\frac{C}{c_{0}}
$$

Then, (6.1) yields

$$
\begin{equation*}
\frac{d\left(\frac{\mathcal{I}(\tilde{u})}{\lambda^{\theta}}\right)}{d t} \gtrsim-\frac{b^{2 k-1}}{\lambda^{4+\theta}} \tag{6.2}
\end{equation*}
$$

Now, the definition (4.15) of $b$ together with the bootstrap assumptions (4.22) (4.23) (4.24) yield

$$
\begin{equation*}
b \sim \lambda^{1-\alpha} \tag{6.3}
\end{equation*}
$$

In view of (6.2) and (6.3), we obtain

$$
\begin{equation*}
\frac{d\left(\frac{\mathcal{I}(\tilde{u})}{\lambda^{\theta}}\right)}{d t} \gtrsim-b \lambda^{2(1-\alpha)(k-1)-4-\theta} \tag{6.4}
\end{equation*}
$$

This yields after integration between $t$ and $\bar{t}$ using $\mathcal{I}(\tilde{u})=0$ at $t=\bar{t}$ from the well prepared initial data assumption (4.8):

$$
\begin{equation*}
\mathcal{I}(\tilde{u}) \lesssim \lambda(t)^{\theta} \int_{t}^{\bar{t}} b(\tau) \lambda(\tau)^{2(1-\alpha)(k-1)-4-\theta} d \tau \tag{6.5}
\end{equation*}
$$

We now use the bootstrap bounds (4.21) (4.22) (4.23) and the modulation equation (4.35) to estimate:

$$
\left|\frac{\lambda_{s}}{\lambda}+b\right| \lesssim\left|\mathcal{P}_{1}(b, \widetilde{\beta})\right|+b\|\varepsilon\|_{H_{\mu}^{1}}+b^{k} \lesssim \delta^{C} b \text { from which } 0<b \lesssim-\lambda \lambda_{t}
$$

We conclude from (6.5) and the choice of $k$

$$
k>1+\frac{1+\max \left(\frac{\theta}{2}, 1\right)}{1-\alpha}
$$

that

$$
\mathcal{I}(\tilde{u}) \lesssim \lambda(t)^{2}
$$

In view of the coercivity (5.21) of $\mathcal{I}(\tilde{u})$, we obtain

$$
\|\nabla \tilde{u}\|_{L^{2}}^{2}+\frac{1}{\lambda^{2}}\|\tilde{u}\|_{L^{2}}^{2} \lesssim \lambda^{2}
$$

or equivalently

$$
\begin{equation*}
\|\varepsilon\|_{H_{\mu}^{1}} \lesssim \lambda^{2} \tag{6.6}
\end{equation*}
$$

step 2 Control of the modulation parameters.

To conclude the proof of Proposition 4.3, it remains to control the modulation parameters. We first derive an estimate for $\operatorname{Mod}(t)$. Note that in view of (6.4), we obtain the following improvement of (6.6)

$$
\|\varepsilon\|_{H_{\mu}^{1}} \lesssim \lambda^{2+(1-\alpha)\left(k-1-\frac{2}{1-\alpha}\right)} .
$$

Together with (4.35) and (6.3), we deduce

$$
\begin{equation*}
\operatorname{Mod}(t) \lesssim b\|\varepsilon\|_{H_{\mu}^{1}}+b^{k} \lesssim b^{k} \tag{6.7}
\end{equation*}
$$

The control of the modulation parameters is achieved by the following lemma.
Lemma 6.1. Let $k$ satisfying the condition

$$
\begin{equation*}
k>\frac{2}{1-\alpha}+1 \tag{6.8}
\end{equation*}
$$

Let $t_{e}$ defined in Lemma 4.1. Let $\underline{t}$ and $\bar{t}$ such that $t_{e} \leq \underline{t}<\bar{t}<0$. Let $\left(\lambda_{e}, b_{e}, \widetilde{\beta}_{e}, r_{e}, \gamma_{e}\right)$ solution to the exact system (4.1) of modulation equations. Let $(\lambda, b, \widetilde{\beta}, r, \gamma)$ initialized at $t=\bar{t}$ as

$$
\begin{equation*}
(\lambda, b, \widetilde{\beta}, r, \gamma)(\bar{t})=\left(\lambda_{e}, b_{e}, \widetilde{\beta}_{e}, r_{e}, \gamma_{e}\right)(\bar{t}) \tag{6.9}
\end{equation*}
$$

and solution of the following perturbed system of modulation equations on $[\underline{t}, \bar{t}]$ :

$$
\left\{\begin{array}{l}
\frac{\lambda_{s}}{\lambda}+b-\mathcal{P}_{1}(b, \tilde{\beta})=O\left(b^{k}\right)  \tag{6.10}\\
\frac{r_{s}}{\lambda}+2 \beta=O\left(b^{k}\right) \\
\tilde{\beta}_{s}-\mathcal{P}_{2}(b, \tilde{\beta})=O\left(b^{k}\right) \\
b=\frac{2 \beta}{\alpha} \frac{\lambda}{r}, \beta=\beta_{\infty}+\tilde{\beta} \\
\gamma_{s}=1+\beta^{2}+O\left(b^{k}\right)
\end{array}\right.
$$

Then, the following bounds hold on $[\underline{t}, \bar{t}]$ :

$$
\begin{gather*}
b(t)=\frac{1}{1+\alpha}\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{2}{1+\alpha}}|t|^{\frac{1-\alpha}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right)  \tag{6.11}\\
\lambda(t)=\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{1}{1+\alpha}}|t|^{\frac{1}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right)  \tag{6.12}\\
r(t)=g_{\infty}\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{\alpha}{1+\alpha}}|t|^{\frac{\alpha}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right)  \tag{6.13}\\
\widetilde{\beta}(t)=O\left(|t|^{\frac{2(1-\alpha)}{1+\alpha}}\right) \tag{6.14}
\end{gather*}
$$

and

$$
\begin{equation*}
\gamma(t)=\left(1+\beta_{\infty}^{2}\right)\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1-\alpha}{1+\alpha}}\left(\frac{2(1-\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{-\frac{2}{1+\alpha}}|t|^{-\frac{1-\alpha}{1+\alpha}}+O(\log (|t|)) \tag{6.15}
\end{equation*}
$$

The proof of Lemma 6.1 is postponed to Appendix B. We now conclude the proof of Proposition 4.3. The assumptions (6.8) (6.9) (6.10) of Lemma 6.1 are satisfied in view of the choice (4.7) for $k,(6.7)$ and (4.8). Thus, the conclusions of Lemma 6.1 apply. In particular, (6.11) yields (4.26), (6.14) yields (4.27), (6.13) and (6.12) yield (4.28), while (6.6) and (6.12) yield (4.25). This concludes the proof of Proposition 4.3.
6.2. Proof of Theorem 1.2. We are now in position to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\left(t_{n}\right)_{n \geq 1}$ be an increasing sequence of times $t_{n}<0$ such that $t_{n} \rightarrow 0_{-}$. Let $u_{n}$ the solution to (1.1) with initial data at $t=t_{n}$ given by:

$$
\begin{equation*}
u_{n}\left(t_{n}, r\right)=\frac{1}{\lambda_{e}\left(t_{n}\right)^{\frac{2}{p-1}}} Q_{b_{e}\left(t_{n}\right), \widetilde{\beta}_{e}\left(t_{n}\right)}\left(\frac{r-r_{e}\left(t_{n}\right)}{\lambda_{e}\left(t_{n}\right)}\right) e^{i \gamma_{e}\left(t_{n}\right)} \tag{6.16}
\end{equation*}
$$

Let $\underline{t}<0$ be the backwards time provided by Proposition 4.3 which is independent of $n$. We first claim that $u_{n}(\underline{t})$ is compact in $L^{2}$ as $n \rightarrow+\infty$. Indeed, Proposition 4.3 ensures the uniform bound

$$
\begin{equation*}
\forall t \in\left[\underline{t}, t_{n}\right], \quad\left\|u_{n}(t)\right\|_{H^{1}} \lesssim 1 \tag{6.17}
\end{equation*}
$$

This shows that up to a subsequence, $\left(u_{n}(\underline{t})\right)_{n \geq 1}$ is compact in $L^{2}(r<R)$ as $n \rightarrow$ $+\infty$ for all $R>0$. The $L^{2}$ compactness of $u_{n}(\underline{t})$ is now the consequence of a standard localization procedure. Indeed, let a cut-off unction $\chi(x)=0$ for $|x| \leq 1$ and $\chi(x)=1$ for $|x| \geq 2$, then

$$
\left.\left.\left|\frac{d}{d t} \int \chi_{R}\right| u_{n}\right|^{2}|=2| \operatorname{Im}\left(\int \nabla \chi_{R} \cdot \nabla u_{n}\right) \overline{u_{n}}\right) \left\lvert\, \lesssim \frac{1}{R}\right.,
$$

where we used (6.17). Integrating this backwards from $t_{n}$ to $\underline{t}$ and using (6.16) yields:

$$
\lim _{R \rightarrow+\infty} \sup _{n \geq 1}\left\|u_{n}(\underline{t})\right\|_{L^{2}(r>R)}=0
$$

which together with the $L^{2}(r<R)$ compactness of $\left(u_{n}(\underline{t})\right)_{n \geq 1}$ provided by (6.17) implies up to a subsequence:

$$
u_{n}(\underline{t}) \rightarrow u(\underline{t}) \text { in } L^{2} \text { as } n \rightarrow+\infty .
$$

Let then $u \in \mathcal{C}\left([\underline{t}, T), H^{1}\right)$ be the solution to (1.1) with initial data $u(\underline{t})$, then, using the uniform control in $H^{1}$ for $u_{n}$ and the convergence in $L^{2}$ of $u_{n}(\underline{t})$, we obtain $\forall t \in[\underline{t}, \min (T, 0))$,

$$
u_{n}(t) \rightarrow u(t) \text { in } L^{2}
$$

Let $\left(\lambda_{n}(t), b_{n}(t), \gamma_{n}(t), \varepsilon_{n}(t)\right)$ be the geometrical decomposition associated to $u_{n}(t)$

$$
u_{n}=\frac{1}{\lambda_{n}(t)^{\frac{2}{p-1}}}\left(Q_{b_{n}(t)}+\varepsilon_{n}\right)\left(t, \frac{r-r_{n}(t)}{\lambda_{n}(t)}\right) e^{i \gamma_{n}(t)}
$$

then $u$ admits on $[\underline{t}, \min (T, 0))$ a geometrical decomposition of the form

$$
u=\frac{1}{\lambda(t)^{\frac{2}{p-1}}}\left(Q_{b(t), \widetilde{\beta}(t)}+\varepsilon\right)\left(t, \frac{r-r(t)}{\lambda(t)}\right) e^{i \gamma(t)}
$$

with: $\forall t \in[\underline{t}, \min (T, 0))$,

$$
\begin{aligned}
& \lambda_{n}(t) \rightarrow \lambda(t), \quad r_{n}(t) \rightarrow r(t), \quad b_{n}(t) \rightarrow b(t), \quad \widetilde{\beta}_{n}(t) \rightarrow \widetilde{\beta}(t), \quad \gamma_{n}(t) \rightarrow \gamma(t), \\
& \text { and } \varepsilon_{n}(t) \rightarrow \varepsilon(t) \text { in } L^{2} \text { as } n \rightarrow+\infty
\end{aligned}
$$

see [25] for related statements. By passing to the limit in the bounds provided by Proposition 4.3 and Lemma 6.1, we obtain the bounds: $\forall t \in[\underline{t}, \min (T, 0))$,

$$
\begin{gathered}
b(t)=\frac{1}{1+\alpha}\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{2}{1+\alpha}}|t|^{\frac{1-\alpha}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right) \\
\lambda(t)=\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{1}{1+\alpha}}|t|^{\frac{1}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right) \\
r(t)=g_{\infty}\left(\frac{2(1+\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{\alpha}{1+\alpha}}|t|^{\frac{\alpha}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right) \\
\widetilde{\beta}(t)=O\left(|t|^{\frac{2(1-\alpha)}{1+\alpha}}\right) \\
\gamma(t)=\left(1+\beta_{\infty}^{2}\right)\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1-\alpha}{1+\alpha}}\left(\frac{2(1-\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{-\frac{2}{1+\alpha}}|t|^{-\frac{1-\alpha}{1+\alpha}}+O(\log (|t|))
\end{gathered}
$$

and

$$
\|\varepsilon\|_{H_{\mu}^{1}} \lesssim|t|^{\frac{2}{1+\alpha}}
$$

This yields that $u \in \mathcal{C}\left([\underline{t}, 0), H^{1}\right), u$ blows up at time $T=0$. The estimates (1.13), (1.14), (1.15) are now a straightforward consequence of the above estimates for $(b, \lambda, r, \gamma, \varepsilon)$.
This concludes the proof of Theorem 1.2.

## Appendix A. Integration of the exact system of modulation equations

The goal of this Appendix is to prove Lemma 4.1. For convenience, we prove Lemma 4.1 in time $s$, with $\frac{d s}{d t}=\frac{1}{\lambda_{e}^{2}(t)}$. This is done in the following lemma.
Lemma A. 1 (Integration of the exact system of modulation equations in time $s$ ). There exists a universal constant $s_{0} \gg 1$ such that the following holds. Let

$$
\frac{1}{2}<g_{0}<1, \quad \gamma_{0} \in \mathbb{R}
$$

and

$$
\begin{equation*}
b_{0}=\frac{1}{(1-\alpha) s_{0}} . \tag{A.1}
\end{equation*}
$$

Then the solution $\left(\lambda_{e}, b_{e}, \widetilde{\beta}_{e}, r_{e}, \gamma_{e}\right)$ to the dynamical system:

$$
\left(\mathcal{M}_{\infty}\right)=\left\{\begin{array} { l } 
{ \frac { \lambda _ { s } } { \lambda } + b = \mathcal { P } _ { 1 } ( b , \tilde { \beta } ) , }  \tag{A.2}\\
{ \frac { r _ { s } } { \lambda } + 2 \beta = 0 , } \\
{ \hat { \beta } _ { s } = \mathcal { P } _ { 2 } ( b , \tilde { \beta } ) , } \\
{ b = \frac { 2 \beta } { \alpha } \frac { \lambda } { r } , \beta = \beta _ { \infty } + \tilde { \beta } , } \\
{ \gamma _ { s } = 1 + \beta ^ { 2 } }
\end{array} \quad \text { with } \left\{\begin{array}{l}
\frac{r\left(s_{0}\right)}{\lambda^{\alpha}\left(s_{0}\right)}=g_{0} \\
b\left(s_{0}\right)=b_{0} \\
\tilde{\beta}\left(s_{0}\right)=\frac{1}{s_{0}^{2}} \\
\gamma\left(s_{0}\right)=\gamma_{0}
\end{array}\right.\right.
$$

is defined on $\left[s_{0},+\infty\right)$. Moreover, there exists $g_{\infty}>0$ with

$$
\begin{equation*}
g_{\infty}=g_{0}+o_{s_{0} \rightarrow+\infty}(1) \tag{A.3}
\end{equation*}
$$

such that the following asymptotics hold on $\left[s_{0},+\infty\right)$ :

$$
\begin{gather*}
b_{e}(s)=\frac{1}{(1-\alpha) s}+O\left(\frac{|\log s|}{s^{2}}\right), \quad\left|\tilde{\beta}_{e}(s)\right| \lesssim \frac{1}{s^{2}}  \tag{A.4}\\
\lambda_{e}(s)=\left[\frac{\alpha g_{\infty}}{2(1-\alpha) \beta_{\infty} s}\right]^{\frac{1}{1-\alpha}}\left[1+O\left(\frac{\log (s)}{s}\right)\right],  \tag{A.5}\\
r_{e}(s)=g_{\infty}\left[\frac{\alpha g_{\infty}}{2(1-\alpha) \beta_{\infty} s}\right]^{\frac{\alpha}{1-\alpha}}\left[1+O\left(\frac{\log (s)}{s}\right)\right], \tag{A.6}
\end{gather*}
$$

and

$$
\begin{equation*}
\gamma_{e}(s)=\left(1+\beta_{\infty}^{2}\right) s+O(1) . \tag{A.7}
\end{equation*}
$$

We first show how Lemma A. 1 yields the conclusion of Lemma 4.1.
Proof of Lemma 4.1. In view of (A.5), we have

$$
\int_{s_{0}}^{+\infty} \lambda^{2}<+\infty .
$$

Thus, since $\frac{d s}{d t}=\frac{1}{\lambda^{2}(t)}$, the time of existence of the existence of the dynamical system in time $t$ is finite, and we may choose the origin of time $t$ such that the final time is 0 . Then, for all $t_{e} \leq t<0$, we have

$$
-t=\int_{s}^{+\infty} \lambda^{2}
$$

which together with (A.5) yields

$$
\begin{equation*}
\frac{1}{s}=\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1-\alpha}{1+\alpha}}\left(\frac{2(1-\alpha) \beta_{\infty}}{\alpha g_{\infty}}\right)^{\frac{2}{1+\alpha}}|t|^{\frac{1-\alpha}{1+\alpha}}\left(1+O\left(\log (|t|)|t|^{\frac{1-\alpha}{1+\alpha}}\right)\right) \tag{A.8}
\end{equation*}
$$

Injecting (A.8) in (A.4), (A.5) and (A.7) yields the wanted estimates (4.2), (4.3), (4.4), (4.5) and (4.6). This concludes the proof of Lemma 4.1.

We now turn to the proof of Lemma A.1.
Proof of Lemma A.1. step 1 Reformulation and bootstrap bounds.
The local existence of solutions to (A.2) follows from Cauchy Lipschitz. To control the solution on large positive times, let us introduce the auxiliary function:

$$
g=\frac{r}{\lambda^{\alpha}}
$$

which from (A.2) satisfies:

$$
\begin{equation*}
\frac{d g}{d s}=-\alpha\left(\frac{\lambda_{s}}{\lambda}+b\right) \frac{r}{\lambda^{\alpha}}=-\alpha \mathcal{P}_{1}(b, \tilde{\beta}) g \tag{A.9}
\end{equation*}
$$

We view equivalently (A.2) as a system on $(b, g, \tilde{\beta})$ with from direct computation the equivalent system of equations:

$$
\left\{\begin{array} { l } 
{ \frac { d g } { d s } = - \alpha \mathcal { P } _ { 1 } ( b , \tilde { \beta } ) g }  \tag{A.10}\\
{ b _ { s } + ( 1 - \alpha ) b ^ { 2 } = \frac { b } { \beta } \mathcal { P } _ { 2 } ( b , \tilde { \beta } ) + b \mathcal { P } _ { 1 } ( b , \tilde { \beta } ) , } \\
{ \tilde { \beta } _ { s } = \mathcal { P } _ { 2 } ( b , \tilde { \beta } ) , } \\
{ b = \frac { 2 \beta } { \alpha } \frac { \lambda } { r } , \quad \beta = \beta _ { \infty } + \tilde { \beta } }
\end{array} \quad \text { with } \quad \left\{\begin{array}{l}
g\left(s_{0}\right)=g_{0} \\
b\left(s_{0}\right)=b_{0} \\
\tilde{\beta}\left(s_{0}\right)=\frac{1}{s_{0}^{2}}
\end{array}\right.\right.
$$

We bootstrap the following a priori bounds on the solution which are consistent with the initial data:

$$
\begin{equation*}
\forall s_{0} \leq s \leq \bar{s}, \quad|g(s)| \leq 1+2 g_{0}, \quad|\tilde{\beta}(s)| \leq \frac{1}{s^{\frac{3}{2}}}, \quad\left|b(s)-\frac{1}{(1-\alpha) s}\right| \leq \frac{(\log s)^{2}}{s^{2}} \tag{A.11}
\end{equation*}
$$

step 2 Closing the bootstrap.
We claim that the bounds (A.11) can be improved on $\left[s_{0}, \bar{s}\right]$ provided $s_{0}$ has been chosen large enough. Indeed, let us close the $b$ bound. From (A.10), (A.11):

$$
\left|-\frac{d}{d s}\left(\frac{1}{b}\right)+1-\alpha\right| \lesssim \frac{\left|\frac{b}{\beta} \mathcal{P}_{2}(b, \tilde{\beta})+b \mathcal{P}_{1}(b, \tilde{\beta})\right|}{b^{2}} \lesssim \frac{b\left(b^{2}+|\tilde{\beta}|^{2}\right)}{b^{2}} \lesssim \frac{1}{s}
$$

and thus using the boundary condition on $b$ at $s_{0}$ and the initialization (A.1):

$$
\left|\frac{1}{b(s)}-(1-\alpha) s\right| \lesssim\left|\frac{1}{b\left(s_{0}\right)}-(1-\alpha) s_{0}\right|+\int_{s_{0}}^{s} \frac{d \sigma}{\sigma} \lesssim \log s
$$

from which using $s \geq s_{0} \gg 1$ :

$$
\begin{equation*}
\left|b(s)-\frac{1}{(1-\alpha) s}\right| \lesssim \frac{\log s}{s^{2}} . \tag{A.12}
\end{equation*}
$$

Next, we consider $\widetilde{\beta}$. Since

$$
\mathcal{P}_{2}(b, \widetilde{\beta})=-2 b \widetilde{\beta}+O\left(b^{3}+\widetilde{\beta}^{3}\right)
$$

we obtain in view of (A.11),

$$
\widetilde{\beta}_{s}=-2 b \widetilde{\beta}+O\left(\frac{1}{s^{3}}\right)
$$

which we rewrite using (A.11) (A.12):

$$
\left|\frac{d}{d s}\left(s^{\frac{2}{1-\alpha}} \tilde{\beta}\right)\right| \lesssim s^{\frac{2}{1-\alpha}}\left[\frac{\log s}{s^{2}}|\tilde{\beta}|+\frac{1}{s^{3}}\right] \lesssim s^{\frac{2}{1-\alpha}-3}
$$

We integrate using the boundary condition (A.10) and $\frac{2}{1-\alpha}-2>0$ :

$$
\left|s^{\frac{2}{1-\alpha}} \tilde{\beta}(s)\right| \lesssim\left|s_{0}^{\frac{2}{1-\alpha}} \tilde{\beta}_{0}\right|+s^{\frac{2}{1-\alpha}-2}-s_{0}^{\frac{2}{1-\alpha}-2} \lesssim s^{\frac{2}{1-\alpha}-2}
$$

and thus

$$
\begin{equation*}
|\widetilde{\beta}(s)| \lesssim \frac{1}{s^{2}} \tag{A.13}
\end{equation*}
$$

We now close the $g$ bound in brute force from (A.12), (A.13), (A.10) which yield:

$$
\left|\frac{d g}{d s}\right| \lesssim \frac{1+2 g_{0}}{s^{3}}
$$

and thus in view of the initialization (A.10)

$$
\begin{equation*}
|g(s)| \leq g_{0}+C \frac{1+g_{0}}{s_{0}^{2}} \leq \frac{1}{2}+\frac{3}{2} g_{0} \tag{A.14}
\end{equation*}
$$

for $s_{0}$ large enough. The bounds (A.12), (A.13), (A.14) improve (A.11) and thus from a standard continuity argument, the bounds (A.12), (A.13), (A.14) hold on $\left[s_{0},+\infty\right)$ and the solution is global.
step 3. Conclusion.
The bounds (A.12), (A.13) being now global, (A.4) is proved. We moreover conclude from (A.10):

$$
\int_{s_{0}}^{+\infty}\left|\frac{d g}{d s}\right| d s \lesssim \int_{s_{0}}^{+\infty} \frac{d s}{s^{3}}=o_{s_{0} \rightarrow+\infty}(1)
$$

and hence there exists $g_{\infty}$ satisfying (A.3) such that:

$$
\begin{equation*}
\forall s \geq s_{0}, \quad\left|g(s)-g_{\infty}\right| \lesssim \frac{1}{s^{2}} \tag{A.15}
\end{equation*}
$$

This yields from (A.10), (A.15), (A.11):

$$
\lambda(s)=\frac{\alpha b}{2 \beta} r=\frac{\alpha}{2(1-\alpha) \beta_{\infty} s} g_{\infty} \lambda^{\alpha}\left[1+O\left(\frac{\log (s)}{s}\right)\right]
$$

from which:

$$
\lambda(s)=\left[\frac{\alpha g_{\infty}}{2(1-\alpha) \beta_{\infty} s}\right]^{\frac{1}{1-\alpha}}\left[1+O\left(\frac{\log (s)}{s}\right)\right]
$$

Together with (A.15), we obtain

$$
r(s)=g \lambda^{\alpha}=g_{\infty}\left[\frac{\alpha g_{\infty}}{2(1-\alpha) \beta_{\infty} s}\right]^{\frac{\alpha}{1-\alpha}}\left[1+O\left(\frac{\log (s)}{s}\right)\right]
$$

Finally, it only remains to estimate $\gamma$. In view of (A.2) and (A.13), we have

$$
\frac{d \gamma}{d s}=1+\beta_{\infty}^{2}+O\left(\frac{1}{s^{2}}\right)
$$

which after integration between $s_{0}$ and $s$ yields

$$
\gamma(s)=\left(1+\beta_{\infty}^{2}\right) s+O(1)
$$

This concludes the proof of Lemma A.1.

## Appendix B. Stability of the modulation equations

The goal of this Appendix is to prove Lemma 6.1. Recall that $t_{e}$ is defined in Lemma 4.1 and that $\underline{t}$ and $\bar{t}$ are such that $t_{e} \leq \underline{t}<\bar{t}<0$. We introduce the auxiliary functions

$$
g=\frac{r}{\lambda^{\alpha}} \text { and } g_{e}=\frac{r_{e}}{\lambda_{e}^{\alpha}} .
$$

We define

$$
\underline{b}=b-b_{e}, \quad \underline{g}=g-g_{e} \text { and } \underline{\widetilde{\beta}}=\widetilde{\beta}-\widetilde{\beta}_{e}
$$

From the initial conditions (6.9), we have

$$
\begin{equation*}
(\underline{b}, \underline{g}, \underline{\widetilde{\beta}})(\bar{t})=(0,0,0) . \tag{B.1}
\end{equation*}
$$

Let $t_{0}$ such that $\underline{t} \leq t_{0}<\bar{t}$ and such that the following bootstrap assumptions ${ }^{15}$ hold

$$
\begin{equation*}
\forall t_{0} \leq t \leq \bar{t}, \quad|\underline{g}(t)|+|\underline{\tilde{\beta}}(t)|+|\underline{b}(t)| \leq|t|^{\frac{2}{1+\alpha}} . \tag{B.2}
\end{equation*}
$$

Next, recall (A.2)

$$
\begin{equation*}
\frac{d g_{e}}{d s}=-\alpha \mathcal{P}_{1}\left(b_{e}, \tilde{\beta}_{e}\right) g_{e} \tag{B.3}
\end{equation*}
$$

Also, the modulation equations for $r$ and $\lambda$ and the choice of $b$ in (6.10) implies

$$
\begin{equation*}
\frac{d g}{d s}=\left(\frac{r_{s}}{\lambda}+2 \beta\right) g \frac{\lambda}{r}-\alpha g\left(\frac{\lambda_{s}}{\lambda}+b\right)=-\alpha \mathcal{P}_{1}(b, \tilde{\beta}) g+O\left(b^{k}\right) \tag{B.4}
\end{equation*}
$$

We have

$$
\begin{equation*}
\frac{d s}{d t}=\frac{1}{\lambda^{2}}=\left(\frac{2 \beta}{\alpha b g}\right)^{\frac{2}{1-\alpha}} \text { and } \frac{d s_{e}}{d t}=\frac{1}{\lambda_{e}^{2}}=\left(\frac{2 \beta_{e}}{\alpha b_{e} g_{e}}\right)^{\frac{2}{1-\alpha}} \tag{B.5}
\end{equation*}
$$

In view of the dynamical system (6.10) for $(b, \widetilde{\beta})$, the dynamical system (B.4) for $g$, the dynamical system (A.2) for $\left(b_{e}, g_{e}, \widetilde{\beta}_{e}\right)$, the dynamical system (B.3) for $g_{e}$, the definition of $(\underline{b}, \underline{g}, \underline{\widetilde{\beta}}),(\mathrm{B} .5)$, and the bootstrap bound (B.2), we obtain the following dynamical system for $(\underline{b}, \underline{g}, \underline{\widetilde{\beta}})$ on $t_{0} \leq t \leq \bar{t}$

$$
\begin{align*}
& \underline{b}_{t}-\frac{2 \alpha}{1+\alpha} \frac{\underline{b}}{|t|}=O\left(|t|^{-\frac{2 \alpha}{1+\alpha}}(|\underline{b}|+|\underline{\widetilde{\beta}}|+|\underline{g}|)+|t|^{k \frac{1-\alpha}{1+\alpha}-\frac{2}{1+\alpha}}\right)  \tag{B.6}\\
& \underline{\widetilde{\beta}}_{t}+\frac{2}{1+\alpha} \frac{\widetilde{\beta}}{|t|}=O\left(|t|^{-\frac{2 \alpha}{1+\alpha}}(|\underline{b}|+|\underline{\widetilde{\beta}}|+|\underline{g}|)+|t|^{k \frac{1-\alpha}{1+\alpha}-\frac{2}{1+\alpha}}\right) \tag{B.7}
\end{align*}
$$

and

$$
\begin{equation*}
\underline{g}_{t}=O\left(|t|^{-\frac{2 \alpha}{1+\alpha}}(|\underline{\mid}|+|\underline{\widetilde{\beta}}|+|\underline{g}|)+|t|^{k \frac{1-\alpha}{1+\alpha}-\frac{2}{1+\alpha}}\right) . \tag{B.8}
\end{equation*}
$$

Integrating (B.8) between $t$ and $\bar{t}$ and using (B.1) and (6.8) yields

$$
\begin{equation*}
|\underline{g}(t)| \lesssim \int_{t}^{\bar{t}}|\tau|^{-\frac{2 \alpha}{1+\alpha}}(|\underline{b}|+|\underline{\widetilde{\beta}}|+|\underline{g}|) d \tau+|t|^{k \frac{1-\alpha}{1+\alpha}+1-\frac{2}{1+\alpha}} \tag{B.9}
\end{equation*}
$$

Integrating (B.6) between $t$ and $\bar{t}$ and using (B.1) and (6.8) yields

$$
\begin{equation*}
|\underline{b}(t)| \lesssim|t|^{-\frac{2 \alpha}{1+\alpha}} \int_{t}^{\bar{t}}(|\underline{b}|+|\underline{\widetilde{\beta}}|+|\underline{g}|) d \tau+|t|^{k \frac{1-\alpha}{1+\alpha}+1-\frac{2}{1+\alpha}} . \tag{B.10}
\end{equation*}
$$

Integrating (B.7) between $t$ and $\bar{t}$ and using (B.1) and (6.8) yields

$$
\begin{equation*}
|\underline{\widetilde{\beta}}(t)| \lesssim|t|^{\frac{2}{1+\alpha}} \int_{t}^{\bar{t}}|\tau|^{-2}(|\underline{b}|+|\underline{\widetilde{\beta}}|+|\underline{g}|) d \tau+|t|^{k \frac{1-\alpha}{1+\alpha}+1-\frac{2}{1+\alpha}} . \tag{B.11}
\end{equation*}
$$

[^12]In view of (B.9), (B.10) and (B.11), we obtain

$$
\begin{align*}
|\underline{b}|+|\underline{\widetilde{\beta}}|+|\underline{g}| \lesssim & \int_{t}^{\bar{t}}\left(|\tau|^{-\frac{2 \alpha}{1+\alpha}}+|t|^{-\frac{2 \alpha}{1+\alpha}}+|t|^{\frac{2}{1+\alpha}}|\tau|^{-2}\right)(|\underline{b}|+|\underline{\widetilde{\beta}}|+|\underline{g}|) d \tau \\
& +|t|^{\frac{1-\alpha}{1+\alpha}+1-\frac{2}{1+\alpha}} . \tag{B.12}
\end{align*}
$$

Injecting the bootstrap assumption (B.2), noticing that the integral is convergent, and then reiterating finally yields

$$
\begin{equation*}
|\underline{b}|+|\underline{\widetilde{\beta}}|+|\underline{g}| \lesssim|t|^{k \frac{1-\alpha}{1+\alpha}+1-\frac{2}{1+\alpha}} \tag{B.13}
\end{equation*}
$$

which is an improvement of the bootstrap assumption (B.2) in view of (6.8). Thus, (B.13) holds on $[\underline{t}, \bar{t}]$. Now, the wanted estimate (6.11) for $b$ and (6.14) for $\widetilde{\beta}$ follow from (B.13), (6.8), and the estimate (4.2) for $b_{e}$ and (4.5) for $\widetilde{\beta}_{e}$. Also, the wanted estimate (6.12) for $\lambda$ and (6.13) for $r$ follow from the definition of $b$ and $g$, (B.13), (6.8), and the estimate (4.4) for $r_{e}$ and (4.3) for $\lambda_{e}$.

Finally, we derive the wanted estimate for $\gamma$. In view of the dynamical system (6.10) for $\gamma$, the dynamical system (A.2) for $\gamma_{e}$, (B.5), and the estimate (B.13), we obtain the following dynamical system for $\gamma-\gamma_{e}$

$$
\begin{equation*}
\left(\gamma-\gamma_{e}\right)_{t}=O\left(|t|^{k \frac{1-\alpha}{1+\alpha}+1-\frac{5-\alpha}{1+\alpha}}\right) \tag{B.14}
\end{equation*}
$$

Now, recall that $\gamma(\bar{t})=\gamma_{e}(\bar{t})$ from (6.9), so that integrating (B.14) between $t$ and $\bar{t}$ and using (6.8) yields

$$
\begin{equation*}
\left|\gamma-\gamma_{e}\right| \lesssim|t|^{k \frac{1-\alpha}{1+\alpha}+2-\frac{5-\alpha}{1+\alpha}} \tag{B.15}
\end{equation*}
$$

Now, the wanted estimate for $\gamma$ (6.15) follows from (B.15), (6.8), and the estimate for $\gamma_{e}$ (4.6). This concludes the proof of Lemma 6.1.

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[^0]:    ${ }^{1}$ for data $u_{0} \in \Sigma=H^{1} \cap\left\{x u \in L^{2}\right\}$.

[^1]:    
    ${ }^{3}$ If the rough profile $Q e^{-i \beta_{\infty} y}$ in (1.13) is replaced by the precise profile of $(3.32)$, then the convergence in (1.13) is in $H^{1}\left(\mathbb{R}^{N}\right)$ instead of $L^{2}\left(\mathbb{R}^{N}\right)$.

[^2]:    ${ }^{4}$ even though a similar structure could be exhibited which would probably be relevant for stability issues, and in particular a finite order expansion is enough to close the analysis.

[^3]:    ${ }^{5}$ defined on $y>-\frac{2 \beta}{\alpha b}$.

[^4]:    ${ }^{6}$ related to the universal coercivity constant in (1.26).

[^5]:    $7_{\text {see }}[28]$ for further details.

[^6]:    8 this is consistent with (1.8) and can be proved in $\Sigma$ without the radial assumption.

[^7]:    ${ }^{9}$ defined on $y>-\frac{2 \beta}{\alpha b}$.

[^8]:    ${ }^{10}$ note that solutions are given up to an element of the kernel, but adjusting this element is irrelevant.

[^9]:    ${ }^{11}$ to handle the case when $|z| \gg 1$.

[^10]:    ${ }^{12}$ Recall that $\mu=\left(1+\frac{\alpha b}{2 \beta} y\right)^{N-1}$ and thus, $y>-\frac{\delta}{b}$ implies $\mu \gtrsim 1$.
    ${ }^{13}$ let us recall that $p>1$ but $p<2$ is allowed in our range of parameters.

[^11]:    14 using the smallness of $b$ and the exponential localization of $Q$, see for example [20]

[^12]:    15 which are consistent with (B.1).

