

Higher-Order Total Directional Variation: Analysis*

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Abstract. We analyze a new notion of total anisotropic higher-order variation which, differently from total generalized variation in [K. Bredies, K. Kunisch, and T. Pock, *SIAM J. Imaging Sci.*, 3 (2010), pp. 492–526], quantifies for possibly nonsymmetric tensor fields their variations at arbitrary order weighted by possibly inhomogeneous, smooth elliptic anisotropies. We prove some properties of this total variation and of the associated spaces of tensors with finite variations. We show the existence of solutions to a related regularity-fidelity optimization problem. We also prove a decomposition formula which appears to be helpful for the design of numerical schemes, as shown in a companion paper, where several applications to image processing are studied.

Key words. anisotropic total variation, higher-order total variation, variational model

AMS subject classifications. 47A52, 49M30, 49N45, 65J22, 94A08

DOI. 10.1137/19M1239210

1. Introduction. Total variation (TV) regularization is one of the most prominent regularization approaches, successfully applied in a variety of imaging problems. Indeed, since [23], TV played a crucial role for image denoising, image deblurring, inpainting, magnetic resonance image (MRI) reconstruction, and many others; see [11]. Extensions of TV regularization are TV-type regularizers that feature higher-order derivatives [12, 13, 21, 24, 26, 9]—in particular, accommodating for more complex image structures and countering certain TV artifacts such as staircasing—as well as TV regularizers that encode directional information—so as to enhance the quality of the smoothing results along preferred directions; e.g., [4, 27, 5, 15, 25, 18, 20, 19, 17, 16, 14]. Very general anisotropies have also been studied, as in [1], where it is shown that a fairly general class of metrics, possibly discontinuous, yields a well-defined notion of first-order anisotropic TV.

*Received by the editors January 17, 2019; accepted for publication (in revised form) January 6, 2020; published electronically March 24, 2020.

<https://doi.org/10.1137/19M1239210>

Funding: The work of the first author was supported by the EPSRC grant EP/L016516/1 for the University of Cambridge, Cambridge Centre for Analysis DTC. The work of the second author was supported by the French National Research Agency (ANR) research grant MIRIAM ANR-14-CE27-0019 and by the European Union Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant 777826 (NoMADS). The work of the third author was supported by EPSRC grants EP/M00483X/1, EP/K009745/1, and EP/N014588/1; the Leverhulme Trust project “Breaking the non-convexity barrier”; the Alan Turing Institute grant TU/B/000071; the CHiPS Horizon 2020 RISE project grant; and the Isaac Newton Institute and the Cantab Capital Institute for the Mathematics of Information.

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In this paper we consider a new class of TV-type regularizers that we have recently introduced in [22] and called *total directional variation* (TDV). These regularizers extend the higher-order TV of [9] (the so-called total generalized variation (TGV); see below) and the *directional total generalized variation* (DTGV) of [14] (which promotes smoothness along a single, constant direction) to higher-order TV regularization with spatially varying directional smoothing. This is done by means of weighting derivatives with 2-tensors; see below. In [22] we propose the TDV regularizer, discuss its discretization and numerical solution, and demonstrate its performance on a range of imaging applications, such as image denoising, wavelet-based zooming, and digital elevation map interpolation with applications to atomic force microscopy data. In this paper we give a theoretical analysis of the TDV regularizer in the continuum.

Let Ω be a bounded Lipschitz domain. We address the analysis of the higher-order TDV defined for every tensor-valued function $\mathbf{u} : \Omega \rightarrow \mathcal{T}^\ell(\mathbb{R}^d)$, with $\mathcal{T}^\ell(\mathbb{R}^d)$ the vector space of ℓ -tensors in $\mathbb{R}^d$ and $\ell \in \mathbb{N}$, as

$$(1.1) \quad \text{TDV}_{\alpha}^{\mathbf{Q},\ell}(\mathbf{u}, \mathcal{M}) := \sup \left\{ \int_{\Omega} \mathbf{u} \cdot \text{div}_{\mathcal{M}}^{\mathbf{Q}} \Psi \, d\mathbf{x}, \Psi \in \mathcal{Y}_{\mathcal{M},\alpha}^{\mathbf{Q},\ell} \right\},$$

where $\mathbf{Q}$ is the order of regularization, $\mathcal{M}$ is a collection of weighting fields acting on each derivative order, α is the vector regularization parameter, and

$$(1.2) \quad \mathcal{Y}_{\mathcal{M},\alpha}^{\mathbf{Q},\ell} = \left\{ \Psi : \Psi \in C_c^{\mathbf{Q}}(\Omega, \mathcal{T}^{\ell+\mathbf{Q}}(\mathbb{R}^d)), \left\| \text{div}_{\mathcal{M}}^j \Psi \right\|_{\infty} \leq \alpha_j, \forall j = 0, \dots, \mathbf{Q} - 1 \right\},$$

with $\text{div}_{\mathcal{M}}^j$ the $\mathcal{M}$ -anisotropic divergence operator of order j ; see sections 3 and 4 for the precise definitions. The higher-order TDV extends the classical notion of isotropic TGV to the (smooth) elliptic anisotropic case.

1.1. Related works. The use of modified TV regularizers in imaging processing has increased in recent decades with the aim to enhance the local information in images. We refer the reader to the introduction of the complementary part of this work [22] for a detailed review.

For our purposes it is useful to recall the TGV [9, 8, 7] which appears in many image processing tasks. It is defined for a derivative order $\mathbf{Q} \geq 1$ as

$$(1.3) \quad \text{TGV}_{\alpha}^{\mathbf{Q},\ell}(\mathbf{u}) = \sup \left\{ \int_{\Omega} \mathbf{u} \cdot \text{div}^{\mathbf{Q}} \Psi \, d\mathbf{x} \mid \Psi \in C_c^{\mathbf{Q}}(\Omega, \text{Sym}^{\ell+\mathbf{Q}}(\mathbb{R}^d)), \left\| \text{div}^j \Psi \right\|_{\infty} \leq \alpha_j, \forall j = 0, \dots, \mathbf{Q} - 1 \right\},$$

where $\text{Sym}^{\ell+\mathbf{Q}}(\mathbb{R}^d)$ is the space of symmetric tensors, $\alpha = (\alpha_0, \dots, \alpha_{\mathbf{Q}-1})$ is a weight vector of positive real numbers, $\text{div} \Psi = \text{trace}(\nabla \otimes \Psi)$, and $\text{div}^j \Psi = \text{trace}^j(\nabla^j \otimes \Psi)$ [9, equation (2.1)].

In [14], the directional version of (1.3) is presented for a fixed and single global direction only and for an imaging function $u : \Omega \rightarrow \mathbb{R}$: There, the continuous DTV and DTGV are defined as

$$(1.4) \quad \text{DTV}(u) = \sup \left\{ \int_{\Omega} u \, \text{div} \tilde{\Psi} \, d\mathbf{x} \mid \Psi \in C_c^1(\Omega, \mathbb{R}^2), \tilde{\Psi}(\mathbf{x}) \in E^{a,\theta}(\mathbf{0}), \forall \mathbf{x} \in \Omega \right\},$$

$$(1.5) \quad \text{DTGV}_{\alpha}^Q(u) = \sup \left\{ \int_{\Omega} u \operatorname{div}^Q \tilde{\Psi} \, d\mathbf{x} \mid \left\| \operatorname{div}^q \tilde{\Psi} \right\|_{\infty} \leq \alpha_q, \forall q = 0, \dots, Q-1 \right\},$$

where $\tilde{\Psi}(\mathbf{x}) = \mathbf{R}_{\theta} \mathbf{\Lambda}_a \Psi(\mathbf{x})$ for $\Psi \in B_1(\mathbf{0})$ and $a \in (0, 1]$, with $\mathbf{R}_{\theta}$ a rotation matrix and $\mathbf{\Lambda}_a = \operatorname{diag}(1, a)$ a contraction matrix, and $\tilde{\Psi}(\mathbf{x}) \in E^{a, \theta}(\mathbf{0})$, where $E^{a, \theta}(\mathbf{0})$ is the closed elliptical set defined as $E^{a, \theta}(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x}^T \mathbf{R}_{\theta} \mathbf{\Lambda}_{1/a}^2 \mathbf{R}_{\theta}^T \mathbf{x} \leq 1\}$.

1.2. Motivation of the paper. We are interested in the analysis of the regularizer proposed in [22] that generalizes (1.4)–(1.5) for handling nonconstant smoothing directions in the domain $\Omega \subset \mathbb{R}^2$. In particular, we study the total directional variation $\text{TDV}_{\alpha}^{Q, \ell}(\mathbf{u}, \mathcal{M})$ (for a fixed-order Q and a collection of weighting fields $\mathcal{M}$) of a $\mathcal{T}^{\ell}(\mathbb{R}^d)$ -valued function u . We analyze the space BDV_{α}^Q of $\mathcal{T}^{\ell}(\mathbb{R}^d)$ -valued functions whose total directional variation $\text{TDV}_{\alpha}^{Q, \ell}(\mathbf{u}, \mathcal{M})$ is finite. We exhibit an equivalent representation for $\text{TDV}_{\alpha}^{Q, \ell}(\mathbf{u}, \mathcal{M})$, and we prove the existence of solutions to the $\text{TDV}_{\alpha}^{Q, \ell} - L^2$ problem.

We show that the theoretical results for $\text{TGV}_{\alpha}^{Q, \ell}$, shown in [9, 7, 8] for symmetric tensor fields and isotropic derivative operators, naturally extend to the case of possibly nonsymmetric tensor fields and elliptic anisotropic derivative operators. A key for this extension is provided by Lemmas 3.1 and 3.4 and by Definition 3.2, which gives a suitable notion of weighted divergence for possibly nonsymmetric tensor fields.

1.3. Organization of the paper. The paper is organized as follows: We introduce the preliminary notation in section 2 and the higher-order TDV regularizers in section 3, in section 4 we discuss the space of functions of bounded directional variation, in section 5 we show the equivalent decomposition of TDV_{α}^Q with respect to a collection of fields $\mathcal{M}$, and in section 6 we prove the existence of solutions for the $\text{TDV}_{\alpha}^Q - L^2$ problem.

2. Preliminaries. In this section we introduce the notation of tensors and function spaces considered for the definition and analysis of TDV.

2.1. Tensors. Following [9], let $\mathcal{T}^{\ell}(\mathbb{R}^d)$ and $\text{Sym}^{\ell}(\mathbb{R}^d)$ be the vector spaces of ℓ -tensor and symmetric ℓ -tensors in $\mathbb{R}^d$, respectively, defined as

$$\mathcal{T}^{\ell}(\mathbb{R}^d) := \left\{ \xi : \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{\ell\text{-times}} \rightarrow \mathbb{R}, \text{ such that } \xi \text{ is } \ell\text{-linear} \right\},$$

$$\text{Sym}^{\ell}(\mathbb{R}^d) := \left\{ \xi : \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{\ell\text{-times}} \rightarrow \mathbb{R}, \text{ such that } \xi \text{ is } \ell\text{-linear and symmetric} \right\},$$

where $\xi \in \mathcal{T}^{\ell}(\mathbb{R}^d)$ is symmetric if $\xi(\mathbf{a}_1, \dots, \mathbf{a}_{\ell}) = \xi(\mathbf{a}_{\pi(1)}, \dots, \mathbf{a}_{\pi(\ell)})$ for all permutations π of $\{1, \dots, \ell\}$. By convention, $\mathcal{T}^0(\mathbb{R}^d) = \text{Sym}^0(\mathbb{R}^d)$ is identified with $\mathbb{R}$, and every element of $\mathcal{T}^1(\mathbb{R}^d) = \text{Sym}^1(\mathbb{R}^d)$ can be identified with a vector of $\mathbb{R}^d$ acting on $\mathbb{R}^d$ through the scalar product. We have $\mathcal{T}^{\ell}(\mathbb{R}^d) \equiv \text{Sym}^{\ell}(\mathbb{R}^d)$ only for $\ell = 0, 1$. For example, $\mathcal{T}^2(\mathbb{R}^d)$ can be identified with the space of general $d \times d$ real matrices, whereas $\text{Sym}^2(\mathbb{R}^d)$ can be identified with the space of symmetric $d \times d$ real matrices. We have the following operations on $\mathcal{T}^{\ell}(\mathbb{R}^d)$ (assuming that $a_k \in \mathbb{R}^d \forall k$):

- $\otimes$ is the tensor product for $\xi_1 \in \mathcal{T}^{\ell_1}(\mathbb{R}^d)$, $\xi_2 \in \mathcal{T}^{\ell_2}(\mathbb{R}^d)$, with $\xi_1 \otimes \xi_2 \in \mathcal{T}^{\ell_1+\ell_2}(\mathbb{R}^d)$:

$$(\xi_1 \otimes \xi_2)(\mathbf{a}_1, \dots, \mathbf{a}_{\ell_1+\ell_2}) = \xi_1(\mathbf{a}_1, \dots, \mathbf{a}_{\ell_1}) \xi_2(\mathbf{a}_{\ell_1+1}, \dots, \mathbf{a}_{\ell_1+\ell_2});$$

- $\text{trace}(\xi) \in \mathcal{T}^{\ell-2}(\mathbb{R}^d)$ is the trace of $\xi \in \mathcal{T}^{\ell}(\mathbb{R}^d)$, with $\ell \geq 2$, defined by

$$\text{trace}(\xi)(\mathbf{a}_1, \dots, \mathbf{a}_{\ell-2}) = \sum_{i=1}^d \xi(\mathbf{e}_i, \mathbf{a}_1, \dots, \mathbf{a}_{\ell-2}, \mathbf{e}_i),$$

where $\mathbf{e}_i$ is the i th standard basis vector;

- $(\cdot)^\sim$ is the operator such that if $\xi \in \mathcal{T}^{\ell}(\mathbb{R}^d)$, then

$$\xi^\sim(\mathbf{a}_1, \dots, \mathbf{a}_{\ell}) = \xi(\mathbf{a}_{\ell}, \mathbf{a}_1, \dots, \mathbf{a}_{\ell-1});$$

- $(\cdot)^{\bar{}}$ is the operator such that if $\xi \in \mathcal{T}^{\ell}(\mathbb{R}^d)$, then

$$\bar{\xi}(\mathbf{a}_1, \dots, \mathbf{a}_{\ell}) = \xi(\mathbf{a}_{\ell}, \dots, \mathbf{a}_1);$$

- let $\xi, \eta \in \mathcal{T}^{\ell}(\mathbb{R}^d)$; then $\mathcal{T}^{\ell}(\mathbb{R}^d)$ is equipped with the scalar product defined as

$$\xi \cdot \eta = \sum_{p \in \{1, \dots, d\}^{\ell}} \xi(\mathbf{e}_{p_1}, \dots, \mathbf{e}_{p_{\ell}}) \eta(\mathbf{e}_{p_1}, \dots, \mathbf{e}_{p_{\ell}});$$

- a Frobenius-type norm for $\xi \in \mathcal{T}^{\ell}(\mathbb{R}^d)$ is given by $|\xi| = \sqrt{\xi \cdot \xi}$.

2.2. Spaces. Let $\Omega \subset \mathbb{R}^d$ be a fixed open domain. We define the Lebesgue spaces of $\mathcal{T}^{\ell}(\mathbb{R}^d)$ -valued tensor fields as

$$L^p(\Omega, \mathcal{T}^{\ell}(\mathbb{R}^d)) = \left\{ \xi : \Omega \rightarrow \mathcal{T}^{\ell}(\mathbb{R}^d) \text{ is measurable and } \|\xi\|_p < \infty \right\},$$

with

$$\|\xi\|_p = \left(\int_{\Omega} |\xi(\mathbf{x})|^p \, d\mathbf{x} \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty \quad \text{and} \quad \|\xi\|_{\infty} = \text{ess sup}_{\mathbf{x} \in \Omega} |\xi(\mathbf{x})|.$$

Also, $L^p_{\text{loc}}(\Omega, \mathcal{T}^{\ell}(\mathbb{R}^d))$ is defined as usual: Since the vector norm in $\mathcal{T}^{\ell}(\mathbb{R}^d)$ is a scalar product, the duality holds: $L^p(\Omega, \mathcal{T}^{\ell}(\mathbb{R}^d))^* = L^{p^*}(\Omega, \mathcal{T}^{\ell}(\mathbb{R}^d))$, with $1/p + 1/p^* = 1$ for $1 \leq p < \infty$.

We now introduce the derivative for tensors and its weighted version. In what follows, the elements of $\xi : \Omega \rightarrow \mathcal{T}^{\ell}(\mathbb{R}^d)$ are described via the shortened notation

$$(\xi(\cdot))_{i_1, \dots, i_{\ell}} := \xi(\cdot)(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_{\ell}}).$$

Definition 2.1. Let $\nabla = (\partial_1, \dots, \partial_d)^T$ be the derivative operator and $\xi : \Omega \rightarrow \mathcal{T}^{\ell}(\mathbb{R}^d)$ a differentiable tensor-valued function. The Q th order (unweighted) derivative of ξ is defined as $(\nabla^Q \otimes \xi) : \Omega \rightarrow \mathcal{T}^{\ell+Q}(\mathbb{R}^d)$ with

$$((\nabla^Q \otimes \xi)(\cdot))_{j_1, \dots, j_Q, i_1, \dots, i_{\ell}} = ((D^Q \xi(\cdot))_{j_1, \dots, j_Q})_{i_1, \dots, i_{\ell}},$$

where $D^Q \xi : \Omega \rightarrow \mathcal{L}^Q(\mathbb{R}^d, \mathcal{T}^Q(\mathbb{R}^d))$ denotes the Fréchet derivative of ξ and $\mathcal{L}^Q(\mathbb{R}^d, \mathcal{T}^Q(\mathbb{R}^d))$ the space of Q -linear and continuous mappings from $\mathbb{R}^Q$ onto $\mathcal{T}^{\ell}(\mathbb{R}^d)$.

Definition 2.2. Let ξ, ∇, D be as above and $\eta : \Omega \rightarrow \mathcal{T}^2(\mathbb{R}^d)$. For $Q = 1$, the derivative operator weighted by η is defined as

$$\eta \nabla = \left(\sum_{k=1}^d \eta_{j,k} \partial_k \right)_{j=1}^d,$$

and the first-order derivative of ξ weighted by η is defined as $(\eta \nabla \otimes \xi) : \Omega \rightarrow \mathcal{T}^{\ell+1}(\mathbb{R}^d)$, with

$$((\eta \nabla \otimes \xi)(\cdot))_{j,i_1,\dots,i_\ell} = \left((\eta D \xi(\cdot))_j \right)_{i_1,\dots,i_\ell}.$$

For the Q th order case, i.e., whenever each derivative order is weighted by the corresponding element of a collection $(\eta^q)_{q=1}^Q$, with each $\eta^q : \Omega \rightarrow \mathcal{T}^2(\mathbb{R}^d)$,

$$((\eta^Q \nabla \otimes \dots \otimes \eta^1 \nabla \otimes \xi)(\cdot))_{j_1,\dots,j_Q,i_1,\dots,i_\ell} = \left((\eta^Q D (\dots (\eta^1 D \xi(\cdot)))) \right)_{j_1,\dots,j_Q}{}_{i_1,\dots,i_\ell}.$$

We denote the Banach space of Q -times continuously differentiable $\mathcal{T}^\ell(\mathbb{R}^d)$ -valued tensor fields as $C^Q(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, where $(\nabla^Q \otimes u) : \Omega \rightarrow \mathcal{T}^{Q+\ell}(\mathbb{R}^d)$ and

$$\|u\|_{\infty,Q} = \max_{\ell=0,\dots,Q} \sup_{x \in \Omega} \left| \nabla^\ell \otimes u(x) \right|.$$

The space of fields in $C^Q(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ with compact support is denoted by $C_c^Q(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ and its completion under the supremum norm by $C_0^Q(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$. The space of Radon measures on $\Omega \subset \mathbb{R}^d$ is denoted by $\mathcal{M}$, and, by the Riesz representation theorem, we identify

$$\mathcal{M}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \equiv C_0(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))^*,$$

and we have

$$\|\cdot\|_{\mathcal{M}} = \sup \left\{ \langle \cdot, \Psi \rangle \mid \Psi \in C_0(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)), \|\Psi\|_\infty \leq 1 \right\}.$$

$\mathcal{D}'(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ denotes the space of $\mathcal{T}^\ell(\mathbb{R}^d)$ -valued distributions on Ω , and $\mathcal{D}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) = C_c^\infty(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ is the associated space of test functions.

2.3. Notation. In what follows, we deal with derivatives of order up to $Q \in \mathbb{N}^*$. Since the weighting of each derivative order is the core operation of this work, we make use of a collection of smooth weighting tensor fields $\mathcal{M} = (\mathbf{M}_j)_{j=1}^Q$, where for all $j \in \{1, \dots, Q\}$, $\mathbf{M}_j : \Omega \rightarrow \mathcal{T}^2(\mathbb{R}^d)$ and $\forall x \in \Omega$, $\mathbf{M}_j(x)$ can be identified with a positive definite $d \times d$ matrix. When $Q = 1$ or when only one derivative is involved, we will refer directly to a unique weighting tensor field $\mathbf{M}$.

3. Higher-order TDV. For making sense of the distributional formulation of higher-order directional variation in (1.1) we need an integration by parts formula for the weighted derivative of tensors in Definition 2.2. Namely, we consider

$$\int_{\Omega} (\mathbf{M} \nabla \otimes \mathbf{A}) \cdot \Psi \, dx,$$

with $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain, $\mathbf{M} \in C^1(\Omega, \mathcal{T}^2(\mathbb{R}^d))$, $\mathbf{A} \in C^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, and $\Psi \in C^1(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$. We next explore the transfer of $\mathbf{M}$ on Ψ .

Lemma 3.1. *Let Ω , $\mathbf{M}$, $\mathbf{A}$, and Ψ as above. Then*

$$(3.1) \quad \int_{\Omega} (\mathbf{M} \nabla \otimes \mathbf{A}) \cdot \Psi \, d\mathbf{x} = \int_{\Omega} (\nabla \otimes \mathbf{A}) \cdot \text{trace}(\mathbf{M} \otimes \Psi^{\sim}) \, d\mathbf{x} \text{ for all } \mathbf{M}, \mathbf{A}, \Psi.$$

Proof. Using Einstein notation, we have

$$\begin{aligned} (\mathbf{M} \nabla \otimes \mathbf{A}) \cdot \Psi &= \mathbf{M}_{j,k} \partial_k \mathbf{A}_{i_1, \dots, i_{\ell}} \Psi_{j, i_1, \dots, i_{\ell}} \\ &= \partial_k \mathbf{A}_{i_1, \dots, i_{\ell}} \mathbf{M}_{j,k} \Psi_{j, i_1, \dots, i_{\ell}} \\ &= (\nabla \otimes \mathbf{A})_{k, i_1, \dots, i_{\ell}} \mathbf{M}_{j,k} (\Psi^{\sim})_{i_1, \dots, i_{\ell}, j} \\ &= (\nabla \otimes \mathbf{A})_{k, i_1, \dots, i_{\ell}} (\mathbf{M} \otimes \Psi^{\sim})_{j, k, i_1, \dots, i_{\ell}, j} \\ &= (\nabla \otimes \mathbf{A})_{k, i_1, \dots, i_{\ell}} (\text{trace}(\mathbf{M} \otimes \Psi^{\sim}))_{k, i_1, \dots, i_{\ell}} \\ &= (\nabla \otimes \mathbf{A}) \cdot \text{trace}(\mathbf{M} \otimes \Psi^{\sim}). \end{aligned}$$

Therefore, we get

$$\int_{\Omega} (\mathbf{M} \nabla \otimes \mathbf{A}) \cdot \Psi \, d\mathbf{x} = \int_{\Omega} (\nabla \otimes \mathbf{A}) \cdot \text{trace}(\mathbf{M} \otimes \Psi^{\sim}) \, d\mathbf{x}. \quad \blacksquare$$

Definition 3.2. *Let $\mathbf{M}$ and Ψ as above. We define the $\mathbf{M}$ -divergence of Ψ as*

$$(3.2) \quad \text{div}_{\mathbf{M}}(\Psi) = \text{trace}(\nabla \otimes [\text{trace}(\mathbf{M} \otimes \Psi^{\sim})]^{\sim}).$$

Remark 3.3. For $\mathbf{M} = \mathbf{I}$ the divergence in (3.2) is $\text{div}(\Psi) = \text{trace}(\nabla \otimes \Psi^{\sim})$. When Ψ is a symmetric tensor, since $\Psi^{\sim} = \Psi$, we retrieve $\text{div}(\Psi) = \text{trace}(\nabla \otimes \Psi)$ of [9, equation (2.1)].

The next lemma provides an integration by parts formula which justifies the definition of the $\mathbf{M}$ -divergence operator.

Lemma 3.4. *Let Ω , $\mathbf{M}$, $\mathbf{A}$, and Ψ as above. Then*

$$(3.3) \quad \int_{\Omega} (\mathbf{M} \nabla \otimes \mathbf{A}) \cdot \Psi \, d\mathbf{x} = \int_{\partial\Omega} (\nu \otimes \mathbf{A}) \cdot \text{trace}(\mathbf{M} \otimes \Psi^{\sim}) \, d\mathcal{H}^{d-1} - \int_{\Omega} \mathbf{A} \cdot \text{div}_{\mathbf{M}} \Psi \, d\mathbf{x},$$

where ν is the outward unit normal on $\partial\Omega$ and $\text{div}_{\mathbf{M}} \Psi$ as in (3.2).

Proof. We know from Lemma 3.1 that

$$\int_{\Omega} (\mathbf{M} \nabla \otimes \mathbf{A}) \cdot \Psi \, d\mathbf{x} = \int_{\Omega} (\nabla \otimes \mathbf{A}) \cdot \text{trace}(\mathbf{M} \otimes \Psi^{\sim}) \, d\mathbf{x}.$$

Let $\Phi := \text{trace}(\mathbf{M} \otimes \Psi^{\sim}) \in \mathcal{T}^{\ell+1}(\mathbb{R}^d)$. From the Gauss–Green theorem, in Einstein notation,

$$\begin{aligned} \int_{\Omega} (\nabla \otimes \mathbf{A}) \cdot \Phi \, d\mathbf{x} &= \int_{\Omega} \partial_k \mathbf{A}_{i_1, \dots, i_{\ell}} \Phi_{k, i_1, \dots, i_{\ell}} \, d\mathbf{x} \\ &= \int_{\partial\Omega} \nu_k \mathbf{A}_{i_1, \dots, i_{\ell}} \Phi_{k, i_1, \dots, i_{\ell}} \, d\mathcal{H}^{d-1} - \int_{\Omega} \mathbf{A}_{i_1, \dots, i_{\ell}} \partial_k (\Phi^{\sim})_{i_1, \dots, i_{\ell}, k} \, d\mathbf{x}. \end{aligned}$$

Now, by remarking that

$$\nu_k \mathbf{A}_{i_1, \dots, i_\ell} = (\nu \otimes \mathbf{A})_{k, i_1, \dots, i_\ell}$$

and

$$\partial_k (\Phi^\sim)_{i_1, \dots, i_\ell, k} = (\nabla \otimes \Phi^\sim)_{k, i_1, \dots, i_\ell, k} = (\text{trace}(\nabla \otimes \Phi^\sim))_{i_1, \dots, i_\ell},$$

we conclude

$$\begin{aligned} \int_{\Omega} (\mathbf{M} \nabla \otimes \mathbf{A}) \cdot \Psi \, dx &= \int_{\partial\Omega} (\nu \otimes \mathbf{A}) \cdot \text{trace}(\mathbf{M} \otimes \Psi^\sim) \, d\mathcal{H}^{d-1} \\ &\quad - \int_{\Omega} \mathbf{A} \cdot \text{trace}(\nabla \otimes [\text{trace}(\mathbf{M} \otimes \Psi^\sim)]^\sim) \, dx. \end{aligned}$$

■

Remark 3.5. For $\Psi \in C_c^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, in Lemma 3.4 the integral on $\partial\Omega$ vanishes:

$$\int_{\Omega} (\mathbf{M} \nabla \otimes \mathbf{A}) \cdot \Psi \, dx = - \int_{\Omega} \mathbf{A} \cdot \text{div}_{\mathbf{M}} \Psi \, dx.$$

With the notion of weighted divergence $\text{div}_{\mathbf{M}}$ of a $(\ell+1)$ -tensor field in place, we can talk about *weak derivatives*, similarly to [7, Definition 2.4].

Definition 3.6. Let $\mathbf{M} \in C^1(\Omega, \mathcal{T}^2(\mathbb{R}^d))$. We say that $\mathbf{A} \in L_{loc}^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ has a weak $\mathbf{M}$ -weighted derivative if there exists $\boldsymbol{\eta} \in L_{loc}^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ such that

$$\int_{\Omega} \boldsymbol{\eta} \cdot \Psi \, dx = - \int_{\Omega} \mathbf{A} \cdot \text{div}_{\mathbf{M}} \Psi \, dx$$

for all $\Psi \in C_c^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$. We write $\boldsymbol{\eta} = \mathbf{M} \nabla \otimes \mathbf{A}$ in this case.

We can now define the TDV of order Q for $\mathbf{u} \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$.

Definition 3.7. Let $\Omega \subset \mathbb{R}^d$, $\mathbf{u} \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, $Q \in \mathbb{N}$, $\mathcal{M} := (\mathbf{M}_j)_{j=1}^Q$ be a collection of fields in $C^\infty(\Omega, \mathcal{T}^2(\mathbb{R}^d))$ and $\boldsymbol{\alpha} := (\alpha_0, \dots, \alpha_{Q-1})$ be a positive weight vector. The TDV of order Q , associated with $\mathcal{M}$ and $\boldsymbol{\alpha}$, is defined as

$$(3.4) \quad \text{TDV}_{\boldsymbol{\alpha}}^{Q, \ell}(\mathbf{u}, \mathcal{M}) := \sup_{\Psi} \left\{ \int_{\Omega} \mathbf{u} \cdot \text{div}_{\mathcal{M}}^Q \Psi \, dx \mid \text{for all } \Psi \in \mathcal{Y}_{\mathcal{M}, \boldsymbol{\alpha}}^{Q, \ell} \right\},$$

where

$$(3.5) \quad \mathcal{Y}_{\mathcal{M}, \boldsymbol{\alpha}}^{Q, \ell} = \left\{ \Psi : \Psi \in C_c^Q(\Omega, \mathcal{T}^{\ell+Q}(\mathbb{R}^d)), \left\| \text{div}_{\mathcal{M}}^j \Psi \right\|_{\infty} \leq \alpha_j, \forall j = 0, \dots, Q-1 \right\}$$

and the weighted divergence of order $j \in [0, Q]$ is defined recursively, from Lemma 3.4, as

$$(3.6) \quad \begin{aligned} \text{div}_{(\cdot)}^0(\Psi) &:= \Psi && \text{if } j = 0, \\ \text{div}_{(\mathbf{M}_Q)}^1(\Psi) &:= \text{trace}(\nabla \otimes [\text{trace}(\mathbf{M}_Q \otimes \Psi^\sim)]^\sim) && \text{if } j = 1, \\ \text{div}_{(\mathbf{M}_{Q-j+1}, \dots, \mathbf{M}_Q)}^j(\Psi) &:= \text{div}_{(\mathbf{M}_{Q-j+1})}^1 \left(\text{div}_{(\mathbf{M}_{Q-j+2}, \dots, \mathbf{M}_Q)}^{j-1}(\Psi) \right) && \text{if } j = 2, \dots, Q. \end{aligned}$$

Thus, the Q th weighted divergence w.r.t. $\mathcal{M}$ is $\text{div}_{\mathcal{M}}^Q(\Psi) := \text{div}_{\mathbf{M}_1}^1(\text{div}_{\mathbf{M}_2}^1(\dots(\text{div}_{\mathbf{M}_Q}^1(\Psi))))$.

Remark 3.8. For $\mathcal{M} = (\mathbf{I})_{j=1}^Q$, where $\mathbf{I}$ is the identity matrix, $\text{TDV}_{\boldsymbol{\alpha}}^{Q, \ell}(\mathbf{u}, \mathcal{M})$ coincides with extension to $\mathcal{T}^\ell(\mathbb{R}^d)$ tensors of the nonsymmetric total generalized variation $\neg\text{symTGV}_{\boldsymbol{\alpha}}^{Q, \ell}(\mathbf{u})$ defined (for $\ell = 0$) in [9, Remark 3.10].

4. Tensor fields of bounded directional variation. In what follows, we introduce the space of bounded directional variation $\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$, which is the natural space for the TDV regularizer. We also state some results about the kernel of the weighted derivatives. To do so, we will treat the discussion of these spaces for first- and higher-order derivatives separately so as to build a recursion rule for tensors of bounded directional variation with weighted derivatives of any order $Q > 0$.

4.1. First-order derivative. As said, when $Q = 1$, the collection $\mathcal{M}$ is made by one smooth tensor field only, namely, $\mathbf{M}$: therefore, we will use $\mathbf{M}$ within this section. We will always assume that $\mathbf{M}(\mathbf{x})$ can be identified with a positive definite matrix at every point of Ω .

Remark 4.1. For $Q = 1$, when $\text{TDV}_\alpha^{1,\ell}(\mathbf{u}) < \infty$ in (3.4), the weak weighted derivative is a Radon measure on Ω with values in $\mathcal{T}^{\ell+1}(\mathbb{R}^d)$.

Definition 4.2. The TDV of a $\mathcal{T}^\ell(\mathbb{R}^d)$ -valued function $\mathbf{u}$ w.r.t. the field $\mathbf{M}$ is defined as the Radon norm of $\mathbf{M} \nabla \otimes \mathbf{u}$ and indicated as

$$(4.1) \quad \text{TDV}_1^{1,\ell}(\mathbf{u}, \mathbf{M}) = \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))}.$$

Definition 4.3. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathbf{M} \in C^\infty(\Omega, \mathcal{T}^2(\mathbb{R}^d))$ such that $\mathbf{M}(\mathbf{x})$ is a positive definite matrix at every $\mathbf{x} \in \Omega$. The space of $\mathcal{T}^\ell(\mathbb{R}^d)$ -valued tensor functions $\mathbf{u}$ of bounded directional variation of order 1 with respect to the field $\mathbf{M}$ is defined as

$$\text{BDV}^1(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d)) = \left\{ \mathbf{u} \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \mid \mathbf{M} \nabla \otimes \mathbf{u} \in \mathcal{M}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)) \right\}.$$

For simplicity, we denote $\text{BDV}(\mathbf{u}, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d)) = \text{BDV}^1(\mathbf{u}, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$.

Remark 4.4. Since $\mathcal{T}^\ell(\mathbb{R}^d) \equiv \text{Sym}^\ell(\mathbb{R}^d)$ for $\ell = 0, 1$, it is easily seen that

$$\text{BDV}(\Omega, \mathbf{I}, \mathcal{T}^0(\mathbb{R}^d)) \equiv \text{BV}(\Omega, \mathbb{R}) \quad \text{and} \quad \text{BDV}(\Omega, \mathbf{I}, \mathcal{T}^1(\mathbb{R}^d)) \equiv \text{BV}(\Omega, \mathbb{R}^d),$$

with $\text{BV}(\Omega, \mathbb{R})$, $\text{BV}(\Omega, \mathbb{R}^d)$ the spaces of scalar-valued and vector-valued functions of bounded variation, respectively [2].

We now prove that tensor fields of bounded directional variation can be approximated by smooth functions, similarly to [7, Proposition 4.13]. To do so, we first need to show that the weighted gradient is closed, similarly to [7, Proposition 4.2].

Proposition 4.5. Let $p \in [1, \infty]$. If $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ and $\mathbf{M} \nabla \otimes \mathbf{u}_j \rightharpoonup \boldsymbol{\eta}$ in $L^p(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$, then $\mathbf{M} \nabla \otimes \mathbf{u} = \boldsymbol{\eta}$; i.e., the weighted gradient is closed in the distributional sense. The statement remains true for weak-* convergence in $\mathcal{M}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ and $\mathcal{M}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$, respectively.

Proof. The proof is omitted since it is just a notational adaptation of [7, Proposition 4.2]. ■

Similarly to [7, Proposition 4.13], we can approximate functions of bounded directional deformation with smooth functions.

Proposition 4.6. *Let Ω be a bounded domain. The set $C^\infty(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \cap \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ is dense in $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ in the sense that for each $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ there exists an approximating sequence $\{\mathbf{u}_j\}_{j \in \mathbb{N}} \subset C^\infty(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ that converges strictly to $\mathbf{u}$, i.e.,*

$$(4.2) \quad \begin{cases} \mathbf{u}_j \rightarrow \mathbf{u} & \text{in } L^1(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d)), \\ \mathbf{M} \nabla \otimes \mathbf{u}_j \xrightarrow{*} \mathbf{M} \nabla \otimes \mathbf{u} & \text{in } \mathcal{M}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)), \\ \|\mathbf{M} \nabla \otimes \mathbf{u}_j\|_{\mathcal{M}} \rightarrow \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}}. \end{cases}$$

If the support of $\mathbf{u}$ is compact in Ω , then $(\mathbf{u}_j)_{j \in \mathbb{N}}$ can be chosen such that each $\mathbf{u}_j$ is in $C_c^\infty(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$.

Proof. The proof is based on a standard use of mollifiers so as to obtain a sequence $(\mathbf{u}_j)_{j \in \mathbb{N}}$ in $C^\infty(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ satisfying the first and the third convergence in (4.2). The boundedness of $(\mathbf{M} \nabla \otimes \mathbf{u}_j)_{j \in \mathbb{N}}$ in $\mathcal{M}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ implies that there exists a subsequence (not relabeled) weakly-* converging to $\mathbf{M} \nabla \otimes \mathbf{u}$ since the operator $\mathbf{M} \nabla$ is closed by Proposition 4.5. ■

We are now going to discuss some results about the kernel of the weighted derivative operator $\ker(\mathbf{M} \nabla)$. In order to do so, we will define a continuous projection map R onto $\ker(\mathbf{M} \nabla)$ so as to prove the coercivity estimate for the TDV in (4.4).

Remark 4.7. Being $\ker(\mathbf{M} \nabla)$ the space of polynomials of vanishing first weighted derivative, it is in $L^\infty(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ because Ω is bounded; therefore,

$$\ker(\mathbf{M} \nabla)^\perp = \left\{ \mathbf{f} \in L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \mid \int_{\Omega} \mathbf{f} \cdot \mathbf{u} = 0 \text{ d}\mathbf{x} \text{ for all } \mathbf{u} \in \ker(\mathbf{M} \nabla) \right\}$$

is a closed subspace of $L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$.

Remark 4.8. Note also that $\ker(\mathbf{M} \nabla) \equiv \ker(\nabla)$ since the field $\mathbf{M}$ is assumed everywhere invertible.

Proposition 4.9. *There exists a continuous projection $R : L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \rightarrow L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ such that*

$$\text{Im}(R) = \ker(\mathbf{M} \nabla) = \ker(\nabla) \quad \text{and} \quad \ker(R) = \ker(\mathbf{M} \nabla)^\perp = \ker(\nabla)^\perp.$$

Proof. The proof is an easy adaptation of the proof given at the beginning of [8, Appendix A]. We observe that $\ker(\mathbf{M} \nabla)$ is finite-dimensional; therefore,

$$L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) = \ker(\mathbf{M} \nabla) \oplus \ker(\mathbf{M} \nabla)^\perp,$$

and since both subspaces are closed, then the open mapping theorem implies that there exists a continuous projection R such that

$$(4.3) \quad R : L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \rightarrow L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$$

with $\text{Im}(R) = \ker(\mathbf{M} \nabla)$ and $\ker(R) = \ker(\mathbf{M} \nabla)^\perp$; see [10, Example 1, page 38]. As a consequence, the adjoint projection R^* is a continuous projection in $L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ onto $\ker(\mathbf{M} \nabla)^{\perp\perp} = \ker(\mathbf{M} \nabla)$. ■

The following Sobolev–Korn inequality holds, similarly to [7, Corollary 4.20], which will be proved for the general case $Q \geq 1$ in Proposition 6.7.

Lemma 4.10. For any continuous projection R onto $\ker(\mathbf{M} \nabla)$ as in (4.3), there exists a constant $C > 0$, depending only on Ω , R , and $\mathbf{M}^{-1}$, such that it holds for each $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ that

$$(4.4) \quad \|\mathbf{u} - R\mathbf{u}\|_{d/(d-1)} \leq C \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}}.$$

Proof. We first need to prove $\|\mathbf{u} - R\mathbf{u}\|_1 \leq C \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}}$. This follows by the same proof in [7, Theorem 4.19] with minor notational changes. From the continuous embedding of BDV into $L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ proved later in Theorem 6.4 (with (6.3) in place), we have

$$\|\mathbf{u} - R\mathbf{u}\|_{d/(d-1)} \leq \tilde{C} (\|\mathbf{u} - R\mathbf{u}\|_1 + \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}}) \leq C \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}}. \quad \blacksquare$$

Definition 4.11. Let $B_\varepsilon(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 \leq \varepsilon\}$ be the ℓ^2 -closed ε -ball centered at $\mathbf{0} \in \mathbb{R}^d$ and $B_{\mathbf{M}, \varepsilon}(\mathbf{0}) = \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{M}\mathbf{x}\|_2 \leq \varepsilon\}$ be the $\mathbf{M}$ -anisotropic closed ε -ball centered at $\mathbf{0} \in \mathbb{R}^d$.

Similarly to [8, Lemma A.1], we can prove the following lemma.

Lemma 4.12. The closure of the set

$$U = \left\{ -\operatorname{div}_{\mathbf{M}} \Psi \mid \Psi \in C_c^\infty(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)), \|\Psi\|_\infty \leq 1 \right\}$$

in $L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \cap \ker(\mathbf{M} \nabla)^\perp$ contains $\mathbf{0}$ as an interior point.

Proof. We have to check that the functional $F : L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \rightarrow [0, \infty]$ is coercive:

$$F(\mathbf{u}) = \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}} + I_{\{\mathbf{0}\}}(R^* \mathbf{u}).$$

Here R is the continuous projection map defined in (4.3), and I_Z is the indicator function of this set, i.e., $I_Z(\mathbf{x}) = 0$ if $\mathbf{x} \in Z$ and $I_Z(\mathbf{x}) = \infty$ otherwise.

Let $(\mathbf{u}_j)_j \in L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ with $\|\mathbf{u}_j\|_{d/(d-1)} \rightarrow \infty$. We can distinguish two cases: either $F(\mathbf{u}_j) = \infty$ or $F(\mathbf{u}_j) < \infty$, which is the case for $\mathbf{u}_j \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d)) \cap \ker(R^*)$.

When $F(\mathbf{u}_j) < \infty$, $R^* \mathbf{u}_j = 0$, and the Sobolev–Korn inequality in (4.4) gives

$$\|\mathbf{u}_j\|_{d/(d-1)} \leq C \|\mathbf{M} \nabla \otimes \mathbf{u}_j\|_{\mathcal{M}} = CF(\mathbf{u}_j)$$

for a constant $C > 0$, independently of j . This means that $F(\mathbf{u}_j) \rightarrow \infty$ and the coercivity is proved. Thus, the Fenchel conjugate of F

$$F^* : L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \rightarrow]-\infty, \infty]$$

is continuous at $\mathbf{0}$ [6, Theorem 4.4.10]. Since $\ker(R^*) = \operatorname{Im}(\mathbf{I} - R^*)$, we have

$$\begin{aligned} F^*(\Psi) &= \sup_{\mathbf{u} \in \ker(R^*)} \langle \Psi, \mathbf{u} \rangle - \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}} \\ &= \sup_{\mathbf{u} \in L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))} \langle \Psi, \mathbf{u} - R^* \mathbf{u} \rangle - \|\mathbf{M} \nabla \otimes (\mathbf{u} - R^* \mathbf{u})\|_{\mathcal{M}} \\ &= \sup_{\mathbf{u} \in L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))} \langle \Psi - R\Psi, \mathbf{u} \rangle - \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}} \\ &= I_U^{**}(\Psi - R\Psi) \\ &= I_{\bar{U}}(\Psi - R\Psi). \end{aligned}$$

The continuity in $\mathbf{0}$ implies that there exists $\varepsilon > 0$ such that the anisotropic ball $B_{\mathbf{M},\varepsilon}$ induced by $\mathbf{M}$ is such that $B_{\mathbf{M},\varepsilon}(\mathbf{0}) \subset (I - R)^{-1}(\bar{U})$. Thus, for each $\Psi \in L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \cap \ker(\mathbf{M} \nabla)^\perp$ with $\|\Psi\|_d \leq \varepsilon$, we have $\Psi = \Psi - R\Psi \in \bar{U}$, showing that $\mathbf{0}$ is an interior point. ■

We can now prove that a distribution $\mathbf{u}$ is in $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ as soon as the weighted derivative $\mathbf{M} \nabla$ is a Radon measure, similarly to [8, Theorem 2.6].

Theorem 4.13. *Let $\Omega \in \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathbf{u} \in \mathcal{D}'(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ be a distribution such that $\mathbf{M} \nabla \otimes \mathbf{u} \in \mathcal{M}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$ in the distributional sense for a positive definite field $\mathbf{M} \in C^\infty(\Omega, \mathcal{T}^2(\mathbb{R}^d))$. Then $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$.*

Proof. Let $\mathbf{u} \in \mathcal{D}'(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ be such that $\mathbf{M} \nabla \otimes \mathbf{u} \in \mathcal{M}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$ in the distributional sense. We need to prove that $\mathbf{u} \in L^1(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$.

Let $X = L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \cap \ker(\mathbf{M} \nabla)^\perp$, which is a Banach space with the induced norm. Let $\delta > 0$ and U from Lemma 4.12 such that $B_{\mathbf{M},\delta}(\mathbf{0})$ exists and $B_{\mathbf{M},\delta}(\mathbf{0}) \subset \bar{U} \subset X$. We define also the following sets:

$$K_1 = \left\{ \Psi \in C_c^\infty(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)) \mid \|\Psi\|_\infty \leq \delta^{-1}, \|\text{div}_{\mathbf{M}} \Psi\|_d \leq 1 \right\},$$

$$K_2 = \left\{ \Psi \in C_c^\infty(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)) \mid \|\Psi\|_\infty \leq \delta^{-1} \right\}.$$

Straightforwardly, we have $K_1 \subset K_2$. By testing $\mathbf{u}$ with $-\text{div}_{\mathbf{M}} \Psi$ and $\Psi \in K_1$, since $\mathbf{M} \nabla \otimes \mathbf{u} \in \mathcal{M}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$, we get by density

$$\begin{aligned} \sup_{\Psi \in K_1} \langle \mathbf{u}, -\text{div}_{\mathbf{M}} \Psi \rangle &\leq \sup_{\Psi \in K_2} \langle \mathbf{u}, -\text{div}_{\mathbf{M}} \Psi \rangle = \sup_{\substack{\Psi \in C_0(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)) \\ \|\Psi\|_\infty \leq \delta^{-1}}} \langle \mathbf{M} \nabla \otimes \mathbf{u}, \Psi \rangle \\ &= \delta^{-1} \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}}. \end{aligned}$$

One can show that $\overline{\{-\text{div}_{\mathbf{M}} \Psi \mid \Psi \in K_1\}} = \overline{B_{\mathbf{M},1}(\mathbf{0})} \in X$ and thus

$$\sup_{\Psi \in K_1} \langle \mathbf{u}, -\text{div}_{\mathbf{M}} \Psi \rangle = \|\mathbf{u}\|_{X^*};$$

i.e., $\mathbf{u}$ can be extended to an element in X^* . Also, X is a closed subspace of $L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, and by the Hahn–Banach theorem $\mathbf{u}$ can be extended to $\mathbf{v} \in L^d(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))^* = L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$. Thus, $\mathbf{v} \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ with the distribution $\mathbf{u} - \mathbf{v} \in \ker(\mathbf{M} \nabla)$, and we have

$$\langle \mathbf{u} - \mathbf{v}, -\text{div}_{\mathbf{M}} \Psi \rangle = \langle \mathbf{u} - \mathbf{u}, -\text{div}_{\mathbf{M}} \Psi \rangle = \mathbf{0} \quad \text{for each } \Psi \in C_c^\infty(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$$

since $-\text{div}_{\mathbf{M}} \Psi \in X$. Thus, $\mathbf{u} - \mathbf{v}$ is a polynomial of degree less than ℓ , $(\mathbf{u} - \mathbf{v}) \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ and $\mathbf{u} = \mathbf{v} + (\mathbf{u} - \mathbf{v}) \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, leading to $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$. ■

4.2. Higher-order derivatives. When Q order of derivatives are involved, we deal with the collection of tensor fields $\mathcal{M} = (\mathbf{M}_j)_{j=1}^Q$. For a distribution $\mathbf{u} \in \mathcal{D}'(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, we get from Theorem 4.13

$$\mathbf{M}_Q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes \mathbf{u} \in \mathcal{M}(\Omega, \mathcal{T}^{\ell+Q}(\mathbb{R}^d)),$$

which implies

$$\mathbf{M}_{Q-1} \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes \mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}_Q, \mathcal{T}^{\ell+Q-1}(\mathbb{R}^d));$$

thus, we have

$$\|\mathbf{M}_Q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes \mathbf{u}\|_{\mathcal{M}} = \text{TDV}^{1, \ell+Q-1}(\mathbf{M}_{Q-1} \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes \mathbf{u}, \mathbf{M}_Q).$$

Definition 4.14. The TDV of order Q of a $\mathcal{T}^\ell(\mathbb{R}^d)$ -valued function $\mathbf{u}$ w.r.t. the collection of fields $\mathcal{M}$ is defined as the Radon norm of $\mathbf{M}_Q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes \mathbf{u}$ and indicated as

$$(4.5) \quad \text{TDV}_1^{Q, \ell}(\mathbf{u}, \mathcal{M}) = \|\mathbf{M}_Q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes \mathbf{u}\|_{\mathcal{M}(\Omega, \mathcal{T}^{\ell+Q}(\mathbb{R}^d))}.$$

Definition 4.15. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathcal{M} = (\mathbf{M}_j)_{j=1}^Q$ be a collection of smooth tensor fields such that $\mathbf{M}_j \in C^\infty(\Omega, \mathcal{T}^2(\mathbb{R}^d))$ for each $j = 1, \dots, Q$. The space of $\mathcal{T}^\ell(\mathbb{R}^d)$ -valued tensor functions $\mathbf{u}$ of bounded directional variation of order Q with respect to the collection of fields $\mathcal{M}$ is defined as

$$\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d)) = \left\{ \mathbf{u} \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \mid \mathbf{M}_Q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes \mathbf{u} \in \mathcal{M}(\Omega, \mathcal{T}^{\ell+Q}(\mathbb{R}^d)) \right\}.$$

In particular, the spaces are nested, and the larger is Q , the smaller is the space. The space $\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ is endowed with the following norm:

$$\|\mathbf{u}\|_{\text{BDV}^Q} = \|\mathbf{u}\|_1 + \|\mathbf{M}_Q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes \mathbf{u}\|_{\mathcal{M}}.$$

Remark 4.16. For fixed ℓ, Q and by changing the weights α , $\text{TDV}_\alpha^{Q, \ell}$ yields equivalent norms and hence the same space. Thus, we can omit the weights in $\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$.

4.3. Properties.

Proposition 4.17. Given $\mathcal{M} = (\mathbf{M}_j)_{j=1}^Q$, $\text{TDV}_\alpha^{Q, \ell}(\cdot, \mathcal{M})$ is a continuous seminorm on $\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ with finite-dimensional kernel $\ker(\mathbf{M}_Q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla)$.

Proof. Positive homogeneity is ensured by definition of $\text{TDV}_\alpha^{Q, \ell}$: from the linearity of the integral, we have

$$\text{TDV}_\alpha^{Q, \ell}(\lambda \mathbf{u}, \mathcal{M}) = |\lambda| \text{TDV}_\alpha^{Q, \ell}(\mathbf{u}, \mathcal{M}).$$

For the triangular inequality, take $\mathbf{u}_1, \mathbf{u}_2 \in \text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$, and let $\Psi \in \mathcal{Y}_{\mathcal{M}, \alpha}^{Q, \ell}$. Then

$$\text{TDV}_\alpha^{Q, \ell}(\mathbf{u}_1 + \mathbf{u}_2, \mathcal{M}) \leq \sup_{\Psi} \int_{\Omega} (\mathbf{u}_1 + \mathbf{u}_2) \cdot \text{div}_{\mathcal{M}}^Q(\Psi) \, d\mathbf{x} \leq \text{TDV}_\alpha^{Q, \ell}(\mathbf{u}_1, \mathcal{M}) + \text{TDV}_\alpha^{Q, \ell}(\mathbf{u}_2, \mathcal{M}).$$

For the continuity, let $\mathbf{u}_1, \mathbf{u}_2 \in \text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$. Then it holds, exactly as in the BV case, that

$$\left| \text{TDV}_\alpha^{Q, \ell}(\mathbf{u}_1, \mathcal{M}) - \text{TDV}_\alpha^{Q, \ell}(\mathbf{u}_2, \mathcal{M}) \right| \leq \text{TDV}_\alpha^{Q, \ell}(\mathbf{u}_1 - \mathbf{u}_2, \mathcal{M}) \leq \|\mathbf{u}_1 - \mathbf{u}_2\|_{\text{BDV}^Q}.$$

By definition of $\text{TDV}_\alpha^{Q, \ell}(\mathbf{u}_1, \mathcal{M})$, we have $\text{TDV}_\alpha^{Q, \ell}(\mathbf{u}_1, \mathcal{M}) = 0$ if and only if

$$\int_{\Omega} \mathbf{u} \cdot \text{div}_{\mathcal{M}}^Q \Psi \, d\mathbf{x} = 0 \quad \text{for each } \Psi \in C_c^Q(\Omega, \mathcal{T}^{\ell+Q}(\mathbb{R}^d)),$$

which is equivalent to $\mathbf{u} \in \ker(\mathbf{M}_Q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla)$ in the weak sense. Therefore, $\text{TDV}_\alpha^{Q,\ell}$ is a seminorm and BDV^Q a normed linear space. From Remark 4.7, $\ker(\mathbf{M}_j \nabla)$ on $\mathcal{D}'(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$ has finite dimension for each $j = 0, \dots, Q-1$; then $\ker(\mathbf{M}_Q \nabla)$ considered on $\mathcal{D}'(\Omega, \mathcal{T}^{\ell+Q}(\mathbb{R}^d))$ is finite-dimensional, and therefore $\ker(\mathbf{M}_Q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla)$ on $\mathcal{D}'(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ is finite-dimensional. ■

Proposition 4.18. $\text{TDV}_\alpha^{Q,\ell}(\cdot, \mathcal{M})$ is convex and lower semicontinuous on

$$\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d)).$$

Proof. Fix $Q, \ell \in \mathbb{N}$, let $\mathcal{M}$ be a collection of fields in $\mathcal{T}^2(\mathbb{R}^d)$, and let $\Psi \in \mathcal{Y}_{\mathcal{M}, \alpha}^{Q,\ell}$. Then for any $\mathcal{M}$ and α , we take $\mathbf{u}_1, \mathbf{u}_2 \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ and $t \in [0, 1]$. Thus,

$$\begin{aligned} \text{TDV}_\alpha^{Q,\ell}(t\mathbf{u}_1 + (1-t)\mathbf{u}_2, \mathcal{M}) &= \sup_{\Psi} \int_{\Omega} (t\mathbf{u}_1 + (1-t)\mathbf{u}_2) \cdot \text{div}_{\mathcal{M}}^Q(\Psi) \, dx \\ &\leq t \sup_{\Psi} \int_{\Omega} \mathbf{u}_1 \cdot \text{div}_{\mathcal{M}}^Q(\Psi) \, dx + (1-t) \sup_{\Psi} \int_{\Omega} \mathbf{u}_2 \cdot \text{div}_{\mathcal{M}}^Q(\Psi) \, dx \\ &= t \text{TDV}_\alpha^{Q,\ell}(\mathbf{u}_1, \mathcal{M}) + (1-t) \text{TDV}_\alpha^{Q,\ell}(\mathbf{u}_2, \mathcal{M}). \end{aligned}$$

Hence, $\text{TDV}_\alpha^{Q,\ell}$ is convex. For the lower semicontinuity, let $(\mathbf{u}_j)_{j \in \mathbb{N}}$ be a Cauchy sequence in $\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ such that $\mathbf{u}_j \rightarrow \mathbf{u} \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$. From the definition of $\text{TDV}_\alpha^{Q,\ell}$, we have

$$\int_{\Omega} \mathbf{u} \cdot \text{div}_{\mathcal{M}}^Q \Psi \, dx = \lim_{j \rightarrow \infty} \int_{\Omega} \mathbf{u}_j \cdot \text{div}_{\mathcal{M}}^Q \Psi \, dx \leq \liminf_{j \rightarrow \infty} \text{TDV}_\alpha^{Q,\ell}(\mathbf{u}_j, \mathcal{M}) \quad \text{for any } \Psi \in \mathcal{Y}_{\mathcal{M}, \alpha}^{Q,\ell}.$$

Then, taking the supremum, we have $\text{TDV}_\alpha^{Q,\ell}(\mathbf{u}, \mathcal{M}) \leq \liminf_{j \rightarrow \infty} \text{TDV}_\alpha^{Q,\ell}(\mathbf{u}_j, \mathcal{M})$. ■

Similarly to [9, Proposition 3.5], the space $\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ is a Banach space when equipped with a suitable norm.

Proposition 4.19. $\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$, endowed with the norm

$$\|\mathbf{u}\|_{\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))} = \|\mathbf{u}\|_{L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))} + \text{TDV}_\alpha^{Q,\ell}(\mathbf{u}, \mathcal{M}),$$

is a Banach space.

Proof. We have already proved in Proposition 4.18 the lower semicontinuity of $\text{TDV}_\alpha^{Q,\ell}$. As in [9], let $(\mathbf{u}_j)_{j \in \mathbb{N}}$ be a Cauchy sequence in $\text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$. Then it is easy to see that $(\mathbf{u}_j)$ is a Cauchy sequence in $L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ and a limit $\mathbf{u} \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ exists. Thus, by lower semicontinuity, we have

$$\text{TDV}_\alpha^{Q,\ell}(\mathbf{u}, \mathcal{M}) \leq \liminf_{j \rightarrow \infty} \text{TDV}_\alpha^{Q,\ell}(\mathbf{u}_j, \mathcal{M}).$$

So, $\mathbf{u} \in \text{BDV}^Q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$, and we need only to check that $\mathbf{u}$ is the limit in the corresponding norm: Being $(\mathbf{u}_j)_{j \in \mathbb{N}}$ a Cauchy sequence, we can choose $\varepsilon > 0$ and an index j^* such that, for all $j > j^*$, we have

$$\text{TDV}_\alpha^{Q,\ell}(\mathbf{u}_{j^*} - \mathbf{u}_j, \mathcal{M}) \leq \varepsilon.$$

Letting $j \rightarrow \infty$, the lower semicontinuity of $\text{TDV}_{\alpha}^{\mathbf{Q},\ell}$ on $L^1(\Omega, \mathcal{T}^{\ell}(\mathbb{R}^d))$ gives

$$\text{TDV}_{\alpha}^{\mathbf{Q},\ell}(\mathbf{u}_{j^*} - \mathbf{u}, \mathcal{M}) \leq \liminf_{j \rightarrow \infty} \text{TDV}_{\alpha}^{\mathbf{Q},\ell}(\mathbf{u}_{j^*} - \mathbf{u}_j, \mathcal{M}) \leq \varepsilon,$$

and this implies that $\mathbf{u}_j \rightarrow \mathbf{u}$ in $\text{BDV}^{\mathbf{Q}}(\Omega, \mathcal{M}, \mathcal{T}^{\ell}(\mathbb{R}^d))$. ■

5. Equivalent representation. We are going to interpret the dual definition of the regularizer $\text{TDV}_{\alpha}^{\mathbf{Q},\ell}(\mathbf{u}, \mathcal{M})$ in terms of iterated Fenchel duality following the proof given in [8]. First, we prove the following preliminary result, similarly to [8, Lemma 3.4].

Lemma 5.1. *Let $j \geq 1$, and let $\mathbf{z}_{j-1} \in C_0^{j-1}(\Omega, \mathcal{T}^{\ell+j-1}(\mathbb{R}^d))^*$, $\mathbf{z}_j \in C_0^j(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))^*$ be distributions of order $j-1$ and j , respectively. Then*

$$(5.1) \quad \|\mathbf{M}_j \nabla \mathbf{z}_{j-1} - \mathbf{z}_j\|_{\mathcal{M}} = \sup_{\substack{\Psi \in C_c^{\infty}(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d)), \\ \|\Psi\|_{\infty} \leq 1}} \{ \langle \mathbf{z}_{j-1}, \text{div}_{\mathbf{M}_j} \Psi \rangle + \langle \mathbf{z}_j, \Psi \rangle \},$$

with the right-hand side being finite if and only if $\mathbf{M}_j \nabla \mathbf{z}_{j-1} - \mathbf{z}_j \in \mathcal{M}(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$ in the distributional sense.

Proof. In the distributional sense, we have, for all $\Psi \in C_c^{\infty}(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$,

$$\langle \mathbf{z}_j - \mathbf{M}_j \nabla \mathbf{z}_{j-1}, \Psi \rangle = \langle \mathbf{z}_j, \Psi \rangle + \langle \mathbf{z}_{j-1}, \text{div}_{\mathbf{M}_j} \Psi \rangle.$$

Since $C_c^{\infty}(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$ is dense in $C_0(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$, the distribution $\mathbf{z}_j - \mathbf{M}_j \nabla \mathbf{z}_{j-1}$ can be extended to an element in $C_0(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))^* = \mathcal{M}(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$ if and only if the supremum in (5.1) is finite, in which case it coincides with the Radon norm by definition. ■

Finally, we are ready to show the *minimum representation* of $\text{TDV}_{\alpha}^{\mathbf{Q},\ell}$, similarly to [8, Theorem 3.5].

Proposition 5.2. *Let $\mathbf{u} \in L_{loc}^1(\Omega, \mathcal{T}^{\ell}(\mathbb{R}^d))$, $\text{TDV}_{\alpha}^{\mathbf{Q},\ell}(\mathbf{u}, \mathcal{M})$ be defined as in Definition 3.7 and $\mathcal{M} = (\mathbf{M}_j)_{j=1}^{\mathbf{Q}}$ be a collection of positive definite tensor fields such that $\mathbf{M}_j \in \mathcal{T}^2(\mathbb{R}^d)$ for all j . Then it holds that*

$$(5.2) \quad \text{TDV}_{\alpha}^{\mathbf{Q},\ell}(\mathbf{u}, \mathcal{M}) = \min_{\substack{\mathbf{z}_j \in \text{BDV}(\Omega, \mathbf{M}_{j+1}, \mathcal{T}^{\ell+j}(\mathbb{R}^d)) \\ j=1, \dots, \mathbf{Q}-1, \\ \mathbf{z}_0 = \mathbf{u}, \mathbf{z}_{\mathbf{Q}} = \mathbf{0}}} \sum_{j=1}^{\mathbf{Q}} \alpha_{\mathbf{Q}-j} \|\mathbf{M}_j \nabla \mathbf{z}_{j-1} - \mathbf{z}_j\|_{\mathcal{M}},$$

with the minimum being finite if and only if $\mathbf{z}_j \in \text{BDV}(\Omega, \mathbf{M}_{j+1}, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$ for each $j = 0, \dots, \mathbf{Q}-1$, with $\mathbf{z}_0 = \mathbf{u}$ and $\mathbf{z}_{\mathbf{Q}} = \mathbf{0}$.

Proof. Let $\mathbf{u} \in L_{loc}^1(\Omega, \mathcal{T}^{\ell}(\mathbb{R}^d))$ be such that $\text{TDV}_{\alpha}^{\mathbf{Q},\ell}(\mathbf{u}, \mathcal{M}) < \infty$. In order to make use of the Fenchel–Rockafellar duality, we introduce the following Banach spaces:

$$\begin{aligned} X &= C_0^1(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)) \times \dots \times C_0^{\mathbf{Q}}(\Omega, \mathcal{T}^{\ell+\mathbf{Q}}(\mathbb{R}^d)), \\ Y &= C_0^1(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)) \times \dots \times C_0^{\mathbf{Q}-1}(\Omega, \mathcal{T}^{\ell+\mathbf{Q}-1}(\mathbb{R}^d)). \end{aligned}$$

Let $\mathbf{z} = (z_1, \dots, z_{Q-1}) \in Y$ be the primal variable, $\mathbf{w} = (w_1, \dots, w_Q) \in X$ be the dual variables, and $\mathcal{K} \in \mathcal{L}(X, Y)$ be the linear operator defined as

$$\mathcal{K} = \begin{pmatrix} -\mathbf{I} & -\operatorname{div}_{\mathbf{M}_2} & \mathbf{0} & \dots & \dots & \dots & \dots & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & -\operatorname{div}_{\mathbf{M}_3} & \mathbf{0} & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \mathbf{0} & -\mathbf{I} & -\operatorname{div}_{\mathbf{M}_{j+1}} & \mathbf{0} & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ddots & \dots & \dots & \dots & \mathbf{0} & -\mathbf{I} & -\operatorname{div}_{\mathbf{M}_Q} \end{pmatrix}$$

such that

$$\mathcal{K} \begin{pmatrix} w_1 \\ \vdots \\ w_Q \end{pmatrix} = \begin{pmatrix} -w_1 - \operatorname{div}_{\mathbf{M}_2} w_2 \\ \vdots \\ -w_j - \operatorname{div}_{\mathbf{M}_{j+1}} w_{j+1} \\ \vdots \\ -w_{Q-1} - \operatorname{div}_{\mathbf{M}_Q} w_Q \end{pmatrix}.$$

Let the proper, convex, and lower semicontinuous functionals

$$F : X \rightarrow]-\infty, \infty], \quad F(\mathbf{w}) = -\langle \mathbf{u}, \operatorname{div}_{\mathbf{M}_1} \mathbf{w}_1 \rangle + \sum_{j=1}^Q I_{\{\|\cdot\|_\infty \leq \alpha_{Q-j}\}}(w_j),$$

$$G : Y \rightarrow]-\infty, \infty], \quad G(\mathbf{z}) = I_0(\mathbf{z}),$$

where I_Z is the indicator function of this set, i.e., $I_Z(\mathbf{z}) = 0$ if $\mathbf{z} \in Z$ and $I_Z(\mathbf{z}) = \infty$ otherwise. Then the following identity holds from [Definition 3.7](#):

$$\operatorname{TDV}_{\alpha}^{\mathbf{Q}, \ell}(\mathbf{u}, \mathcal{M}) = \sup_{\mathbf{w} \in X} -F(\mathbf{w}) - G(\mathcal{K}\mathbf{w}).$$

Next, we want to obtain the following result:

$$(5.3) \quad \operatorname{TDV}_{\alpha}^{\mathbf{Q}, \ell}(\mathbf{u}, \mathcal{M}) = \min_{\mathbf{w}^* \in Y^*} F^*(-\mathcal{K}^* \mathbf{w}^*) + G^*(\mathbf{w}^*).$$

This follows from [\[3, Corollary 2.3\]](#) once we show that

$$Y = \bigcup_{\lambda > 0} \lambda (\operatorname{dom}(G) - \mathcal{K} \operatorname{dom}(F)).$$

Indeed, let $\mathbf{z} \in Y$, and define recursively

$$\begin{aligned} \mathbf{w}_Q &= \mathbf{0} && \in C_0^Q(\Omega, \mathcal{T}^{\ell+Q}(\mathbb{R}^d)), \\ \mathbf{w}_{Q-1} &= \mathbf{z}_{Q-1} - \operatorname{div}_{\mathbf{M}_Q} \mathbf{w}_Q && \in C_0^{Q-1}(\Omega, \mathcal{T}^{\ell+Q-1}(\mathbb{R}^d)), \\ &\vdots && \\ \mathbf{w}_j &= \mathbf{z}_j - \operatorname{div}_{\mathbf{M}_{j+1}} \mathbf{w}_{j+1} && \in C_0^j(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d)), \\ &\vdots && \\ \mathbf{w}_1 &= \mathbf{z}_1 - \operatorname{div}_{\mathbf{M}_2} \mathbf{w}_2 && \in C_0^1(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)). \end{aligned}$$

Hence, $\mathbf{w} \in X$ and $-\mathcal{K}\mathbf{w} = \mathbf{z} \in Y$. Moreover, for $\lambda > 0$ large enough, we have

$$\|\lambda^{-1}\mathbf{w}\|_\infty \leq \alpha_{Q-j} \quad \text{for all } j = 1, \dots, Q.$$

Therefore, from $\lambda^{-1}\mathbf{w} \in \operatorname{dom}(F)$ and $\mathbf{0} \in \operatorname{dom}(G)$, we get the following representation:

$$\mathbf{z} = \lambda(\mathbf{0} - \mathcal{K}\lambda^{-1}\mathbf{w}).$$

This means that (5.3) holds and the minimum is obtained in Y^* , which can be written as

$$Y^* = \left(C_0^1(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))\right)^* \times \dots \times \left(C_0^1(\Omega, \mathcal{T}^{\ell+Q-1}(\mathbb{R}^d))\right)^*$$

and $\mathbf{z}^* = (\mathbf{z}_1^*, \dots, \mathbf{z}_{Q-1}^*)$, $\mathbf{z}_j^* \in C_0^j(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$, for $1 \leq j \leq Q-1$. Hence, imposing $\mathbf{z}_0^* = \mathbf{u}$ and $\mathbf{z}_Q^* = \mathbf{0}$, from $G^* = \mathbf{0}$ the following chain holds:

$$\begin{aligned} F^*(-\mathcal{K}^*\mathbf{z}^*) + G^*(\mathbf{z}^*) &= \sup_{\mathbf{w} \in X} \left(\langle -\mathcal{K}^*\mathbf{z}^*, \mathbf{w} \rangle + \langle \mathbf{u}, \operatorname{div}_{\mathbf{M}_1} \mathbf{w}_1 \rangle - \sum_{j=1}^Q I_{\{\|\cdot\|_\infty \leq \alpha_{Q-j}\}}(\mathbf{w}_j) \right) \\ &= \sup_{\substack{\mathbf{w} \in X \\ \|\mathbf{w}_j\|_\infty \leq \alpha_{Q-j} \\ j=1, \dots, Q}} \left(\langle \mathbf{u}, \operatorname{div}_{\mathbf{M}_1} \mathbf{w}_1 \rangle + \sum_{j=1}^{Q-1} \langle \mathbf{z}_j^*, \operatorname{div}_{\mathbf{M}_{j+1}} \mathbf{w}_{j+1} + \mathbf{w}_j \rangle \right) \\ &= \sum_{j=1}^Q \alpha_{Q-j} \left(\sup_{\substack{\mathbf{w}_j \in C_0^j(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d)), \\ \|\mathbf{w}_j\|_\infty \leq 1}} \langle \mathbf{z}_{j-1}^*, \operatorname{div}_{\mathbf{M}_j} \mathbf{w}_j \rangle + \langle \mathbf{z}_j^*, \mathbf{w}_j \rangle \right). \end{aligned}$$

From Lemma 5.1, we have that each supremum is finite and

$$\sup_{\substack{\mathbf{w}_j \in C_0^j(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d)), \\ \|\mathbf{w}_j\|_\infty \leq 1}} \langle \mathbf{z}_{j-1}^*, \operatorname{div}_{\mathbf{M}_j} \mathbf{w}_j \rangle + \langle \mathbf{z}_j^*, \mathbf{w}_j \rangle = \|\mathbf{M}_j \nabla \mathbf{z}_{j-1}^* - \mathbf{z}_j^*\|_{\mathcal{M}}$$

if and only if $\mathbf{M}_j \nabla \mathbf{z}_{j-1}^* - \mathbf{z}_j^* \in \mathcal{M}(\Omega, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$, for $j = 1, \dots, Q$. Since $\mathbf{z}_Q^* = \mathbf{0}$, by Theorem 4.13 this means that $\mathbf{z}_{Q-1}^* \in \operatorname{BDV}(\Omega, \mathbf{M}_Q, \mathcal{T}^{\ell+Q-1}(\mathbb{R}^d))$, so

$$\mathbf{z}_{Q-1}^* \in \mathcal{M}(\Omega, \mathcal{T}^{\ell+Q-1}(\mathbb{R}^d)).$$

By induction, we have $\mathbf{z}_j^* \in \text{BDV}(\Omega, \mathbf{M}_{j+1}, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$ for each $j = 0, \dots, Q$, so we can take the minimum in (5.3) over all BDV-tensor fields, obtaining (5.2): Such minimum is finite if $\mathbf{u} \in \text{BDV}(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$. ■

Remark 5.3. Let $\alpha_j = (\alpha_0, \dots, \alpha_j)$ be such that $\alpha_j \subseteq \alpha = (\alpha_0, \dots, \alpha_{Q-1})$, and let $\mathcal{M}_{Q-j}^- = (\mathbf{M}_{Q-j+1}, \dots, \mathbf{M}_Q)$ be a subset of $\mathcal{M} = (\mathbf{M}_1, \dots, \mathbf{M}_Q)$ such that $\mathcal{M}_{Q-j}^- \subseteq \mathcal{M}$. Then the regularizer $\text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}, \mathcal{M})$ can be expressed recursively as

$$\begin{aligned} \text{TDV}_{\alpha_0}^{1,\ell+Q-1}(\mathbf{z}_{Q-1}, \mathcal{M}_{Q-1}^-) &= \alpha_0 \|\mathbf{M}_Q \nabla \otimes \mathbf{z}_{Q-1}\|_{\mathcal{M}}, \\ &\vdots \\ \text{TDV}_{\alpha_j}^{j+1,\ell+Q-j-1}(\mathbf{z}_{Q-j}, \mathcal{M}_{Q-j-1}^-) &= \min_{\mathbf{z}_{Q-j}} \left(\alpha_j \|\mathbf{M}_{Q-j} \nabla \otimes \mathbf{z}_{Q-j-1} - \mathbf{z}_{Q-j}\|_{\mathcal{M}} \right. \\ &\quad \left. + \text{TDV}_{\alpha_{j-1}}^{j,\ell+Q-j}(\mathbf{z}_{Q-j}, \mathcal{M}_{Q-j}^-) \right), \\ &\vdots \\ \text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}, \mathcal{M}_0^-) &= \min_{\mathbf{z}_1} \left(\alpha_{Q-1} \|\mathbf{M}_1 \nabla \otimes \mathbf{u} - \mathbf{z}_1\|_{\mathcal{M}} + \text{TDV}_{\alpha_{Q-2}}^{Q-1,\ell+1}(\mathbf{z}_1, \mathcal{M}_1^-) \right), \end{aligned}$$

where $\mathbf{z}_j \in \text{BDV}(\Omega, \mathbf{M}_{j+1}, \mathcal{T}^{\ell+j}(\mathbb{R}^d))$.

Remark 5.4. As in [8, Remark 3.8], the minimum representation is monotonic with respect to the weights. Indeed, let $\alpha, \beta \in \mathbb{R}_+^Q$ with $\alpha_j \leq \beta_j$ for each $j = 0, \dots, Q-1$. Then

$$\text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}, \mathcal{M}) \leq \text{TDV}_{\beta}^{Q,\ell}(\mathbf{u}, \mathcal{M}).$$

6. Existence of TDV-regularized solutions. In this section we prove the existence of solutions to TDV-regularized problems of the type

$$(6.1) \quad \min_{\mathbf{u} \in L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))} \text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}, \mathcal{M}) + F(\mathbf{u}),$$

where $F : L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \rightarrow \mathbb{R}$ is a fidelity term. Next, we will follow [8] so as to check that the same results hold in our weighted case, and we will proceed often by induction on Q . We proceed by proving the embedding theorems and the existence of a minimizer for (6.1).

6.1. Embeddings. We state some results in view of the embedding Theorems 6.4 and 6.5. The following Sobolev–Korn type of inequality holds for smooth tensor fields with compact support, similarly to [7, Theorem 4.8].

Lemma 6.1. *Let $\mathbf{u} \in C_c^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ and $\mathbf{M} \in L^\infty(\Omega, \mathcal{T}^2(\mathbb{R}^d))$ be a field of invertible matrices for every $\mathbf{x} \in \Omega$ such that $S = \sup_{\mathbf{x}} \|(\mathbf{M}(\mathbf{x}))^{-1}\|_2 < \infty$. Then there exists a constant C depending only on Ω, ℓ , and S such that*

$$\|\mathbf{u}\|_{d/(d-1)} \leq C \|\mathbf{M} \nabla \otimes \mathbf{u}\|_1.$$

Proof. Let $\|\cdot\|$ be the operator norm. We have the desired inequality, where the first one is due to the standard Sobolev inequality for tensor-valued functions,

$$\|\mathbf{u}\|_{d/(d-1)} \leq C_1 \|\nabla \otimes \mathbf{u}\|_1 = C_1 \|(\mathbf{M}(\cdot))^{-1} \mathbf{M}(\cdot) \nabla \otimes \mathbf{u}\|_1 \leq C_1 S \|\mathbf{M} \nabla \otimes \mathbf{u}\|_1,$$

and the conclusion follows with $C := C_1 S$. ■

The following lemma states a result similar to [8, Lemma 3.9].

Lemma 6.2. *For each $Q \geq 1$, $\ell \geq 0$ there exists a constant $C_1 > 0$ depending only on Ω , Q and ℓ such that for each $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ and $\bar{\mathbf{w}} \in \ker(\text{TDV}_\alpha^{Q, \ell+1}) \subset L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$,*

$$\|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}} \leq C_1 (\|\mathbf{u}\|_1 + \|\mathbf{M} \nabla \otimes \mathbf{u} - \bar{\mathbf{w}}\|_{\mathcal{M}}).$$

Proof. We argue by contradiction. Suppose that there exists Q and ℓ such that the bound does not hold. Then there exist $(\mathbf{u}_j)_{j \in \mathbb{N}}$ and $(\bar{\mathbf{w}}_j)_{j \in \mathbb{N}}$, with each $\mathbf{u}_j \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ and $\bar{\mathbf{w}}_j \in \ker(\text{TDV}_\alpha^{Q, \ell+1})$, such that

$$\|\mathbf{M} \nabla \otimes \mathbf{u}_j\|_{\mathcal{M}} = 1 \quad \text{and} \quad \|\mathbf{u}_j\|_1 + \|\mathbf{M} \nabla \otimes \mathbf{u}_j - \bar{\mathbf{w}}_j\|_{\mathcal{M}} \leq j^{-1}.$$

Thus, $(\bar{\mathbf{w}}_j)_{j \in \mathbb{N}}$ is bounded with respect to the norm $\|\cdot\|_{\mathcal{M}}$ in the finite-dimensional space $\ker(\text{TDV}_\alpha^{Q, \ell+1})$. Therefore, there exists a subsequence relabeled as $(\bar{\mathbf{w}}_j)_{j \in \mathbb{N}}$ and converging to $\bar{\mathbf{w}} \in \ker(\text{TDV}_\alpha^{Q, \ell+1})$ in the $L^1(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$ norm, and thus $\mathbf{M} \nabla \otimes \mathbf{u}_j \rightarrow \bar{\mathbf{w}}$. Moreover, $\mathbf{u}_j \rightarrow \mathbf{0}$ in $L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ implies that $\mathbf{M} \nabla \otimes \mathbf{u}_j \rightarrow \mathbf{0}$ in $\mathcal{M}$ by closedness of the gradient and this contradicts $\|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}} = 1$. ■

We can also define the zero extension $E\mathbf{u}$ of a function $\mathbf{u}$ of bounded directional variation. Such zero extension has bounded directional variation, as can be proved adapting [7, Corollary 4.15] based on [7, Theorem 4.12].

Corollary 6.3. *Let Ω a bounded Lipschitz domain and $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$. Then the zero extension $E\mathbf{u}$ is in $\text{BDV}(\mathbb{R}^d, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$. In addition, there exists $C > 0$ such that, for all $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$,*

$$(6.2) \quad \|E\mathbf{u}\|_1 + \|\mathbf{M} \nabla \otimes (E\mathbf{u})\|_{\mathcal{M}} \leq C (\|\mathbf{u}\|_1 + \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}}).$$

Proof. It follows by adapting the proof of [7, Corollary 4.15, Theorem 4.12]. ■

In the next theorem, we prove the continuous embedding of the space BDV into $L^{d/(d-1)}$, similarly to [7, Theorem 4.16].

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there is a continuous injection*

$$\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d)) \hookrightarrow L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)).$$

Proof. In this proof we follow [7, Theorem 4.16], with the notational changes $\mathbf{M} \nabla$, $\mathcal{T}^\ell(\mathbb{R}^d)$, and $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ in place of the symmetrized gradient $\mathcal{E}$, $\text{Sym}^\ell(\mathbb{R}^d)$, and $\text{BD}(\Omega, \text{Sym}^\ell(\mathbb{R}^d))$, respectively, and $d \geq 2$. If $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d)) \cap C_c^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, then Lemma 6.1 gives the result. In the general case $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$, its zero extension $E\mathbf{u}$ can be approximated by a sequence of strictly converging continuously differentiable, compactly supported functions $(\mathbf{u}_j)_{j \in \mathbb{N}}$ by applying Proposition 4.6 to a bounded domain Ω' such that $\bar{\Omega} \subset \subset \Omega'$. According to the estimate in Lemma 6.1, we have for each j ,

$$\|\mathbf{u}_j\|_{d/(d-1)} \leq C \|\mathbf{M} \nabla \otimes \mathbf{u}_j\|_1 = C \|\mathbf{M} \nabla \otimes \mathbf{u}_j\|_{\mathcal{M}} \leq C (\|\mathbf{u}_j\|_1 + \|\mathbf{M} \nabla \otimes \mathbf{u}_j\|_{\mathcal{M}}).$$

Now $\mathbf{u}_j \rightarrow E\mathbf{u}$ in $L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, and by the lower semicontinuity of the $L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ -norm, the strict convergence in BDV , and Corollary 6.3, we get

$$(6.3) \quad \|\mathbf{u}\|_{d/(d-1)} = \|E\mathbf{u}\|_{d/(d-1)} \leq C (\|\mathbf{u}\|_1 + \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}}). \quad \blacksquare$$

Now, we show that the embedding in Theorem 6.4 is compact for $1 \leq p < d/(d-1)$, similarly to [7, Theorem 4.17].

Theorem 6.5. *Let Ω be a bounded Lipschitz domain, $1 \leq p < d/(d-1)$, and $(\mathbf{u}_j)_{j \in \mathbb{N}}$ be a bounded sequence in $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$. Then a subsequence $(\mathbf{u}_{j_\ell})_{\ell \in \mathbb{N}}$ converges in $L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$.*

Proof. We aim to prove the compact embedding $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d)) \hookrightarrow L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, i.e., by fixing Ω' such that $\bar{\Omega} \subset \subset \Omega'$ and $\mathbf{u} \in C_c^2(\mathbb{R}^d, \mathcal{T}^\ell(\mathbb{R}^d))$ with support in Ω' ; then

$$\int_{\mathbb{R}^d} |\mathbf{u}(\bar{\mathbf{x}} + \mathbf{h}) - \mathbf{u}(\bar{\mathbf{x}})| \, d\bar{\mathbf{x}} \leq C |\mathbf{h}|^s \|\mathbf{M} \nabla \otimes \mathbf{u}\|_1$$

for some $s > 0$ and all $\mathbf{h} \in \mathbb{R}^d$, $|\mathbf{h}| \leq 1$ with a constant C independent of $\mathbf{u}$. This part follows by the same argument as in the first part of the proof of [7, Theorem 4.17].

Let $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}, \mathcal{T})$ be arbitrary. The zero extension $E\mathbf{u} \in \text{BDV}(\Omega', \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ has compact support in Ω' , and thus there exists a smooth sequence $(\mathbf{u}_j)_{j \in \mathbb{N}}$ in $C_c^\infty(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ such that $\mathbf{u}_j \rightarrow E\mathbf{u}$ in $L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ and $\|\mathbf{M} \nabla \otimes \mathbf{u}_j\|_1 \rightarrow \|\mathbf{M} \nabla \otimes E\mathbf{u}\|_{\mathcal{M}}$ as $j \rightarrow \infty$. Thus,

$$\int_{\mathbb{R}^d} |E\mathbf{u}(\bar{\mathbf{x}} + \mathbf{h}) - E\mathbf{u}(\bar{\mathbf{x}})| \, d\bar{\mathbf{x}} \leq C |\mathbf{h}|^s \|\mathbf{M} \nabla \otimes E\mathbf{u}\|_{\mathcal{M}} \leq C |\mathbf{h}|^s (\|\mathbf{u}\|_1 + \|\mathbf{M} \nabla \otimes \mathbf{u}\|_{\mathcal{M}}).$$

For a bounded sequence in $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$, we have $(E\mathbf{u}_j)_{j \in \mathbb{N}}$ relatively compact; thus, there exist $\mathbf{u} \in L^1(\mathbb{R}^d, \mathcal{T}^\ell(\mathbb{R}^d))$ and a subsequence $(E\mathbf{u}_{j_\ell})_{\ell \in \mathbb{N}}$ with $E\mathbf{u}_{j_\ell} \rightarrow \mathbf{u}$. Also, $\mathbf{u}_{j_\ell} \rightarrow \mathbf{u}|_\Omega$ in $L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, proving the compact embedding $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ in $L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$.

For the general case $1 \leq p < d/(d-1)$, it follows from Theorem 6.4 that $(\mathbf{u}_j)_{j \in \mathbb{N}}$ is bounded in $L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, and the result follows from an application of Young's inequality as in the proof of [7, Theorem 4.17]. ■

Every bounded sequence in $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ admits a subsequence which converges in the weak-* sense, while strict convergence implies weak-* convergence. The embeddings above allow to reinterpret weak-* sequences in $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ as follows:

- weakly converging sequences in $L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ (weak-* for $d = 1$);
- strongly converging sequences in $L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ for any $p \in [1, d/(d-1)[$, continuously.

Also, $C^\infty(\bar{\Omega}, \mathcal{T}^\ell(\mathbb{R}^d))$ is dense in $\text{BDV}(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ with respect to strict convergence.

6.2. Existence. In what follows, we prove the coercivity for $\text{TDV}_\alpha^{\mathbf{Q}, \ell}$ in view of satisfying the conditions of the Tonelli–Weierstraß theorem for the minimization problem (6.1).

Definition 6.6. *For each $\mathbf{Q} \geq 1$ and $\ell \geq 0$, let $R_{\mathbf{Q}, \ell}$ be a linear, continuous, and onto projection such that*

$$R_{\mathbf{Q}, \ell} : L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \rightarrow \ker(\mathbf{M}_\mathbf{Q} \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla).$$

Note that $R_{\mathbf{Q}, \ell}$ defined as above always exists since $\ker(\text{TDV}_\alpha^{\mathbf{Q}, \ell}) = \ker(\mathbf{M}_\mathbf{Q} \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla)$ is finite-dimensional.

The following coercivity estimate holds, similarly to [8, Proposition 3.11].

Proposition 6.7. *For each $Q \geq 1$ and $\ell \geq 0$, there exists a constant $C > 0$ such that, for all $\mathbf{u} \in L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$,*

$$\|\mathbf{M}_1 \nabla \otimes \mathbf{u}\|_{\mathcal{M}} \leq C \left(\|\mathbf{u}\|_1 + \text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}, \mathcal{M}) \right) \quad \text{and} \quad \|\mathbf{u} - R_{Q,\ell} \mathbf{u}\|_{d/(d-1)} \leq C \text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}, \mathcal{M}).$$

Proof. Exactly as in the proof of [8, Proposition 3.11], we proceed by induction on Q . Let $Q = 1$ and $\ell \geq 0$. Then the first inequality is trivial, while the second follows from the Sobolev–Korn inequality of Lemma 4.4.

For the induction step, we fix $\ell \geq 0$, $\alpha = (\alpha_0, \dots, \alpha_Q)$ with $\alpha_i > 0$, Ω and $R_{Q+1,\ell}$, and we assume that both conclusions of the proposition hold for $\tilde{\alpha} = (\alpha_0, \dots, \alpha_{Q-1})$ and any $\ell' \in \mathbb{N}$.

We first show that the estimate for $\|\mathbf{M}_1 \nabla \otimes \mathbf{u}\|_{\mathcal{M}}$ holds when $\mathbf{u} \in \text{BDV}(\Omega, \mathbf{M}_1, \mathcal{T}^\ell(\mathbb{R}^d))$ (otherwise, the estimate is obvious since $\text{TDV}_{\alpha}^{Q+1,\ell}(\mathbf{u}, \mathcal{M}) = +\infty$). Using the map $R_{Q,\ell+1}$, Lemma 6.2, the continuous embeddings

$$\text{BDV}(\Omega, \mathbf{M}_1, \mathcal{T}^{\ell+1}(\mathbb{R}^d)) \hookrightarrow L^{d/(d-1)}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)) \hookrightarrow L^1(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$$

and the induction hypotheses, we get for $\mathbf{w} \in \text{BDV}(\Omega, \mathbf{M}_1, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$ the estimates

$$\begin{aligned} \|\mathbf{M}_1 \nabla \otimes \mathbf{u}\|_{\mathcal{M}} &\leq C_1 (\|\mathbf{M}_1 \nabla \otimes \mathbf{u} - R_{Q,\ell+1} \mathbf{w}\|_{\mathcal{M}} + \|\mathbf{u}\|_1) \\ &\leq C_2 (\|\mathbf{M}_1 \nabla \otimes \mathbf{u} - \mathbf{w}\|_{\mathcal{M}} + \|\mathbf{w} - R_{Q,\ell+1} \mathbf{w}\|_{d/(d-1)} + \|\mathbf{u}\|_1) \\ &\leq C_3 (\|\mathbf{M}_1 \nabla \otimes \mathbf{u} - \mathbf{w}\|_{\mathcal{M}} + \text{TDV}_{\alpha}^{Q,\ell+1}(\mathbf{w}, \mathcal{M}) + \|\mathbf{u}\|_1) \\ &\leq C_4 (\alpha_Q \|\mathbf{M}_1 \nabla \otimes \mathbf{u} - \mathbf{w}\|_{\mathcal{M}} + \text{TDV}_{\alpha}^{Q,\ell+1}(\mathbf{w}, \mathcal{M}) + \|\mathbf{u}\|_1) \end{aligned}$$

for suitable $C_1, C_2, C_3, C_4 > 0$. By taking the minimum over all $\mathbf{w} \in \text{BDV}(\Omega, \mathbf{M}_1, \mathcal{T}^\ell(\mathbb{R}^d))$ we get

$$\|\mathbf{M}_1 \nabla \otimes \mathbf{u}\|_{\mathcal{M}} \leq C_4 \left(\|\mathbf{u}\|_1 + \text{TDV}_{\alpha}^{Q+1,\ell}(\mathbf{u}, \mathcal{M}) \right)$$

via the minimum representation in Remark 5.3.

For the coercivity estimate, assume that it is not true, i.e., there exists $(\mathbf{u}_j)_{j \in \mathbb{N}}$ such that each $\mathbf{u}_j \in L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ and

$$\|\mathbf{u}_j - R_{Q+1,\ell} \mathbf{u}_j\|_{d/(d-1)} = 1 \quad \text{and} \quad \text{TDV}_{\alpha}^{Q+1,\ell}(\mathbf{u}_j, \mathcal{M}) \leq j^{-1}.$$

Since $\ker(\text{TDV}_{\alpha}^{Q+1,\ell}(\mathbf{u}_j, \mathcal{M})) = \text{Im}(R_{Q+1,\ell})$, then for each j it holds that

$$\text{TDV}_{\alpha}^{Q+1,\ell}(\mathbf{u}_j - R_{Q+1,\ell} \mathbf{u}_j, \mathcal{M}) = \text{TDV}_{\alpha}^{Q+1,\ell}(\mathbf{u}_j, \mathcal{M}).$$

Also, since the first estimate holds, then

$$\|\mathbf{M}_1 \nabla \otimes (\mathbf{u}_j - R_{Q+1,\ell} \mathbf{u}_j)\|_{\mathcal{M}} \leq C_4 \left(\text{TDV}_{\alpha}^{Q+1,\ell}(\mathbf{u}_j, \mathcal{M}) + \|\mathbf{u}_j - R_{Q+1,\ell} \mathbf{u}_j\|_1 \right),$$

and $(\mathbf{u}_j - R_{Q+1,\ell} \mathbf{u}_j)_{j \in \mathbb{N}}$ is bounded in $\text{BDV}(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$ by the continuous embedding. By the compact embedding there exists a subsequence of $(\mathbf{u}_j - R_{Q+1,\ell} \mathbf{u}_j)_{j \in \mathbb{N}}$, not relabeled,

converging to $\mathbf{u}^* \in L^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ with $R_{Q+1,\ell}\mathbf{u}^* = 0$ since $R_{Q+1,\ell}(\mathbf{u}_j - R_{Q+1,\ell}\mathbf{u}_j) = 0$ for all j . Moreover, the lower semicontinuity leads to

$$0 \leq \text{TDV}_{\alpha}^{Q+1,\ell}(\mathbf{u}^*, \mathcal{M}) \leq \liminf_{j \rightarrow \infty} \text{TDV}_{\alpha}^{Q+1,\ell}(\mathbf{u}_j, \mathcal{M}) = 0.$$

This means that $\mathbf{u}^* \in \ker(\text{TDV}_{\alpha}^{Q+1,\ell})$ and $\mathbf{M}_1 \nabla \otimes (\mathbf{u}_j - R_{Q+1,\ell}\mathbf{u}_j) \rightarrow \mathbf{0}$ in $\mathcal{M}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))$ with $(\mathbf{u}_j - R_{Q+1,\ell}\mathbf{u}_j) \rightarrow \mathbf{0}$ in $\text{BDV}(\Omega, \mathbf{M}_1, \mathcal{T}^\ell(\mathbb{R}^d))$ and in $L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ by the continuous embedding. This contradicts $\|\mathbf{u}_j - R_{Q+1,\ell}\mathbf{u}_j\|_{d/(d-1)} = 1$ for all j , and the coercivity holds. ■

The next proposition, similar to [8, Proposition 4.1], proves the coercivity of the minimization problem (6.1).

Proposition 6.8. *Let $p \in [1, \infty[$ with $p \leq d/(d-1)$ and $F : L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \rightarrow]-\infty, \infty]$. If F is bounded from below and there exist an onto projection R as in Definition 6.6 such that for each sequence $(\mathbf{u}_j)_{j \in \mathbb{N}}$ with $\mathbf{u}_j \in L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ it holds that*

$$\|R\mathbf{u}_j\|_{d/(d-1)} \rightarrow \infty \quad \text{and} \quad \left(\|\mathbf{u}_j - R\mathbf{u}_j\|_{d/(d-1)} \right)_{j \in \mathbb{N}} \text{ is bounded} \quad \Rightarrow \quad F(\mathbf{u}_j) \rightarrow \infty,$$

then $\text{TDV}_{\alpha}^{Q,\ell} + F$ is coercive in $L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$.

Proof. Let $(\mathbf{u}_j)_{j \in \mathbb{N}}$ be a sequence such that each $\mathbf{u}_j \in L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, and if $(F(\mathbf{u}_j) + \text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}_j, \mathcal{M}))_{j \in \mathbb{N}}$ is bounded, then $(\mathbf{u}_j)_{j \in \mathbb{N}}$ is bounded. Since F is bounded from below by assumption, then the sequences $(F(\mathbf{u}_j))_{j \in \mathbb{N}}$ and $(\text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}_j, \mathcal{M}))_{j \in \mathbb{N}}$ are bounded too. Thus, the boundedness of $(\text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}_j, \mathcal{M}))_{j \in \mathbb{N}}$ implies that each $\mathbf{u}_j \in L^{d/(d-1)}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ by the continuous embedding in Theorem 6.4. Now, let R be a projection map as in Definition 6.6 such that the hypotheses holds. Thus, there exists a constant $C > 0$ such that

$$\|\mathbf{u}_j - R\mathbf{u}_j\|_{d/(d-1)} \leq C \text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}_j, \mathcal{M}) \quad \text{for all } j \in \mathbb{N},$$

and the sequence $(\|\mathbf{u}_j - R\mathbf{u}_j\|_{d/(d-1)})_{j \in \mathbb{N}}$ is bounded. Note that $(\|R\mathbf{u}_j\|_{d/(d-1)})_{j \in \mathbb{N}}$ is bounded too; otherwise, $(F(\mathbf{u}_j))_{j \in \mathbb{N}}$ results unbounded and contradicts the hypothesis. From the continuous embedding of Lebesgue spaces, $(\mathbf{u}_j)_{j \in \mathbb{N}}$ is bounded in $L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$. ■

We are now ready to prove the following existence theorem, similarly to [7, Theorem 4.2].

Theorem 6.9. *Let $p \in [1, \infty[$ with $p \leq d/(d-1)$, and assume that $F : L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \rightarrow]-\infty, \infty]$ is proper, convex, lower semicontinuous and coercive as in Proposition 6.8. Then there exists a solution to the problem*

$$(6.4) \quad \min_{\mathbf{u} \in L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))} \text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}, \mathcal{M}) + F(\mathbf{u}).$$

Furthermore, if $\mathbf{u} \in \text{BDV}(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$ is such that $F(\mathbf{u}) < \infty$, then the minimum is finite.

Proof. We note immediately that the regularizer $\text{TDV}_{\alpha}^{Q,\ell}(\mathbf{u}, \mathcal{M})$ is finite if and only if $\mathbf{u} \in \text{BDV}(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$; otherwise, it is trivial to prove that a minimizer exists and the minimum is equal to $+\infty$. Thus, assume $F(\mathbf{u}) < \infty$ for some $\mathbf{u} \in \text{BDV}(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d))$, and consider a minimizing sequence $(\mathbf{u}_j)_{j \in \mathbb{N}}$ for $G = F + \text{TDV}_{\alpha}^{Q,\ell}$. Note that such sequence exists

since G is bounded from below. Now, applying the coercivity result in Proposition 6.8 for a $p' \in [p, d/(d-1)]$ and $p' > 1$, there exists a subsequence of $(\mathbf{u}_j)$, weakly convergent to $\mathbf{u}^* \in L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$. Moreover, since G is convex and lower semicontinuous, we get that $\mathbf{u}^*$ is a minimizer by weak lower semicontinuity, and by assuming that G is proper, the minimum is finite. ■

From Theorem 6.9, we can conclude as in [8, Corollary 4.3] that there exists a solution for the minimization problem (6.4) in the context of inverse problems, i.e., when the fidelity term $F(\mathbf{u})$ is defined from a forward operator $\mathcal{S} : L^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \rightarrow Y$, linear and continuous in a normed space Y , and the observed data $\mathbf{u}^\diamond \in Y$ as

$$F(\mathbf{u}) = \frac{1}{q} \|\mathcal{S}\mathbf{u} - \mathbf{u}^\diamond\|_Y^q \quad \text{for } q \in [1, \infty[.$$

Of course, for a strictly convex norm $\|\cdot\|_Y$ the uniqueness of the solution depends on the injectivity of $\mathcal{S}$: In general, uniqueness does not hold since $\text{TDV}_{\alpha}^{\mathcal{Q}, \ell}$ is not strictly convex.

7. Conclusions. In this work, we have introduced and analyzed TDV of arbitrary order, providing a precise framework to extend the notions of TGV [9] and DTV [14]. In particular, we have proven a representation formula for the total directional variation of arbitrary order, which is a key for the design of a primal-dual algorithm which can be used in many imaging applications; see [22].

Acknowledgments. The authors are grateful to Prof. Jan Lellman, University of Lübeck, (Germany), and Dr. Martin Holler, University of Graz (Austria), for the useful discussions.

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