OPEN ACCESS
IOP Publishing
Classical and Quantum Gravity

Class. Quantum Grav. 37 (2020) 065014 (7pp)
https://doi.org/10.1088/1361-6382/ab7074

Sudden Brans–Dicke singularities

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Received 19 September 2019, revised 8 January 2020
Accepted for publication 27 January 2020
Published 20 February 2020

Abstract

We show that cosmological sudden singularities that respect the energy conditions can occur at finite times in Brans–Dicke and more general scalar-tensor theories of gravity. We construct these explicitly in the Friedmann universes. Higher-order versions of these singularities are also possible, including those that arise with when scalar fields have a self-interaction potential of power-law form.

Keywords: cosmology, scalar–tensor gravity, singularities

1. Introduction

Following the realisation that finite-time 'sudden' singularities can arise in general relativistic cosmologies where the scale factor, its first time derivative and the fluid density can remain finite whilst its second derivative and fluid pressure diverge although the strong energy condition remains unviolated [1, 2], there has been extensive study of this possibility and its close relatives. Generalisations were found in [3] and examples appeared in anisotropic cosmologies [4] and higher-order gravity theories with $f(R)$ lagrangians [3]. These are weak singularities in the senses of Tipler [5] and Krolak [6] and their conformal diagrams have been constructed in [7]. Geodesics are unscathed by sudden singularities, [8] and the general behaviour of the Einstein and geodesic equations in their neighbourhood was found in [9–11]. This behaviour appears robust in the presence of quantum particle production [12]. The first examples were existence proofs that required unmotivated pressure-density relations, but more recently generalised singularities of this sort have been found by Barrow and Graham [13] to appear in simple Friedmann universes with a scalar field having power-law self-interaction potential $V(\phi) = V_0\phi^n$, $0 < n < 1$ which always develop a finite-time singularity where the Hubble rate and its first derivative are finite, but its second derivative diverges. For non-integer $n > 1$, there is a class of models with even weaker singularities. Infinities first occur at a finite time in the $(k+2)$th time derivative of the Hubble expansion rate, where $k < n < k+1$ and k is a positive integer [13]. These models inflate but inflation ends in a singular fashion.

In this paper we investigate what happens in Brans–Dicke theory [17], which generalises general relativity by admitting the possibility of spacetime variation in the Newtonian gravitational 'constant', G. We show that sudden singularities can also appear in such theories and we display the effects that they have on on the time-evolution of $G(t)$. Other investigations of the effect finite-time singularities on varying constants have been made [14] in the context of the BSBM theory for varying fine structure 'constant' [15], but not with varying G, although the two could be combined [16].

2. Brans–Dicke sudden singularities

We assume the spacetime metric to be of isotropic and homogeneous form, with expansion scale factor $a(t)$, where t is the comoving proper time coordinate, and r is the comoving radial distance and k takes values 0 or ± 1 depending on the curvature of the space sections of constant time and the speed of light is set to unity:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\sin^2\theta + \mathrm{d}\phi^2)\}. \tag{1}$$

In standard notation, the field equations of Brans–Dicke theory with energy-momentum tensor T_{ab} and Brans–Dicke scalar field $\phi(t)$ and scalar field coupling constant ω are:

$$G_{ab} = \frac{8\pi}{\phi}T_{ab} + \frac{\omega}{\phi^2}(\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi^{,c}\phi_{,c}) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) \tag{2}$$

$$\Box\phi = \frac{8\pi}{3+2\omega}T^a_a \tag{3}$$

$$T^b_{a;b} = 0. \tag{4}$$

The Einstein–Brans–Dicke field equations for Friedman universe with metric (1) containing fluid with pressure p and density ρ are:

$$3\frac{\ddot{a}}{a} = -\frac{8\pi}{(3+2\omega)\phi}\left[(2+\omega)\rho + 3p(1+\omega)\right] - \omega\frac{\dot{\phi}^2}{\phi^2} - \frac{\ddot{\phi}}{\phi} \tag{5}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{8\pi}{(3+2\omega)}(\rho - 3p) \tag{6}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{7}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\rho}{3\phi} - \frac{k}{a^2} - \frac{\dot{\phi}\dot{a}}{\phi a} + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2} \tag{8}$$

$$G = \left(\frac{2\omega+4}{2\omega+3}\right)\phi^{-1}. \tag{9}$$

We are interested first in a sudden singularity occurring where $\phi, \dot{\phi}, a, \dot{a}, \varrho$ are finite, but where $\ddot{\phi}, p, \ddot{a}, \dot{\rho}$ can be infinite. Henceforth, we drop the curvature term (k/a^2) and take the flat geometry with $k=0$ since the curvature turns out to play no essential role in the discussion.

In principle, singularities in second time-derivatives of the Brans–Dicke scalar field, $\ddot{\phi}$, could occur at a different time to those in $\ddot{a}$ and p, but it is easy to show (as we will see below) that all finite-time singularities of this type have to occur at the same time[1], which we label t_s. At such a finite-time sudden singularity we see that all terms in (8) are finite as $t \to t_s$ from below and the dominant divergent terms in the remaining equations give the asymptotic system:

$$3\frac{\ddot{a}}{a} \to -\frac{24\pi p(1+\omega)}{(3+2\omega)\phi} - \frac{\ddot{\phi}}{\phi} \tag{10}$$

$$\ddot{\phi} \to \frac{-24\pi p}{(3+2\omega)} \tag{11}$$

$$\dot{\rho} \to -3\frac{\dot{a}}{a}p. \tag{12}$$

This system of three equations ensures the sudden singularities in a, p and ϕ occur at the same time. From (10) and (11), eliminating p, we have a consistency relation:

$$3\frac{\ddot{a}}{a} \to -\frac{24\pi p(1+\omega)}{(3+2\omega)\phi} - \frac{\ddot{\phi}}{\phi} = \frac{\ddot{\phi}}{2\phi}(3+2\omega) - \frac{3\ddot{\phi}}{2\phi} \tag{13}$$

and so, as $t \to t_s$,

$$\frac{\ddot{a}}{a} \to \frac{\omega\ddot{\phi}}{3\phi}. \tag{14}$$

This requires the singularities in second derivatives of $a(t)$ and $\phi(t)$ to be simultaneous. We pick the following forms for the $a(t)$ and $\phi(t)$ evolution:

$$\phi = \phi_s\left(\frac{t}{t_s}\right)^r - C\left(1-\frac{t}{t_s}\right)^n \tag{15}$$

with $0 < r < 1 < n < 2$, and

$$a(t) = \left(\frac{t}{t_s}\right)^q (a_s - 1) + 1 - \left(1-\frac{t}{t_s}\right)^\lambda \tag{16}$$

with $0 < q < 1 < \lambda < 2$. Hence, as $t \to t_s$ we have

$$a \to a_s + q(1-a_s)(1-\frac{t}{t_s}) \to a_s, \tag{17}$$

$$\phi \to \phi_s(1 - r[1-\frac{t}{t_s}]) \to \phi_s, \tag{18}$$

$$\frac{\ddot{\phi}}{\phi} \to -\frac{Cn(n-1)}{t_s^2}(1-\frac{t}{t_s})^{n-2} \to \infty, \tag{19}$$

$$\frac{\ddot{a}}{a} \to -\frac{\lambda(\lambda-1)}{a_s t_s^2}(1-\frac{t}{t_s})^{\lambda-2} \to \infty. \tag{20}$$

[1] In the general solution with a sudden singularity present we have shown that the form of the solution for the scale factor, equation (16) is generalised so that the constants t_s and a_s become functions of the space coordinates [9]. Therefore, the sudden singularity is no longer simultaneous everywhere but the time evolution on approach to it is the same everywhere.

The consistency condition (14) is satisfied if we take:

$$n = \lambda, \tag{21}$$

$$C = \frac{3}{\omega a_s}. \tag{22}$$

Therefore, the final form of the solution with the required simultaneous sudden singularity in $a(t)$ and $\phi(t)$ is:

$$\phi = \phi_s \left(\frac{t}{t_s}\right)^r - \frac{3}{\omega a_s}\left(1 - \frac{t}{t_s}\right)^n, \tag{23}$$

$$a(t) = \left(\frac{t}{t_s}\right)^q (a_s - 1) + 1 - \left(1 - \frac{t}{t_s}\right)^n, \tag{24}$$

with $1 < n < 2$. We note that $\frac{\ddot{a}}{a} < 0$ and so the strong energy condition is still satisfied. We note also that if we wish to shift the sudden singularities up to appear in derivatives of a and ϕ that are higher than second then this can be arranged choosing the range for n suitably, with $r - 1 < n < r$ in order to have a generalised sudden singularity [3] that creates an infinity in the rth time derivatives of $a(t)$ and $\phi(t)$. Other varieties of sudden singularity (see [25] for a classification of types of finite-time singularity involving infinities in different combinations of cosmological variables in theories other than Brans–Dicke) can also be engineered by suitable choice of these parameters and their ranges.

If at early times, $t \to 0$, the solution behaves like the special Brans–Dicke exact solutions with 'Machian' initial condition $\phi(0) = 0$, then

$$a \propto \left(\frac{t}{t_s}\right)^q, \tag{25}$$

$$\phi \propto \left(\frac{t}{t_s}\right)^r. \tag{26}$$

The relations between q and r that exist for the exact power-law Brans–Dicke solutions of (8) in the presence of a perfect-fluid source with equation of state [18]

$$p = (\gamma - 1)\rho \tag{27}$$

are

$$3\gamma q + r = 2. \tag{28}$$

This is the requirement that $\rho/\phi \propto t^{-2}$, and all terms in the Friedmann-like equation (8) fall as t^{-2} since $\rho \propto a^{-3\gamma}$ from equation (7). For example, in the case of radiation ($\gamma = 4/3$) we have ϕ constant and $a \propto t^{1/2}$, $\rho \propto t^{-2}$, so no variation of G in this solution; in the case of dust ($\gamma = 1$), we have $\phi \propto t^r$ and $a \propto t^{(2-r)/3}$, $\rho \propto t^{r-2}$, where r is arbitrary. The choice of variation in G can be made slow enough by choice of r to ensure agreement with big bang nucleosynthesis if required [20, 21].

However, these 'Machian' solutions are not the general solutions of equations (5)–(8). If we take the general solution of equations (5)–(8), [22, 24], then $\phi(0) \neq 0$ and the ('non-Machian') solution is dominated by the scalar field dynamics, rather than by the matter term, ρ/ϕ, as $t \to 0$. In that case we have $a \propto t^{(1-\beta)/(3-\beta)}$ and $\phi \propto t^{2\beta/3}$ with $\beta \equiv \left(\frac{3}{2\omega+3}\right)^{1/2} > 0$

and approach to the vacuum solution of O'Hanlon and Tupper [23]. For large ω, on approach of the theory to general relativity, we have $\beta \to 0$,and hence,

$$a \propto t^{1/3}; \quad \phi \propto t^{2\beta/3}, \tag{29}$$

and so we have $q = 1/3$ and $r = 2\beta/3$ for the possible early time behaviour in general if the vacuum stresses dominate, as we would expect.

As $t \to t_s$, we have the asymptotic forms

$$a(t) \to a_s + q(1 - a_s)(1 - \frac{t}{t_s}), \tag{30}$$

$$\phi(t) \to \phi_s \left[1 - r(1 - \frac{t}{t_s})\right], \tag{31}$$

$$G(t) \to \left(\frac{4+2\omega}{3+2\omega}\right)\phi^{-1} \to \frac{G_s}{1 - r(1 - \frac{t}{t_s})} \to G_s \left[1 + r(1 - \frac{t}{t_s})\right], \tag{32}$$

$$\frac{\dot{\phi}}{\phi} = -\frac{\dot{G}}{G} \to \frac{r\left[1 - (r-1)(1 - \frac{t}{t_s})\right]}{t_s\left[1 - r(1 - \frac{t}{t_s})\right]} \to \frac{r}{t_s}. \tag{33}$$

So could an observational bound on $\dot{G}/G$ today can tell us how close we could be to t_s? Present-day observations bound r as $\dot{G}/G \sim r/t_0 < 10^{-12} yr^{-1}$ when $t_0 \ll t_s$.The usual power-law fall-off in G tails off to a constant value, G_s which is smaller than the present value by a factor t_0/t_s.

3. More general situations

It is straightforward to see the consequences of generalising from Brans–Dicke theory, where the coupling parameter, ω, is constant, to a scalar–tensor gravity theory where $\omega = \omega(\phi)$, as described in [24, 26]. The essential field equations (5)–(6) are generalised in this case to become [27]:

$$3\frac{\ddot{a}}{a} = -\frac{8\pi}{(3+2\omega)\phi}\left[(2+\omega)\rho + 3p(1+\omega)\right] - \omega\frac{\dot{\phi}^2}{\phi^2} - \frac{\ddot{\phi}}{\phi} - \frac{\omega'\dot{\phi}^2}{2(3+2\omega)\phi} \tag{34}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{8\pi}{(3+2\omega)}(\rho - 3p) - \frac{\dot{\phi}^2\omega'(\phi)}{(3+2\omega)}. \tag{35}$$

We can see that the appearance of the new non-zero $\omega'(\phi)$ terms does not affect the finite-time singularities created by the divergences of the $\ddot{\phi}$ and $\ddot{a}$ terms because the ϕ and $\dot{\phi}$ terms that multiply them tend to constants as $t \to t_s$ at the sudden singularity. Hence, we expect the behaviour at sudden singularities in general scalar–tensor theories to be as described above for Brans–Dicke. Infinities can occur in $\omega(\phi)$ at some finite time (even the present day) but are usually harmless. The $\omega(\phi) \to \infty$ limit is part of the general relativity limit. The other requirement in this limit is that[2] $\omega'/\omega^3 \to 0$ as $\omega \to \infty$. So, for example, if $\omega(\phi) \propto (\phi - \phi_0)^n$,

[2] The requirement is $\left|\frac{\omega'}{(3+2\omega)^2(4+2\omega)}\right| \to 0$ when $\omega \to \infty$. In that limit this reduces to $\omega'/\omega^3 \to 0$.

[26], then we require $n < 0$ for $\omega \to \infty$ as $\phi \to \phi_0$ and $n \leqslant -1/2$ to ensure $\omega'/\omega^3 \to 0$. We have not included the potential term [19] in the general scalar–tensor theory and expect new features will enter with its presence when its form is suitably chosen. The effects will mirror those of power-law scalar field potentials in general relativity found in [13] and lead to infinities in higher than second powers of the scale factor.

When a self-interaction potential, $V(\phi)$ is also present in these theories, it adds terms of the form

$$\frac{1}{2\omega + 3}\left(\phi V'(\phi) - 2V(\phi)\right) \tag{36}$$

to the right-hand side of equation (35). Therefore, we expect the new higher-order singularities found by Barrow and Graham [13] in general relativity with power-law scalar field potentials to occur also for scalar–tensor cosmologies with $V(\phi) \propto \phi^n$, $n \neq 2$.

4. Discussion

We have extended the study of finite-time singularities of 'sudden' type from general relativity and associated $f(R)$ gravity theories to scalar–tensor theories which incorporate varying G. We have constructed the form of these singularities in Brans–Dicke gravity theory and argue that more general scalar–tensor theories introduce no new features in the absence of potentials. We have shown elsewhere [9] that the sudden singularity forms for the scale factor as $t \to t_s$ are generic to general relativistic cosmology in the sense that if the form (16) is generalised, so that the constants become space functions of space and the terms form the leading ones in a series expansion around the singularity, then the resulting solution has the required 9-function spatial arbitrariness required of a general solution of the Einstein equations without an imposed equation of state. We expect the same arguments to hold for the asymptotic form (16) in Brans–Dicke theory and our constructions will form part of the general solution of the field equations in scalar–tensor theories also. In [10], we considered the general behaviour of the geodesics on approach to a general sudden singularity of the form (16). The geodesic behaviour is general but it remains an unresolved question as to whether the fact that geodesics are blind to these singularities, as first shown in [8], indicates something unphysical about them. However, we recognise the existence of finite-time singularities in other areas of physics in flat spacetime and the sudden singularities studied here and elsewehere might just be regarded as the general relativistic or scalar tensor theoretic editions of these in curved spacetime.

Acknowledgment

We acknowledge support from the STFC of the UK.

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