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Flexible polyhedra with two degrees of freedom

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Abstract

Polyhedra were long considered rigid, but following Connelly [4], flexible examples with one degree-of-freedom have been known. Here we show a flexible polyhedron with two degrees-of-freedom, generated using a scheme that could be extended to multiple degrees-of-freedom. A polyhedron that has two degrees of freedom is generated based on a set of three tetrahedral joined along edges. Crinkles are added to turn this underlying object into a polyhedron without removing the flexibility. The parameters defining the polyhedron are adjusted to remove clashes and extend the range of motion. We examine whether the two degrees-of-freedom are linked, to determine whether the range of motion of one hinge limits the range of motion of the other, but for a particular example we find that this is not the case.

Keywords: polyhedron, flexibility, rigidity, finite mechanism

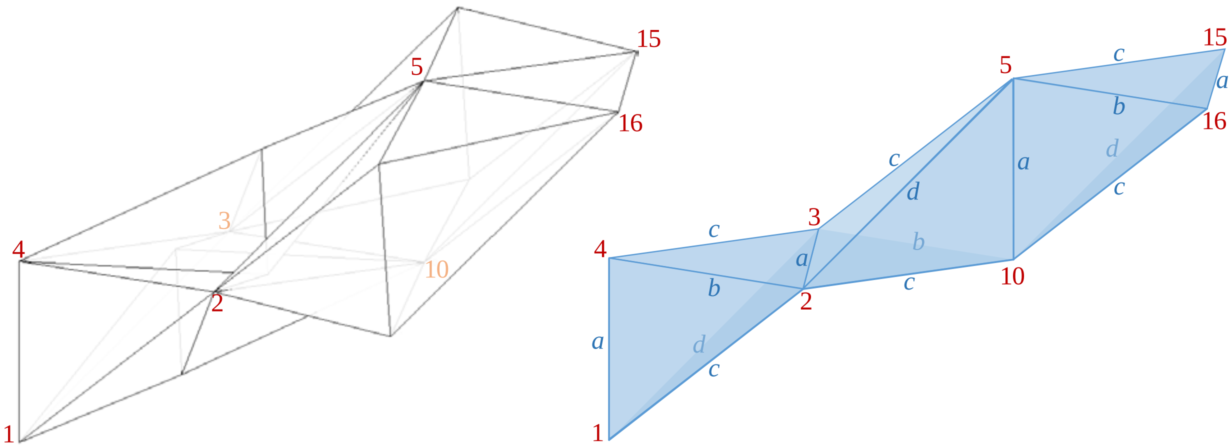

Figure 1: A flexible polyhedron with two degrees of freedom. (a) A flexible polyhedron that rotates around two hinges, 2-3 and 5-10. Each hinge has a range of motion of 14 degrees. (b) The underlying skeleton of the 2-dof flexible polyhedron in (a). This foundation is a chain of three tetrahedra linked freely by two hinges where edges are shared. This skeleton has a C_2 axis through the midpoint of line 2-5 and the midpoint of line 3-10. The first tetrahedron is identical to the last, so that this polyhedron is extendable to a polyhedron with unlimited number of hinges.

1. Introduction

Triangulated polyhedra were long believed to be rigid, and a bar (edge) and joint (node) model automatically satisfies Maxwell's counting rule for the rigidity of frames. In 1766, Euler conjectured that triangulated polyhedra are rigid [3][4][5]. It has since been proven by Cauchy that all convex triangulated polyhedra are rigid [7], and by Gluck that triangulated polyhedra are generically rigid [4], i.e., almost all polyhedra are rigid. However, Connelly showed, in the 1970s, by the simple expedient of a counterexample, that non-convex polyhedra with special geometry can be flexible [4].

Previously, the present authors have considered two different polyhedra based on Connelly's work, the Steffen flexible polyhedron [1], and a new type of triangulated flexible polyhedra based on two tetrahedra sharing an edge, which we will refer to here as the 'two-tetrahedron flexible polyhedron' [2]. Each polyhedron has a single degree of freedom, and we optimized the range of motion of each. By changing edge-lengths, we doubled the range of motion of the standard Steffen flexible polyhedron [1] (the simplest known flexible polyhedron), and found that the two-tetrahedron flexible polyhedron can achieve a better range of motion.

Here we describe how to construct a triangulated polyhedron that has two degrees of freedom, and show an example that has considerable range of motion.

2. Composition of a 2-dof polyhedron

Previously [2], we have described the "two-tetrahedron" flexible polyhedron, based on a skeleton of two tetrahedra linked by a common edge, as shown in Figure 2(a). Here we use the same idea: three tetrahedra jointed along edges, shown in Figure 1(b). The joints of this chain, line 2-3 and line 5-10, are the hinges of the new 2-dof "three-tetrahedron" flexible polyhedron. However, as shown, this object is not a polyhedron, as it does not have a continuous volume. To make the volume continuous, we replace the two coincident edges at the hinge, and their connected faces, with "crinkles"[2], which, because they are based on Bricard octahedra retain the underlying kinematics of the hinge. In previous work [1], we learnt from the Steffen flexible polyhedron that a pair of crinkles around one hinge can "slide" away from each other, hence avoiding clashes, and at the same time make the volume of the polyhedron continuous. Since we have already described the features and derivation of crinkles in previous work [2], in this paper we will only show the "sliding away" idea and then use this idea in the new 2-dof three-tetrahedron flexible polyhedron. For brevity, we will describe an edge, together with its two connected faces, as a "dihedron".

We start by describing the sliding away idea for the two-tetrahedron flexible polyhedron. In Figure 2, a pair of tetrahedra are first shown in (a) and (a'). The two dihedra around the hinge 2-3 are 4-2-5-3 and 1-2-10-3. Their edge lengths are defined as b and c, where b is smaller than c. Then in (b) dihedron 4-2-5-3 is replaced by crinkle A, whose net is shown above, and dihedron 1-2-10-3 is replaced by crinkle B, whose net is shown below. The hinge 2-3 disappears along with the triangles 2-3-4, 2-3-5, 1-2-3 and 2-3-10. The dihedron 4-6-5-8 in crinkle A is identical to the original dihedron 4-2-5-3 but rotated by 180˚. The dihedron 1-9-10-7 in crinkle B is also identical to the original dihedron 1-2-10-3 but rotated by 180˚. Both these two dihedra and the original dihedra are shown in (b') in a side view. Without the original dihedra shown, the "sliding away" of the two new dihedra are clearly seen in (c') and (c) (the viewpoint in (a)-(c) is chosen as such in order to show the gap between the two crinkles). The two crinkles can slide away from each other because the parameters are set as $b < c$. Due to the line symmetry of the crinkles we used, in crinkle A, $l_{34} = l_{56}$, $l_{48} = l_{25}$, $l_{46} = l_{35}$, $l_{24} = l_{58}$; the same applies to crinkle B.

We now extend the ideas above to form a "three-tetrahedron" two-dof flexible polyhedron. The starting point is shown in Figure 1(b), where three identical tetrahedra are shown, each described by four parameters *a–d*. The tetrahedra are joined along two edges/hinge lines, 2-3 and 5-10. There are now four dihedra to be replaced by crinkles (1-2-3-10, 4-2-3-5, 2-5-16-10, 3-5-15-10) to form a continuous volume, as has been done in Figure 1(a). The trick is to find parameters that avoid clashes – both between the pair of crinkles replacing a hinge, as described above for the two-tetrahedron model, but also between crinkles replacing adjacent hinges.

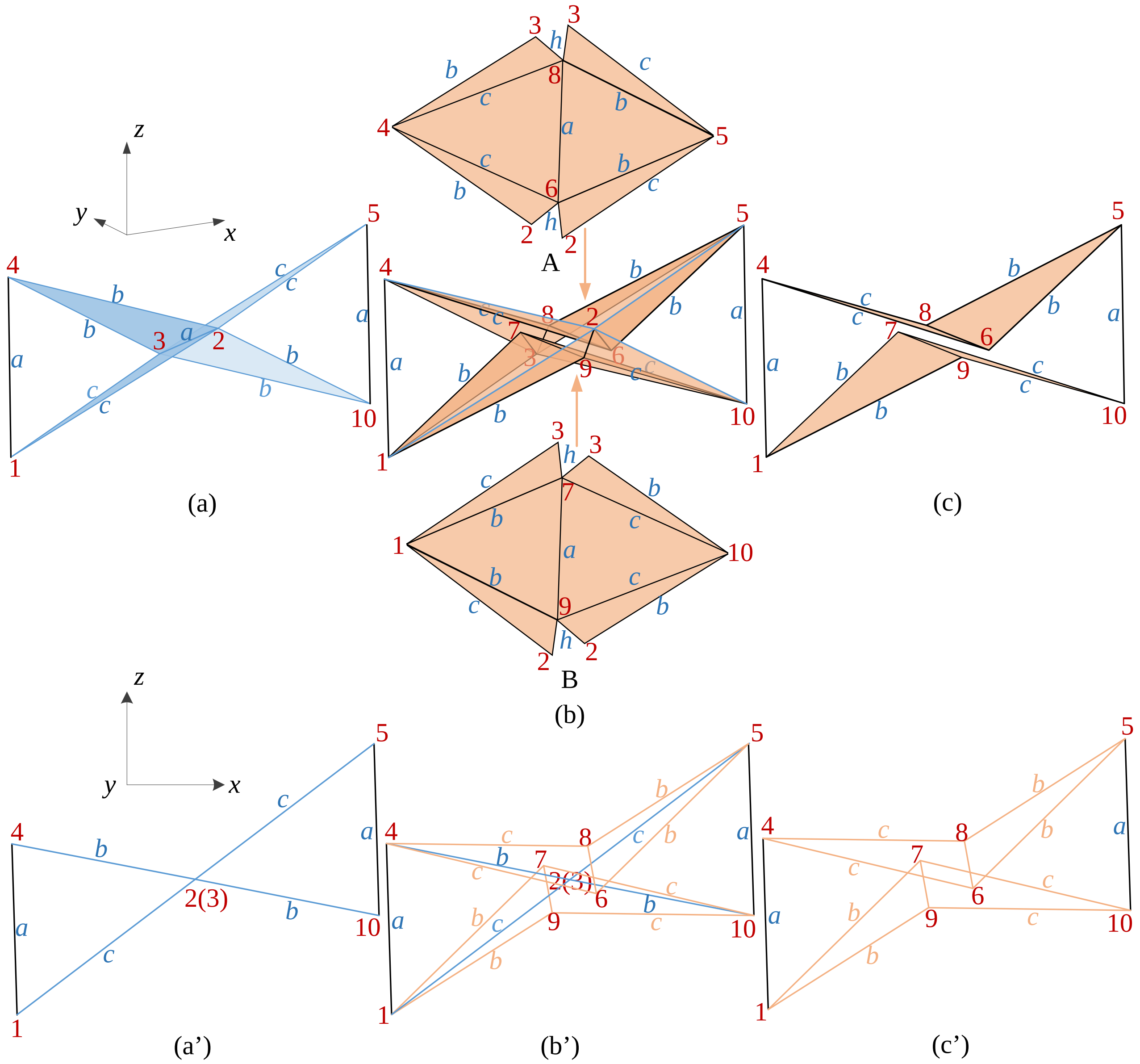

Figure 2: Decomposition of a one-dof two-tetrahedron flexible polyhedron, showing a pair of crinkles sliding away from each other. The viewpoint in (a)-(c) is chosen to clearly show the gap between two crinkles, while the viewpoint in (a')-(c') is along hinge 2-3. (a, a') The underlying skeleton of a new flexible polyhedron – two tetrahedra sharing a common edge. The faces of the two tetrahedra are not shown in order to highlight the two dihedra (shown in blue) and the hinge that will be replaced by two crinkles. (b, b') The pair of dihedra (in blue) are replaced by a pair of crinkles (in orange). The net of each crinkle is shown in A and B. Parameter b is set to be smaller than c, so that the two crinkles can 'slide away' from one another to avoid clashes. (c, c') Only the two dihedra in the middle of the crinkles are now shown, in orange. These are identical to the replaced original dihedra, but rotated about a C_2 axis through the midpoints of line 2-3 and line 4-5.

3. Optimization and results

Our goal is to adjust parameters to first eliminate any clashes, and then to maximize the range of motion. The first model we optimized is shown in Figure 1 with a set of parameters *a*–*d* plus a crinkle-height *h* (not shown). These parameters allow both a global 2-fold rotational symmetry (C_2) to the model, as well as a local 2-fold improper rotational symmetry (C_i) for each pair of tetrahedra joined by a hinge. The global symmetry has a C_2 axis through the midpoint of line 2-5 and the midpoint of line 3-10 in Figure 1(b). The local symmetry is an inversion through the centre point of each hinge (equivalent to a reflection in a mirror, followed by a C_2 rotation about an axis perpendicular to the mirror). Each crinkle can be added in one of two configurations (valley one side, ridge the other), and we choose to add these in a manner consistent with the symmetry. Later, to achieve larger ranges of motion, these symmetry criteria will be relaxed, and the direction of the crinkles might change from hinge to hinge.

The starting point of our optimization has initial parameter values chosen as $a = 1$, $b = 1$, $c = \sqrt{2}$, $d = \sqrt{3}$ and $h = 0.5$. However, there are clashes in the polyhedron with these dimensions which need to be eliminated. By varying all parameter values apart from *a*, a polyhedron without clashes was found with a range of motion about each hinge of 13.7˚. The resultant polyhedron is shown in Figure 3, with parameter values $a = 1$, $b = 1.091$, $c = 1.377$, $d = 1.750$ and $h = 0.534$. The net of this polyhedron is shown in Figure 4 with ridges and valleys drawn.

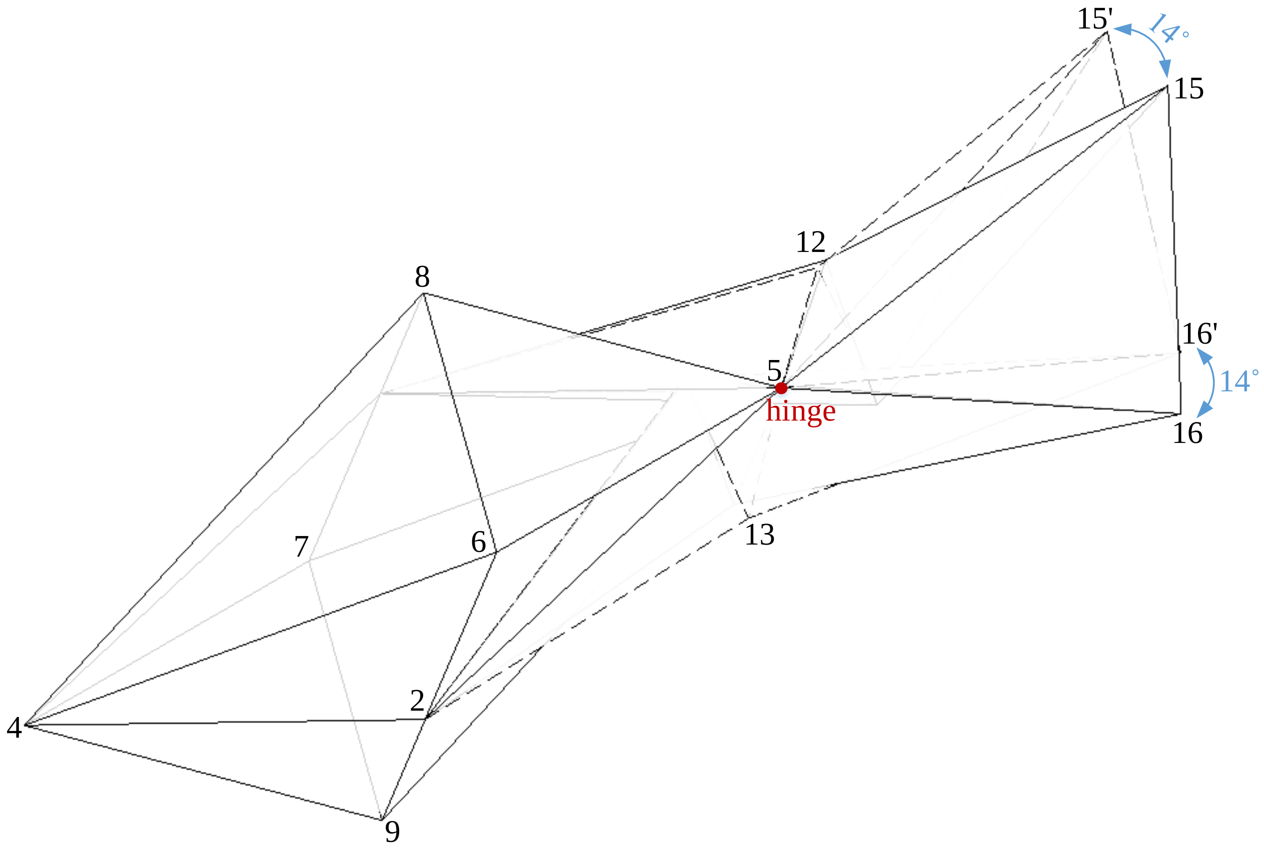

Figure 3: A rotational movement around one hinge in a three-tetrahedron flexible polyhedron. The left-hand two ‘tetrahedra’ are not moved, but the right-most tetrahedron is rotated around hinge 5-10. The two crinkles around this second hinge flex along with the movement of this tetrahedron. This polyhedron has the same set of parameters as the polyhedron in Figure 1, with values $a = 1$, $b = 1.091$, $c = 1.377$, $d = 1.750$ and $h = 0.534$. The full range of motion of each hinge is 13.7˚. The image shows two extreme positions of the rotational range for one hinge.

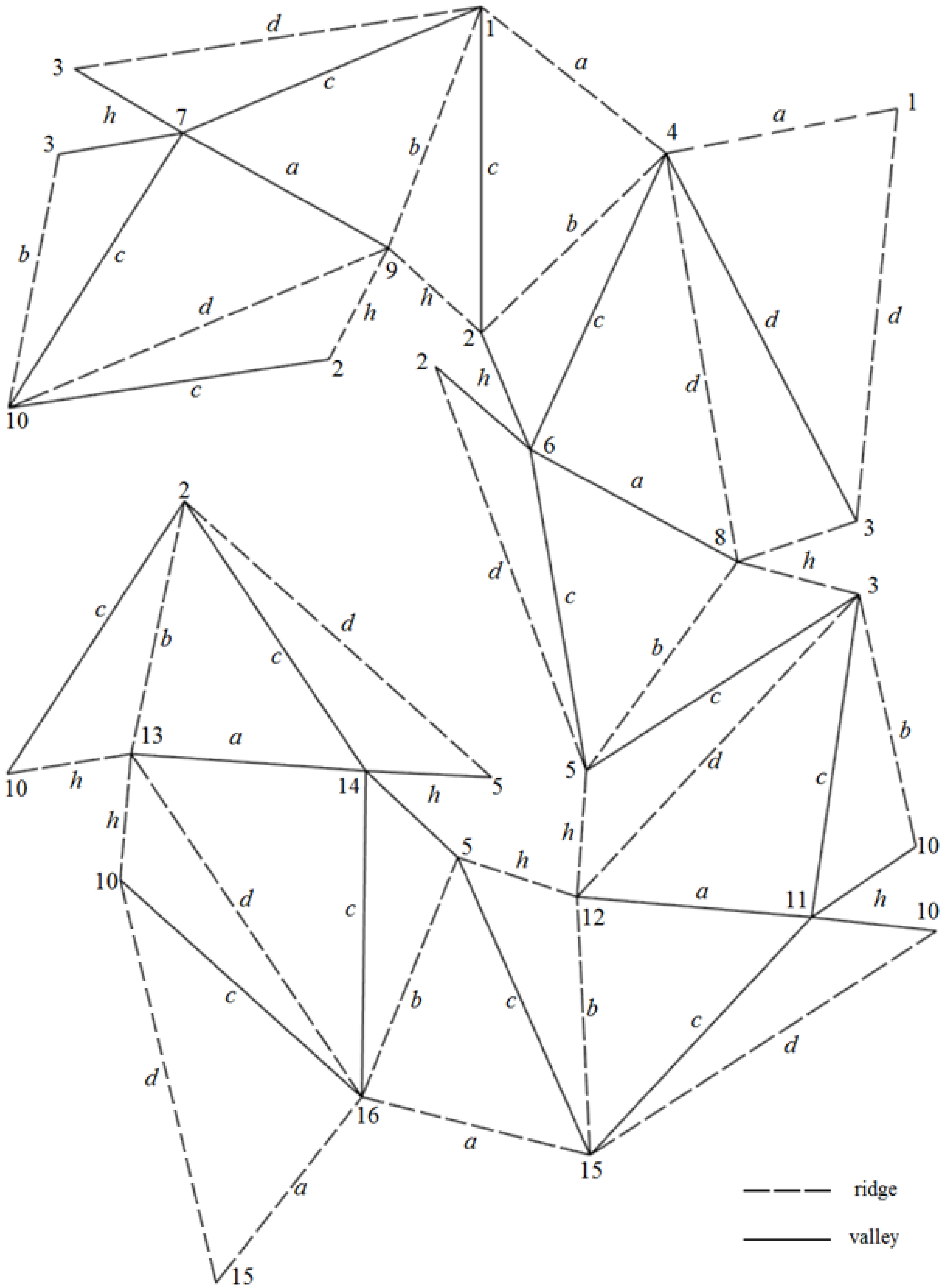

Figure 4: The net of the 2-dof, three-tetrahedron flexible polyhedron with parameter values a = 1, b = 1.091, c = 1.377, d = 1.750 and h = 0.534 and the range of motion of each hinge of 13.7˚. Due the global and local symmetry, all four crinkles in the net are identical, with the ridge and valley directions the same.

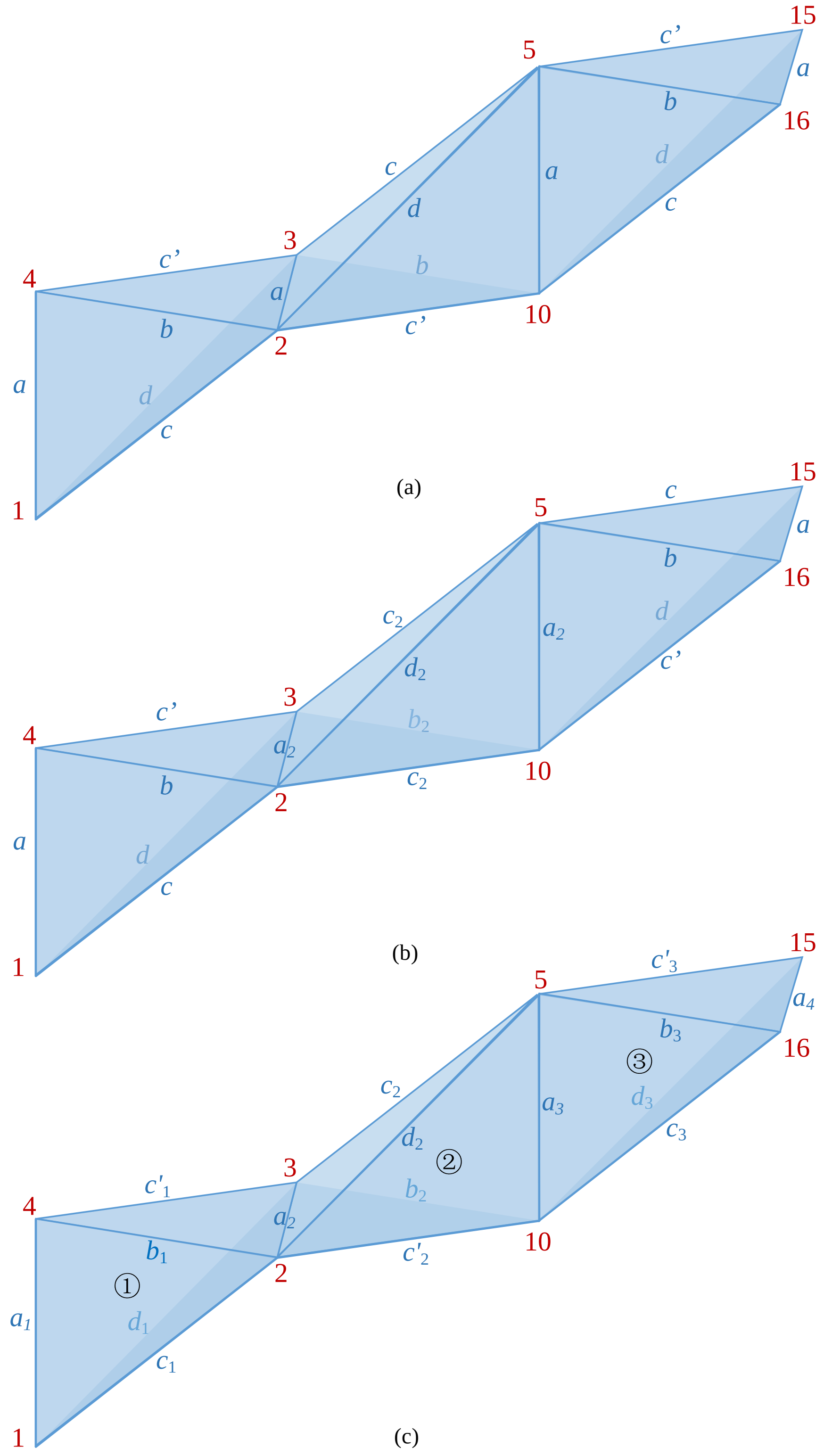

Figure 5: Various parameter schemes for the three-tetrahedra flexible polyhedron. (a) The global symmetry is broken by adding one more parameter *c*', but all tetrahedra are still identical (apart from a reflection) so that the polyhedron is extendable to repetitive *n*-dof systems. (b) The global symmetry is retained, but local symmetry of each hinge is broken by using a different set of parameters for the central tetrahedron. (c) A completely general set of parameters which preserves neither local nor global symmetry; there are now 16 parameters, plus the unshown height of each crinkle.

In order to increase the range of motion, more parameters are introduced. In this process, the global and local symmetry may not be retained. Possible schemes are shown in Figure 5. Here we consider a partial optimization based on the parameters shown in Figure 5(c). For complete generality, in addition to the parameters for the underlying tetrahedra, we would also need to separately specify the four crinkles – each of which could be of ‘Type I’ (based on a Bricard line-symmetric octahedron) or ‘Type II’ (based on a Bricard plane-symmetric octahedron) – see [1] for discussion of the different types of crinkles. In fact, we maintained a single ‘Type I’ crinkle in each case, and specified each with a single height parameter *h*. For initial simplicity, we kept a number of the other parameters equal to one another, as shown in Table 1. Despite these restrictions, it was possible to achieve a considerable range of motion (approximately 37˚ about each hinge).

Table 1: Parameter values that allow a large range of motion in a 2-dof polyhedron

Parameter	$a_1 = a_2 = a_3 = a_4$	$b_1 = b_2 = c'_1 = c'_2$	$c_1 = c_2 = d_1 = d_2$	b_3	d_3	c_3	c'_3	h
Value	6	8	8.5	8.52	7.8	8.9	7.7	1

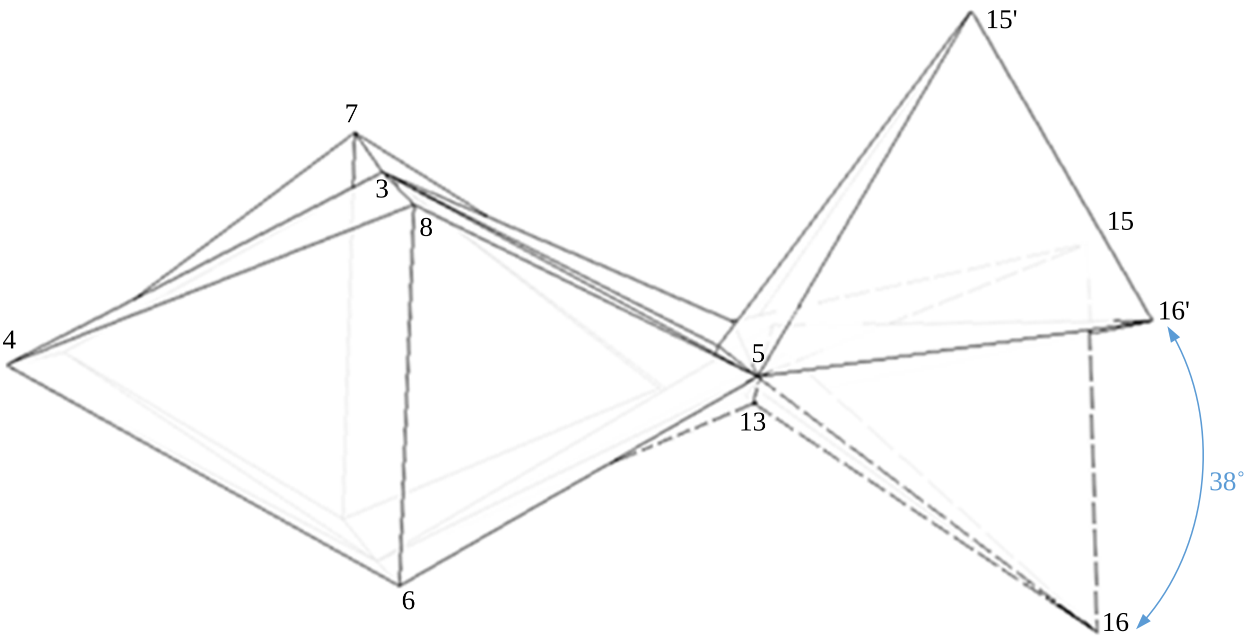

Figure 6: A three-tetrahedron flexible polyhedron showing two positions of the flexing motion about the second hinge. This polyhedron has the set of parameters given in Table 1. The range of motion around the first hinge is 36.5˚, around the second is 38.2˚. Vertices 7 and 8 stick into the page, and vertex 6 sticks out of page – this crinkle direction is the opposite of our choice in Figures 1, 2, 3 and 4, but around hinge 5-10, the crinkle directions are the same. This combination is found to allow large ranges of motion.

The incorporation of crinkles replaces all of the faces of the central tetrahedron 2-3-5-10. Thus, unlike the two-tetrahedron flexible polyhedron, there is now the possibility of clashes between crinkles that are replacing dihedra around adjacent hinges – specifically the crinkles adjacent to edges 2-5, 3-5, 2-10 and 3-10. However, during the optimization, these clashes are found to be much less likely than clashes between the pairs of crinkles replacing one hinge. Thus the potential clashes illustrated in Figure 2, for each of the two hinges, are the primary consideration during the optimization.

3. Independent motions of two hinges

We wanted to check if rotation about one hinge would affect rotation about the other in the 2-dof three-tetrahedron system. Here we use as an example the flexible polyhedron in Figure 6 to show the relationship between the allowed rotations of the two hinges. We first fixed the position of one hinge and tracked the range of motion of the other, and then changed the position of the first hinge and again tracked the full range of motion of the second hinge. Then we followed the same procedure the other way around to measure the range of motion of the first hinge with the second hinge fixed at different positions. Accessible positions of the three-tetrahedron flexible model are shown in Figure 7.

According to the graph, the range of motion of hinge 2-3 is always 36.5˚, no matter the position of hinge 5-10; and the range of motion of hinge 5-10 is always 38.2˚ irrespective of the position of hinge 2-3. This indicates that in this example the rotation of each hinge is not restricted by the motion of the other. The only possible way for the position of one hinge to limit the rotation of the other is through the clashes on the edges 2-5, 3-5, 2-10 and 3-10, i.e. between different pairs of crinkles, but as described in the previous section, these clashes tend not to occur. When we look for a feasible solution or try to optimize a polyhedron for a better range of motion, we can easily avoid clashes on these edges.

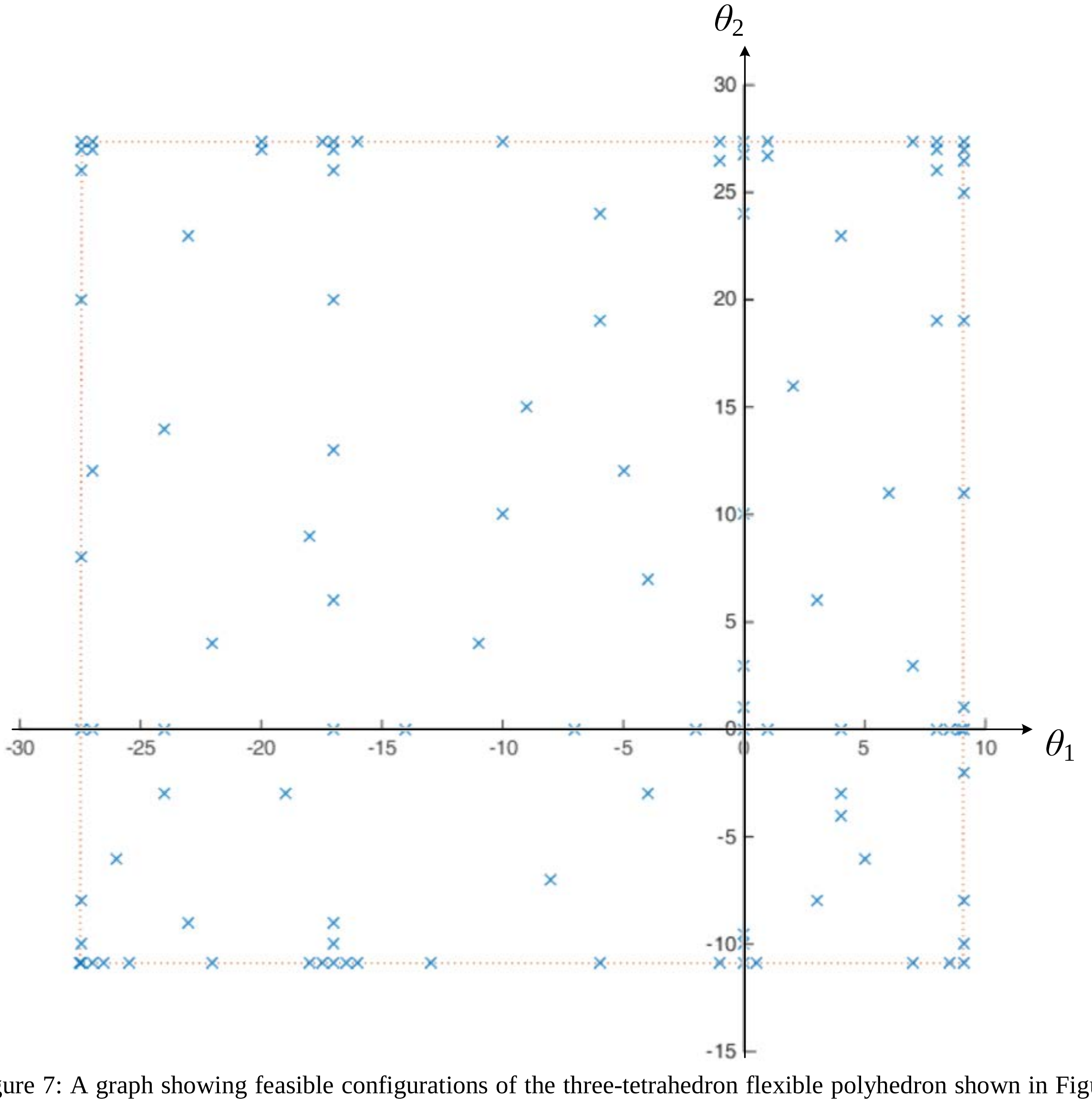

Figure 7: A graph showing feasible configurations of the three-tetrahedron flexible polyhedron shown in Figure 6. The rotation of one hinge θ_1 against the rotation of the other θ_2 are shown: θ_1 is the rotation around line 23; θ_2 is the rotation around the line 5-10. There are 109 results on this graph. They show that the full range of motion of hinge 23 is 36.5˚, and is independent of the rotation of hinge 5-10; the full range of motion of hinge 5-10 is 38.2˚, and is also not limited by the rotation of hinge 2-3.

4. Conclusion

This paper revisited the composition of the 1-dof two-tetrahedron flexible polyhedron, and shows how the same ideas can be used to generate a 2-dof three-tetrahedron flexible polyhedron. Clearly the same ideas can be extended to a longer chain of tetrahedra, to give a flexible polyhedron with as many degrees of freedom as required. Potentially the underlying chain of tetrahedra could form a loop, which could then be transformed to a flexible toroidal polyhedron. If this torus could rotate in a full 'anapole' manner [10] it would give a flexible polyhedron where continuous rotation of components was possible.

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