THE FIBER OF PERSISTENT HOMOLOGY FOR SIMPLICIAL COMPLEXES

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ABSTRACT

We study the inverse problem for persistent homology: For a fixed simplicial complex K, we analyse the fiber of the continuous map PH on the space of filters that assigns to a filter $f:K\to\mathbb{R}$ the total barcode of its associated sublevel set filtration of K. We find that PH is best understood as a map of stratified spaces. Over each stratum of the barcode space the map PH restricts to a (trivial) fiber bundle with fiber a polyhedral complex. Amongst other we derive a bound for the dimension of the fiber depending on the number of distinct endpoints in the barcode. Furthermore, taking the inverse image PH $^{-1}$ can be extended to a monodromy functor on the (entrance path) category of barcodes. We demonstrate our theory on the example of the simplicial triangle giving a complete description of all fibers and monodromy maps. This example is rich enough to have a Möbius band as one of its fibers.

Keywords Persistent Homology, Inverse Problems, Stratifications, Polyhedral Geometry, Entrance Path Category

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Introduction

Topological Data Analysis (TDA) is a rapidly expanding, new area [7, 17, 30] which has been applied to a large variety of data science problems. Its best-known tool, persistent homology, provides a non-linear dimension reduction method which is computable [29, 40] and robust with respect to small perturbations of the underlying data [10]. A growing number of vectorisation methods [1, 6] enable statistical studies of the outcome of persistent homology and combining it with machine learning methods.

It is thus natural to ask how much information can be recovered from persistent homology: Given a particular instance of a persistence module, what can we say about the data set it has been derived from? Any qualitative or quantitative understanding of the information loss would be of great value for future applications, and several approaches to variations of this question have recently appeared [12, 13, 15, 36]. We refer to [32] for a survey of inverse problems for persistent homology.

In this paper we analyse this foundational problem in a general form: For a fixed simplicial complex K, we study the persistent homology map PH (and its inverse) from the real-valued functions on K to the space of barcodes. We are naturally led to study PH as a map of stratified spaces and to extend PH⁻¹ to a functor defined on a natural category of barcodes. This rich structure we expect may also be of interest outside the data science community. Indeed, persistent homology has attracted much recent interest from other branches of mathematics, in particular symplectic topology, stemming from its connection to Morse theory and its close cousin Floer homology [2, 33, 37], and may yet find uses in other areas, see for example [25].

Content and results

Given a fixed finite simplical complex K, let Filt_K be the space of its filters. By definition these are functions $f:K\to I=[0,1]\subset\mathbb{R}$ that are monotonic with respect to face inclusions,

$$\sigma \subseteq \sigma' \Rightarrow f(\sigma) \leqslant f(\sigma')$$
 for all $\sigma, \sigma' \in K$.

Thus each sublevel set $f^{-1}((-\infty,t])$ defines a simplicial subcomplex of K and every f gives rise to a filtration. Persistent homology then defines a continuous map

$$PH : Filt_K \longrightarrow Bar^{\infty}$$
,

where $\operatorname{Bar}^{\infty}$ is the space of total barcodes and PH assigns the union of barcodes in all homological degrees. As K is fixed we restrict our attention to the image

$$Bar_K := PH(Filt_K).$$

In this notation, to understand the information loss of persistent homology is to understand the fiber $PH^{-1}(D)$ at a barcode $D \in Bar_K$. This naturally leads us to a closer analysis of the spaces $Filt_K$ and Bar_K themselves. We will endow them with monoid actions and stratifications, and show that PH is compatible with these extra structures.

Let $\operatorname{End}(I, \leq)$ be the monoid of order preserving continuous maps of the unit interval I that fix the endpoints, and let $\operatorname{Aut}(I, \leq)$ be its subgroup of homeomorphisms. The monoid $\operatorname{End}(I, \leq)$ and hence $\operatorname{Aut}(I, \leq)$ act continuously on Filt_K by post-composition and on Bar_K by moving the endpoints of the bars. As the endpoints of the bars in $\operatorname{PH}(f)$ are a subset of the values of f, the map PH is readily seen to be equivariant with respect to these actions (Lemma 1.5).

For our further analysis it is important that both Filt_K and Bar_K have a natural stratification where each *i*-dimensional stratum is identified with an open simplex

$$\mathring{\Delta}^{i} = \{(x_1, \dots, x_i) \mid 0 < x_1 < \dots < x_i < 1\}$$

such that the coordinates are given by the distinct values in the image of f, and respectively, the distinct endpoints of the bars in D. We identify each such stratum as an $\operatorname{Aut}(I, \leq)$ -orbit, and thus PH, by its equivariance, is a strongly stratified map taking a stratum of filters surjectively onto a stratum of barcodes (Proposition 1.15). In particular, the inverse image $\operatorname{PH}^{-1}(\mathcal{B})$ of any barcode stratum \mathcal{B} is a finite union of filter strata. We then show that PH over a stratum \mathcal{B} is a fiber bundle with fiber a polyhedral complex (Theorem 2.2), and derive some general properties of this fiber. Thus, for example, we show that the dimension of the fiber over a barcode D is bounded by half the difference between the number $\sharp K$ of simplices and the number $\sharp D$ of endpoints in the barcode (Proposition 2.5):

$$\dim \mathrm{PH}^{-1}(D) \leqslant \frac{\sharp K - \sharp D}{2}.$$

Unlike Filt_K, Bar_K is not (the realisation of) a simplical complex: On the boundary of a barcode stratum, viewed as an open simplex, the 1-dimensional subspaces corresponding to bars (x_i, x_j) of length zero are collapsed. Nevertheless,

we are able to describe the attaching (or monodromy) maps of the fiber over \mathcal{B} to the fiber of a lower dimensional stratum $\mathcal{B}'\subseteq\bar{\mathcal{B}}$ in its closure. These attaching maps are homotopic to maps of polyhedra but are not in general homotopic to each other (Proposition 3.12). We find that this structure is most naturally described in terms of the category $\mathbf{Bar}_{\mathbf{K}}$: Its objects are the barcodes in Bar_K and its space of morphisms from D to D' is the subspace of maps $\phi \in \mathrm{End}(\mathrm{I}, \leqslant)$ that send the endpoints of the bars in D surjectively to those of D'. Each of these morphism spaces is discrete up to homotopy (Theorem 3.2). Taking the inverse image then extends to a functor from $\mathbf{Bar}_{\mathbf{K}}$ to the category of topological spaces and continuous maps

$$PH^{-1}: \mathbf{Bar}_{\mathbf{K}} \longrightarrow \mathbf{Top}$$

taking a morphism $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$ to the continuous map $\mathcal{L}_{\phi} : f \in \mathrm{PH}^{-1}(D) \longmapsto \phi \circ f \in \mathrm{PH}^{-1}(D')$, which is indeed well-defined by the equivariance of PH under the action of $\mathrm{End}(\mathrm{I}, \leq)$.

The category \mathbf{Bar}_K , which we were naturally led to consider, is closely related to the entrance path category $\mathbf{Ent}(\mathrm{Bar}_K)$ of the space Bar_K which we prove to be homotopically stratified (Proposition 4.4) in the sense of Quinn [34]. Indeed, we show that descending to the homotopy category, i.e. replacing morphism spaces with the set of their connected components, induces an isomorphism of categories (Proposition 4.7)

$$h\mathbf{Bar}_{\mathbf{K}} \simeq \mathbf{Ent}(\mathbf{Bar}_K).$$

Recall that the entrance path category is the analogue for stratified spaces of the fundamental groupoid, and that functors from the entrance category to the category of sets are in correspondence with branched covers. Taking this analogy one step further by replacing the category of sets with the homotopy category of spaces h**Top**, we may most naturally think of PH as a stratified fiber bundle with polyhedral fibers:

$$PH^{-1} : \mathbf{Ent}(Bar_K) \longrightarrow h\mathbf{Top}.$$

Finally in section 5 we consider variants of our fiber problem and interactions with the symmetries of the underlying simplicial complex. Thus, in section 5.1 we consider the case where filters and barcodes are allowed to take values in the real line \mathbb{R} instead of I. The results on the fiber $\mathrm{PH}^{-1}(D)$ adapt to this situation, with the only difference that the polyhedra in the fiber may now be unbounded. However, we illustrate with examples that the topology of Bar_K is more complicated in the unbounded situation: we show that the bottleneck topology on barcodes does not in general agree with the easy to understand quotient topology, unlike in the bounded case (Proposition 1.17). This is in general no longer true in the unbounded situation. In addition, in section 5.2 we show that the overall analysis holds if instead of all filters we consider only lower star filters. Indeed, the space Low_K of lower star filters is a union of strata in Filt_K . Thus the fiber of PH restricted to the lower star filters is again a polyhedral complex.

In Appendix A, illustrating our theory, we describe in complete detail the case when K is a triangle (with 6 simplices). For each of the 34 barcode strata in Bar_K we describe the fiber with the action of the symmetry group of K and their monodromy maps. While most fibers consist of a set of discrete points, three fibers are homeomorphic to a circle, one is homeomorphic to two copies of the circle, and one is homeomorphic to the Möbius band. As far as we are aware this is the first non-contractible simplicial complex for which the fibers of PH have been studied and also the first example where the fibers are not homotopy discrete.

Related work

Our set-up here is most closely related to that in the work of Cyranka, Mischaikow and Weibel [15] where the authors consider lower star filters on the n-fold subdivided interval and show that the fibers of PH are homotopy discrete.

Previously, Curry in [12] considers the interval with the set of continuous maps. In particular he bounds the connected components of the fiber in terms of the nestings of the intervals in the barcode. Curry with coauthors also studies the higher dimensional example of a sphere in [8] with the set of functions that arise as compositions of an embedding of \mathbb{S}^2 into \mathbb{R}^3 followed by a projection onto the last coordinate.

In general, the persistent homology associated to a single filter cannot determine the underlying simplicial complex or its homotopy type, no more than homology can determine the homotopy type of the underlying space. However, under some conditions a family of such functions might suffice. To understand this question Turner, Mukherjee and Boyer introduced the persistent homology transform (PHT) [36], and proved that indeed under certain circumstances PHT is injective on shapes embedded in \mathbb{R}^3 , see also [14, 19] for a generalisation to higher dimensions. It has even been possible to find algorithmically a left inverse for PHT for some specific classes of sets [4, 5, 18, 26].

There are other persistence based invariants of spaces. One such (stable and computable) invariant for metric graphs has been proposed by Dey, Shi, and Wang [16]. In [31] Oudot and Solomon show that the fiber of this intrinsic transform is generically globally and always locally injective.

Since this paper was submitted, several other properties of the fiber $\mathrm{PH}^{-1}(D)$ have been studied. Motivated by our analysis here, in [23] the first author with Gregory Henselman-Petrusek provides an algorithm and software for the computation of the fiber $\mathrm{PH}^{-1}(D)$. This allows for many more examples to be computed explicitly. Furthermore, with David Beers he analyzes in [22] the fiber of PH for Morse functions on any smooth compact manifold with boundary $\mathcal M$ and shows that each path connected component in the fiber equals the orbit of one of its Morse functions under the action of isotopies of $\mathcal M$. This enables in particular the computation of the homotopy type of each path connected component in the fiber for almost all surfaces. Meanwhile, the second and third author of [15] have extended their analysis in [28] to the n-fold subdivided circle where the fibers are shown to be homotopy equivalent to $\mathbb S^1$.

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1 Stratifications of the spaces of filters and barcodes

We first recall some background theory and define the persistence map PH, the fibers of which are the object of interest. In section 1.2, we introduce the topological monoid $\operatorname{End}(I, \leq)$ of non-decreasing maps of the interval, which provides essential structure: it acts continuously on the spaces of filters and barcodes, and the persistence map PH is equivariant with respect to this action. In section 1.3, we show furthermore that the orbits of the subgroup of homeomorphisms, $\operatorname{Aut}(I, \leq)$, provide stratifications for the space of filters and the space of barcodes and that, due to its equivariance, PH is a strongly stratified map between them. It follows now easily that the fibers over barcodes from the same stratum are pairwise homeomorphic, thus turning the identification of the fiber into a finite problem. In section 1.4 we also show that the image Bar_K of the space of filters under PH has the quotient topology.

1.1 The definition of the persistence map

Let SCpx denote the category of finite (abstract) simplicial complexes and inclusions, and let $K \in \mathbf{SCpx}$ be an arbitrary, non-empty simplicial complex of dimension $d \in \mathbb{N}$, which is fixed throughout the paper. We consider K as a subset of the power set on its vertices. Recall, if $\sigma \in K$ then all its non-empty subsets $\sigma' \subset \sigma$, i.e. its *faces*, are also in K. We write $\sharp K$ for the total number of simplices in the complex K.

We denote by I := [0, 1] the closed unit interval. A typical function on K valued in I is denoted by $f \in I^K$.

Definition 1.1. A filter function, or *filter* for short, on K is a map $f: K \to I$ that is monotonic with respect to face inclusions: For all simplices $\sigma', \sigma \in K$

$$\sigma' \subseteq \sigma \Rightarrow f(\sigma') \leqslant f(\sigma).$$

The set of all filters on K is denoted by $Filt_K$.

The monotonicity condition on filters is equivalent to the property that their sublevel sets are simplicial subcomplexes of K. Thus a filter f gives rise to a filtration $K(f) = \{f^{-1}((-\infty, t])\}_{t \in \mathbb{R}}$, of K which we may think of as a functor from \mathbb{R} (as an ordered set) to the category of simplicial complexes

$$K(f): (\mathbb{R}, \leqslant) \longrightarrow \mathbf{SCpx}.$$

We can then compose this with the functor H_p which takes a simplicial complex to its p-th simplicial homology with coefficients in a fixed field k. This defines the *pth persistent homology functor*

$$H_n(K(f)): (\mathbb{R}, \leq) \longrightarrow \mathbf{Vect}_{\mathbb{R}},$$

which is an instance of a one-parameter, pointwise finite dimensional, finite persistence module, or *persistence module* for short. We denote by **Pers** the category of such persistence modules and natural transformations between them.

Given an interval $J \subseteq \mathbb{R}$, the associated *interval module* $\mathbb{I}_J \in \mathbf{Pers}$ has copies of the field \mathbb{k} over J and zero elsewhere, the copies of \mathbb{k} being connected by identity maps. Given a persistence module $\mathbb{V} \in \mathbf{Pers}$, by the Decomposition Theorem [11], there exists a unique finite multiset \mathcal{J} of intervals such that we have an isomorphism:

$$\mathbb{V} \cong \bigoplus_{J \in \mathcal{J}} \mathbb{I}_J.$$

The finite multiset $\operatorname{Bar}(\mathbb{V})$ of pairs $(\inf J, \sup J) \in (\mathbb{R} \sqcup \{-\infty\}) \times (\mathbb{R} \sqcup \{\infty\})$ for intervals $J \in \mathcal{J}$ appearing in the above decomposition is the so-called *barcode* of the module \mathbb{V} . If $\mathbb{V} = \operatorname{H}_p(K(f))$ is a persistent homology module, then the intervals that occur are all half-open intervals of the form J = [b,d) with restricted values $(b,d) \in I \times (I \sqcup \{\infty\})$. Consequently, we formally define barcodes as follows.

Definition 1.2. A barcode D is a finite multi-set of pairs (b,d) in $I \times (I \sqcup \{\infty\})$, with b < d, called the *intervals* or *bars* of D. An interval (b,d) is *bounded* (resp. *infinite*) if $d < \infty$ (resp. $d = \infty$). The multiplicity of an interval $(b,d) \in D$ is denoted by $D(b,d) \in \mathbb{N}$. The set of all barcodes is denoted by Bar.

We can now define the degree p persistence map as the composition

$$PH_p := Bar(H_p \circ K(.)) : Filt_K \longrightarrow Bar$$

and the (total) persistence map as the product

$$PH := (PH_0, \dots, PH_d) : Filt_K \longrightarrow Bar^{d+1}.$$

We will refer to elements $D = (D_0, \dots, D_d) \in \operatorname{Bar}^{d+1}$ simply as barcodes. We will mainly be interested in barcodes in the image of PH:

$$\operatorname{Bar}_K := \operatorname{PH}(\operatorname{Filt}_K) \subseteq \operatorname{Bar}^{d+1}.$$

The set of filters Filt_K is naturally topologised as a subset of the finite dimensional Euclidean space \mathbb{R}^K .

The standard topology on Bar, and hence on Bar_K , is induced by an (extended) metric which we now recall. A *matching* γ between two barcodes $D, D' \in \operatorname{Bar}$ is a partial injective map γ from intervals of D to those of D'. The $\operatorname{cost} c(\gamma)$ of a matching is the maximum of the following three quantities: (i) the maximum $\|(b,d) - \gamma(b,d)\|_{\infty}$ over intervals $(b,d) \in D$ where γ is defined, (ii) the maximal length $\frac{d-b}{2}$ over intervals $(b,d) \in D$ where γ is not defined, and (iii) the maximal length $\frac{d'-b'}{2}$ over intervals $(b',d') \in D'$ that are not in the image of γ . Here we allow ∞ as a possible value for d, d', and hence also for the maxima; and $\|.\|_{\infty}$ denotes the (extended) supremum norm on $\mathbb{R} \times (\mathbb{R} \cup \{\infty\})$. The *bottleneck distance*, d_b , between D and D' is then defined as:

$$d_b(D, D') := \inf_{\gamma \text{ matching }} c(\gamma).$$

Since our barcodes are finite multisets, d_b defines a true (extended) metric on Bar. We endow Bar with the induced bottleneck topology.

We thus have the following instance of the Stability Theorem [3, 9, 10] in our context.

Theorem 1.3. The degree p persistence map PH_p is Lipschitz continuous, and thus so is the persistence map PH.

For later reference, we record the following elementary fact.

Proposition 1.4. Let $D \in \operatorname{Bar}^{d+1}$ be a non-empty barcode. Then for small enough ϵ , another barcode D' is ϵ -close to D if and only if its intervals satisfy the following:

- for each integer $0 \le p \le d$ and interval $(b,d) \in D_p$, the intervals (b',d') in D'_p satisfying $\|(b',d')-(b,d)\|_{\infty} < \epsilon$ have multiplicities summing up to $D_p(b,d)$, the multiplicity of (b,d) in D_p ;
- the other intervals $(b', d') \in D'$, that is those that are not ϵ -close to intervals in D, are ϵ -small, i.e. $|d' b'| < \epsilon$.

Proof. Take $\epsilon \leqslant \frac{\alpha}{2}$ where α is the minimum of (a) the lengths d-b of intervals $(b,d) \in D$ and (b) all pairwise distances $\|(b,d)-(\overline{b},\overline{d})\|_{\infty}$ for any two geometrically distinct intervals (b,d) and $(\overline{b},\overline{d})$ in D. Note that $\alpha>0$ as by definition all our barcodes are finite, i.e. D has finite support.

1.2 Actions on filters and barcodes, and equivariance of the persistence map

Let $\operatorname{Aut}(I,\leqslant)$ be the space of orientation preserving homeomorphisms of I, and $\operatorname{End}(I,\leqslant)$ be the space of continuous non-decreasing maps that fix the boundary points 0 and 1. We consider them as subspaces of the space of all continuous maps of I to itself with the compact open (or equivalently $||.||_{\infty}$ -metric) topology. In this topology $\operatorname{End}(I,\leqslant)$ is the closure of $\operatorname{Aut}(I,\leqslant)$. For future reference we note that the straight line interpolation

$$\phi_t := (1 - t)\phi + t\phi'$$

between maps $\phi, \phi' \in \text{End}(I, \leq)$ defines a continuous path in $\text{End}(I, \leq)$.

Since the boundary points are fixed by elements in $\operatorname{Aut}(I, \leq)$ and $\operatorname{End}(I, \leq)$, they extend by the identity to automorphisms and endomorphisms of the real line \mathbb{R} and the extended real line $\mathbb{R} \cup \{\pm \infty\}$. When the context requires it, we will tacitly extend our maps without changing notation.

The monoid $\operatorname{End}(I, \leq)$ acts from the left on Filt_K by post-composition:

$$\phi.f := \phi \circ f.$$

It also acts from the left on Bar, and hence diagonally on Bar^{d+1} , by applying ϕ to all the endpoints of the bars in D with the convention that $\phi(\infty) = \infty$ and bars of length zero are suppressed:

$$\phi.D := \{ (\phi(b), \phi(d)) \mid (b, d) \in D \text{ and } \phi(b) \neq \phi(d) \}. \tag{1}$$

Thus $\phi.D$ contains (with multiplicities) an interval $(\phi(b), \phi(d))$ for each interval $(b, d) \in D$ as long as $\phi(b) \neq \phi(d)$, and an interval $(\phi(b), \infty)$ for each interval $(b, \infty) \in D$. Note that ϕ being non-decreasing does not imply that $\phi(x) \ge x$ for all x. In particular, through the action of ϕ , intervals can move to the left, to the right, be contracted or expanded.

A key result used in this work is that the persistence map is equivariant with respect to these actions.

Lemma 1.5 (Equivariance). The persistence map PH is $\operatorname{End}(I, \leq)$ -equivariant: For all $\phi \in \operatorname{End}(I, \leq)$ and $f \in \operatorname{Filt}_K$

$$PH(\phi \circ f) = \phi.PH(f).$$

Proof. We fix a filter $f \in \text{Filt}_K$ and a map $\phi \in \text{End}(I, \leq)$.

Recall that $\mathrm{PH}(f)$ is the union of $\mathrm{PH}_p(f)$ for $p=0,\ldots,d$, and $\mathrm{PH}_p(f)$ is given by the composition $\mathrm{Bar}(\mathrm{H}_p\circ K(f))$. By definition of $K(\phi\circ f)$, for $t\in\mathbb{R}$ we have

$$K(\phi \circ f)(t) = (\phi \circ f)^{-1} ((-\infty, t]) = f^{-1} ((-\infty, \max(\phi^{-1}(\{t\})))) = K(f) (\max(\phi^{-1}(\{t\}))).$$

Note that ϕ is non-decreasing and continuous. Thus the inverse image $\phi^{-1}(\{t\})$ of the point t is a closed, bounded interval and hence contains its maximum. On composition with the singular homology functor H_p this yields

$$H_p \circ K(\phi \circ f)(t) = H_p \circ K(f) (\max(\phi^{-1}(\lbrace t \rbrace))).$$

Hence the barcode $PH_p(\phi \circ f)$ is the barcode of the persistence module $t \mapsto H_p \circ K(f)(\max(\phi^{-1}(\{t\})))$, which rewrites uniquely as a sum of interval modules:

$$\begin{aligned} \operatorname{H}_p \circ K(f) \big(\max(\phi^{-1}(\{t\})) \big) &\simeq \Big[\bigoplus_{(b,d) \in \operatorname{PH}_p(f)} \mathbb{I}_{[b,d)} \Big] \big(\max(\phi^{-1}(\{t\})) \big) \\ &= \bigoplus_{(b,d) \in \operatorname{PH}_p(f)} \mathbb{I}_{[b,d)} \big(\max(\phi^{-1}(\{t\})) \big) \\ &= \bigoplus_{(b,d) \in \operatorname{PH}_p(f)} \mathbb{I}_{[\phi(b),\phi(d))} (t). \end{aligned}$$

The first equality follows from the definition of the barcode $\operatorname{PH}_p(f)$, i.e. $\operatorname{H}_p \circ K(f)$ decomposes as $\bigoplus_{(b,d) \in \operatorname{PH}_p(f)} \mathbb{I}_{[b,d)}$. The second equality holds because pre-composition by the map $t \mapsto \max(\phi^{-1}(\{t\}))$ induces an additive endofunctor on persistence modules. The third one is a consequence of ϕ being non-decreasing, as then $\max \phi^{-1}(\{t\}) \in [b,d)$ is equivalent to $t \in [\phi(b), \phi(d))$.

This yields
$$PH_p(\phi \circ f) = \phi.PH_p(f)$$
, and hence $PH(\phi \circ f) = \phi.PH(f)$.

Remark 1.6. In the above proof we indirectly made use of the following more general categorical framework where both K(f) and $H_p \circ K(f)$ are considered as functors from the category (\mathbb{R}, \leq) of ordered real numbers defining concrete instances of filtrations and persistence modules, that is functors

$$F:(\mathbb{R},\leqslant)\longrightarrow \mathbf{SCpx}$$
 and $\mathbb{V}:(\mathbb{R},\leqslant)\longrightarrow \mathbf{Vect}_{\mathbb{k}}.$

Precomposition with any endofunctor α of (\mathbb{R}, \leq) defines a right action both on filtrations and on persistence modules. Furthermore, composition by any functor $L: \mathbf{SCpx} \to \mathbf{Vect}_{\mathbb{R}}$ defines a map from filtrations to persistence modules, and we have the following general equivariance result due to associativity for composition of functors:

$$L(F.\alpha) = L \circ (F \circ \alpha) = (L \circ F) \circ \alpha = L(F).\alpha.$$

The endofunctors of (\mathbb{R}, \leq) are the (weakly) order preserving maps of \mathbb{R} , that is maps α satisfying: $t' < t \Rightarrow \alpha(t') \leq \alpha(t)$. Note that α does not have to be continuous.

For the proof of Lemma 1.5 we have used the functor $L=\mathrm{H}_p$ and the fact that an element $\phi\in\mathrm{End}(\mathrm{I},\leqslant)$ gives rise to an endofunctor $\alpha:=\max\phi^{-1}$ defined by $t\mapsto\max\phi^{-1}(\{t\})$. We furthermore used that $K(\phi\circ f)=K(f)\circ\max\phi^{-1}$ and $\mathbb{I}_{[\phi(b),\phi(d))}=\mathbb{I}_{[b,d)}\circ\max\phi^{-1}$ to translate the given action on the space of filters and barcodes into the functorial setting.

It is an easy exercise to show that the action of $\operatorname{End}(I,\leqslant)$ on filters is continuous. We next show that the action is also continuous on barcodes.

Proposition 1.7. End (I, \leq) acts continuously on Bar, i.e. it is induced by a continuous map

$$\operatorname{End}(I, \leq) \times \operatorname{Bar} \to \operatorname{Bar}.$$

Proof. We show that the map $\operatorname{End}(I,\leqslant) \times \operatorname{Bar} \to \operatorname{Bar}$ is sequentially continuous. Let $(\phi_n,D_n) \in \operatorname{End}(I,\leqslant) \times \operatorname{Bar}$ be a sequence converging to some (ϕ,D) . Let $\epsilon>0$ be small enough such that the intervals of any barcode that is ϵ -close to D satisfy the alternative of Proposition 1.4. As I is compact, ϕ is uniformly continuous and there exists $\eta>0$ such that $|\phi(d)-\phi(b)|<\epsilon$ whenever $b,d\in I$ satisfy $|d-b|<\eta$. Let n be large enough such that D_n is $\min(\epsilon,\eta)$ -close to D, and moreover $\|\phi_n-\phi\|_{\infty}<\epsilon$.

If $(b_n, d_n) \in D_n$ is a small interval, i.e. $d_n - b_n < \min(\epsilon, \eta)$, then:

$$|\phi_n(d_n) - \phi_n(b_n)| < |\phi_n(d_n) - \phi(d_n)| + |\phi(d_n) - \phi(b_n)| + |\phi(b_n) - \phi_n(b_n)| < 3\epsilon.$$

Therefore $(\phi_n(b_n), \phi_n(d_n))$ is a 3ϵ -small interval in $\phi_n.D_n$.

Else, (b_n, d_n) is $\min(\epsilon, \eta)$ -close to a unique interval (b, d) of D, and then

$$\|(\phi_n(b_n),\phi_n(d_n)) - (\phi(b),\phi(d))\|_{\infty} \leqslant \|(\phi_n(b_n),\phi_n(d_n)) - (\phi(b_n),\phi(d_n))\|_{\infty} + \|(\phi(b_n),\phi(d_n)) - (\phi(b),\phi(d))\|_{\infty} < 2\epsilon.$$

This yields a canonical matching from $\phi_n.D_n$ to $\phi.D$ with cost less than 3ϵ .

1.3 Stratifications of the spaces of filters and barcodes

We introduce a weak notion of stratification in the sense that we will not require (for now) any conditions on how strata are glued together. We will return to this in Section 4.1.

Definition 1.8. A stratification of a topological space X is a filtration $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots$ by a (possibly infinite) sequence of closed subspaces $X_i, i \in \mathbb{N}$, where the sets $X_i \setminus X_{i-1}$ are topological manifolds of dimension i. The path connected components of $X_i \setminus X_{i-1}$ are called *i-strata*, or strata of dimension i. A stratified map between two stratified spaces X and X' is a continuous, filtration preserving map of the underlying spaces. A strongly stratified map is a stratified map that maps any stratum of X surjectively to a stratum of X'.

We will show that Filt_K and Bar_K are stratified spaces with strata given by the $\operatorname{Aut}(I, \leq)$ -orbits, each homeomorphic to an open standard simplex for some i,

$$\mathring{\Delta}^i := \{ (x_1, \dots, x_i) \mid 0 < x_1 < \dots < x_i < 1 \} \subset \mathbb{R}^i.$$

Recall that $\operatorname{Aut}(I, \leq)$ acts continuously on the space of filters Filt_K by post-composition. For $f \in \operatorname{Filt}_K$, we denote the associated orbit by

$$S = S_f := \{ \phi. f \mid \phi \in Aut(I, \leq) \}.$$

Two filters are in the same orbit if they induce the same pre-order on the simplices of K: $\sigma \leqslant \sigma' \iff f(\sigma) \leqslant f(\sigma')$. Inside a given orbit a filter is uniquely determined by the sequence of its values that are not equal to 0 or 1 sorted in increasing order. Varying f by an element in $\operatorname{Aut}(I,\leqslant)$ varies this sequence over the whole open standard simplex. Thus for each orbit $\mathcal S$ the map

$$\mathcal{S} \stackrel{\cong}{\longrightarrow} \mathring{\Delta}^{\dim \mathcal{S}}, \qquad \dim \mathcal{S} := \sharp (\operatorname{Im}(f) \cap (0,1))$$

that sends a filter to the increasing sequence of its distinct values that are not equal to 0 or 1 defines an affine homeomorphism. The inverse map μ is a coordinate chart for the stratum S which in fact extends to the closure,

$$\mu: \Delta^{\dim \mathcal{S}} \xrightarrow{\cong} \overline{\mathcal{S}},$$
 (2)

and \overline{S} is the orbit of $\operatorname{End}(I, \leq)$. We record that the orbits define a stratification.

Proposition 1.9. For each $i \ge 0$, let F_i be the union of $\operatorname{Aut}(I, \le)$ -orbits S with $\dim S \le i$. This defines a stratification of Filt_K . The i-strata are given by the orbits of dimension i.

Remark 1.10. A stratum is simply an equivalence class of filters, where filters are declared equivalent if they induce the same pre-order on simplices. This point of view was already adopted in [24] in the context of persistence differentiation. Equivalently, the stratification is the hyperplane arrangement generated by the equalities $f(\sigma) = f(\sigma')$. It is well-known to be a Whitney stratification, but we will not make use of this richer structure here.

Similarly we construct a stratification of barcodes Bar^{d+1} . For $D \in \operatorname{Bar}^{d+1}$, we consider the associated $\operatorname{Aut}(I, \leq)$ -orbit

$$\mathcal{B} = \mathcal{B}_D := \{ \phi.D \mid \phi \in \operatorname{Aut}(I, \leqslant) \}.$$

The orbits partition the space of barcodes Bar^{d+1} . Within such an orbit, the multiplicities and the nestings of bars are constant, and it is only the consecutive values of the interval endpoints that can vary. Thus for each orbit \mathcal{B} the map

$$\mathcal{B} \xrightarrow{\cong} \mathring{\Delta}^{\dim \mathcal{B}}$$
, $\dim \mathcal{B} = \dim \mathcal{B}_D = \dim D := \sharp \text{ distinct endpoints of } D \text{ that are in } (0,1)$

that sends a barcode to the increasing sequence of its distinct values of interval endpoints that are not equal to 0, 1 or ∞ defines a homeomorphism. The inverse map ν is then a coordinate chart for the stratum:

$$\nu: \mathring{\Delta}^{\dim \mathcal{B}} \xrightarrow{\cong} \mathcal{B}. \tag{3}$$

Remark 1.11. In fact ν is even a local isometry when \mathcal{B} is equipped with the bottleneck distance and $\mathring{\Delta}^{\dim \mathcal{B}}$ with the $\|.\|_{\infty}$ -metric: This is because in a fixed stratum barcodes have a constant number and nestings of bars; hence endpoints can be matched (in an increasing order) and the $\|.\|_{\infty}$ -metric gives us the cost of the induced matching, which will be optimal when the barcodes are close enough.

Proposition 1.12. For each $i \ge 0$, let B_i be the union of the $\operatorname{Aut}(I, \le)$ -orbits \mathcal{B} with $\dim \mathcal{B} \le i$. This defines a stratification of Bar^{d+1} . The i-strata are the orbits of dimension i.

Proof. As orbits provide a partition, the sets B_i for $i \in \mathbb{N}$ form a filtration of Bar^{d+1} . Each of the B_i is also closed as the complement is open: barcodes close to a given barcode D have the same number of bars with endpoints in (0,1) or more as can be deduced from Proposition 1.4.

Furthermore, the complements $B_i \backslash B_{i-1}$ are by definition the union of finitely many (disjoint) orbits $\mathcal{B}_1, \ldots, \mathcal{B}_{l_i}$, each homeomorphic to $\mathring{\Delta}^i$. The lemma below implies that the closure of each \mathcal{B}_j does not intersect any of the \mathcal{B}_k for $k \neq j$. Thus a path D_t of diagrams in $B_i \backslash B_{i-1}$ with $D_0 \in \mathcal{B}_j$ cannot leave the orbit \mathcal{B}_j . On the other hand, each stratum \mathcal{B}_j is path connected since D_0 can be connected to any other diagram $D_1 \in \mathcal{B}_j$ by a linear path $D_t := (t\phi + (1-t)\mathrm{Id}).D_0$ where $\phi \in \mathrm{Aut}(I, \leqslant)$ is such that $D_1 = \phi.D_0$. Hence the path connected components of $B_i \backslash B_{i-1}$ are the orbits of dimension i, and $B_i \backslash B_{i-1}$ is a manifold of dimension i.

Lemma 1.13. Let $D, D' \in Bar^{d+1}$ be two barcodes. Then the following are equivalent:

- (1) There exists a non-decreasing map $\phi \in \text{End}(I, \leq)$ such that $D' = \phi.D$;
- (2) $\mathcal{B}_{D'} \subseteq \overline{\mathcal{B}_D}$, i.e. the stratum containing D' is in the closure of that containing D.

Proof. Assume (1) and let $\phi \in \text{End}(I, \leq)$ be such that $D' = \phi.D$. Consider the paths

$$\phi_t := (1 - t)\operatorname{Id} + t\phi$$
 and $D_t := \phi_t.D.$

For $t \in [0,1)$, $\phi_t \in \operatorname{Aut}(I,\leqslant)$ and hence $D_t \in \mathcal{B}_D$. By continuity of the monoid action, Proposition 1.7, the path of barcodes D_t is continuous in t on the whole interval [0,1]. Consequently, in the limit, $D' = D_1 \in \overline{\mathcal{B}_D}$. If $D'' \in \mathcal{B}_{D'}$ is another barcode from the orbit defined by D' then there exists a $\beta \in \operatorname{Aut}(I,\leqslant)$ with $D'' = \beta.D'$. Consider $\beta.D_t = (\beta \circ \phi_t).D$. By the same argument as above, this is a continuous path of barcodes from D to D'' that is contained entirely in \mathcal{B}_D with the possible exception when t = 1. Hence $D'' \in \overline{\mathcal{B}_D}$, and more generally $\mathcal{B}_{D'} \subseteq \overline{\mathcal{B}}_D$ which is (2).

Conversely, assume (2). If $D' \in \mathcal{B}_D$ then by definition of \mathcal{B}_D there exists a $\phi \in \operatorname{Aut}(I, \leqslant)$ with $D' = \phi.D$ and (1) is satisfied. So we may assume $D' \notin \mathcal{B}_D$ (and hence the entire orbit $\mathcal{B}_{D'}$ is contained in the boundary $\overline{\mathcal{B}_D} \setminus \mathcal{B}_D$). Let $D_n, n \geqslant 0$, be a sequence in \mathcal{B}_D converging to D', and let $\phi_n \in \operatorname{Aut}(I, \leqslant)$ such that $D_n = \phi_n.D$. Then, by the characterisation of the local neighborhoods in Bar, Proposition 1.4 for small enough ϵ and large enough n, the bars in n0 can be matched up (one-to-one) with bars in n0 which collapses the small bars. The number of intervals in n0 is the same as in n0, in particular finite, so for n0 small enough relative to the distances between consecutive endpoints n0, the union of the n0, the union of the n0 do not cover any segment n0. In this case we can use n0 to construct a non-decreasing map n0 end n1. With n2 mith n3 with n4 end n5 map n6. Hence, n6 end n7 is gives (1).

Remark 1.14. The monoid $\operatorname{End}(I, \leq)$ acts coordinate-wise on any simplex Δ^i , and under this action the orbit of any point in the interior $\mathring{\Delta}^i$ is the closed simplex Δ^i . As both μ and ν are compatible with the action restricted to $\operatorname{Aut}(I, \leq)$, they can be extended to equivariant maps from the closed simplex:

$$\nu: \Delta^{\dim \mathcal{B}} \longrightarrow \overline{\mathcal{B}}, \quad \text{with} \quad \nu(\phi.x) := \phi.\nu(x).$$
 (4)

This is easily seen to be well-defined, i.e. given $x, x' \in \mathring{\Delta}^{\dim \mathcal{B}}$ and $\phi, \phi' \in \operatorname{End}(I, \leqslant)$ such that $\phi.x = \phi'.x'$ we have $\phi.\nu(x) = \phi'.\nu(x')$. From the above lemma $\overline{\mathcal{B}}$ is the monoid orbit of $\operatorname{End}(I, \leqslant)$, therefore the extension ν is surjective. Hence $\overline{\mathcal{B}}$ (as a set) can be identified as a quotient of the closed standard simplex. Indeed, ν is a strongly stratified map where the stratification on the standard simplex is the usual one and $\overline{\mathcal{B}}$ is considered a sub-stratified space of Bar^{d+1} . In general ν is not injective on the boundary, see Example 1.19.

With both the stratifications of Filt_K and Bar^{d+1} in place, the persistence map $\mathrm{PH}:\mathrm{Filt}_K\to\mathrm{Bar}^{d+1}$ is then a map of stratified spaces in the following strong way:

Proposition 1.15. The persistence map is a strongly stratified map. Namely, let S be an i-stratum in the space of filters. Then there exists a j-stratum B with $j \leq i$ and PH(S) = B.

Proof. Strata in the spaces of filters and barcodes are the $Aut(I, \leq)$ -orbits with respect to which PH is equivariant by Lemma 1.5. We thus have for all $f \in Filt_K$ and associated stratum S_f

$$PH(\mathcal{S}_f) = \{PH(\phi, f) = \phi.PH(f) \mid \phi \in Aut(I, \leq)\} = \mathcal{B}_{PH(f)}.$$

Furthermore, over a fixed stratum in Bar_K the fibers are all homeomorphic.

Proposition 1.16. Let $\mathcal{B} \subseteq \operatorname{Bar}_K$ be a barcode stratum. The pre-images of PH over elements in \mathcal{B} are pairwise homeomorphic.

Proof. Let $D, D' \in \mathcal{B}$ so that $D' = \phi.D$ for some $\phi \in \operatorname{Aut}(I, \leq)$. By the equivariance of PH, Lemma 1.5, the action (by post-composition) of ϕ on Filt_K restricts to a map from $\operatorname{PH}^{-1}(D)$ to $\operatorname{PH}^{-1}(D') = \operatorname{PH}^{-1}(\phi.D) = \phi.\operatorname{PH}^{-1}(D)$. \square

Therefore Bar_K is a stratified subspace of Bar^{d+1} consisting of the union of strata $\operatorname{PH}(\mathcal{S})$, where $\mathcal{S} \subseteq \operatorname{Filt}_K$ varies over the set of strata of the space of filters. In particular Bar_K is a finite union of strata, finite dimensional and compact, unlike Bar^{d+1} which has infinitely many strata of arbitrarily large dimensions.

1.4 The space Bar_K as a quotient space

We will now show that as a topological space Bar_K is the quotient of the space of filters $Filt_K$ induced by the persistence map.

Proposition 1.17. The quotient topology on $Bar_K = PH(Filt_K)$ induced by PH agrees with the bottleneck topology, that is we have a homeomorphism from the quotient

$$\overline{\mathrm{PH}}: (\mathrm{Filt}_K / \sim) \xrightarrow{\cong} \mathrm{Bar}_K,$$

where \sim is defined by $f \sim f' \Leftrightarrow PH(f) = PH(f')$.

Proof. By Theorem 1.3, $PH : Filt_K \to Bar_K$ is continuous, and hence, by the universal property of the quotient, it induces a continuous bijection \overline{PH} . It remains to prove that the inverse \overline{PH}^{-1} is also continuous, or equivalently that \overline{PH} is open.

Let U be an open set in $(\operatorname{Filt}_K/\sim)$ and let $D\in U$. Then by definition of the quotient topology $\operatorname{PH}^{-1}(U)$ is open and contains $\operatorname{PH}^{-1}(D)$. Since PH is continuous, $\operatorname{PH}^{-1}(D)$ is closed, and being a subset of I^K it is in fact compact. Thus for some $\eta>0$ we have that the η -offset of $\operatorname{PH}^{-1}(D)$ lies in $\operatorname{PH}^{-1}(U)$:

$$PH^{-1}(D)_{\eta} := \{ f \in Filt_K, \exists g \in PH^{-1}(D), \|f - g\|_{\infty} < \eta \} \subseteq PH^{-1}(U).$$

We will show that $\mathrm{PH}^{-1}(D')\subseteq\mathrm{PH}^{-1}(D)_{\eta}$ for D' close enough to D in the bottleneck metric. By the above this implies that $\mathrm{PH}^{-1}(D')\subseteq\mathrm{PH}^{-1}(U)$, which amounts to $D'\in U$, and hence U is an open set in Bar_K .

By Proposition 1.4, for ϵ small enough, bars $(b',d') \in D'$ that are not ϵ -small are matched up with bars of D that are ϵ -close. We consider the following equivalence relation on interval endpoints b',d' of D'. First, we deem equivalent all endpoints that are ϵ -close to the same endpoint x_i of D. Then, we deem equivalent two ϵ -small intervals that overlap: $[b',d'] \cap [b'',d''] \neq \emptyset$ and take the transitive closure of that relation. Since there are at most $\sharp K$ endpoints in D', the endpoints in the same equivalence class span a range of size at most $\sharp K \times \epsilon$. Thus if ϵ has been chosen small enough to start with, then it is impossible to find in the same equivalence class two endpoints of D' that are ϵ -close to distinct endpoints $x_i \neq x_i$ of D.

This allows constructing a map $\phi: I \to I$ such that $\phi.D' = D$ as follows: Over the span of an equivalence class of endpoints we define ϕ as the constant map with value x_i in the case where there is an endpoint b' or d' which is ϵ -close to the endpoint x_i of D, and with an arbitrary value in the span in the case where there is no such endpoint in the equivalence class. We extend ϕ linearly on I. By design ϕ differs from the identity map by at most $\sharp K \times \epsilon$, because the span of each equivalence class has diameter bounded by $\sharp K \times \epsilon$. Hence if we take an arbitrary $f \in \mathrm{PH}^{-1}(D')$, then $g := \phi \circ f$ belongs to $\mathrm{PH}^{-1}(D)$ by equivariance of PH, and g is $(\sharp K \times \epsilon)$ -close to f. Up to shrinking ϵ so that $\sharp K \times \epsilon < \eta$, we have $f \in \mathrm{PH}^{-1}(D)_{\eta}$. Therefore $\mathrm{PH}^{-1}(D') \subseteq \mathrm{PH}^{-1}(D)_{\eta}$.

The top dimensional strata of Filt_K consists of the injective filters $f:K\to I$ that do not take the values 0 or 1. Hence the dimension of the top strata is $\sharp K$. The interval endpoints of a barcode $D=\mathrm{PH}(f)$ form a subset of the values of f and in general $\dim \mathcal{B}_D \leqslant \dim \mathcal{S}_f$. However, when f is injective, each simplex enters the next sublevel set of the filtration by itself and hence induces a change in homology. Thus in particular we see that the dimension of the top dimensional barcode strata in Bar_K is again $\sharp K$. Let

$$\operatorname{Bar}_K^{\operatorname{top}} := \bigcup_{\dim \mathcal{B} = \sharp K} \mathcal{B} = \{\operatorname{PH}(f) \mid f \in \operatorname{Filt}_K \text{ is injective and does not take the values } 0 \text{ or } 1\}.$$

Proposition 1.18. The barcode space Bar_K is the closure of its top dimensional strata, i.e.

$$\operatorname{Bar}_K = \overline{\operatorname{Bar}_K^{\operatorname{top}}}.$$

Proof. Let $D = \operatorname{PH}(f)$ be a barcode in the image. We can always factor f as $f = \phi \circ g$ for some injective filter $g \in \operatorname{Filt}_K$ and a non-decreasing map $\phi \in \operatorname{End}(I, \leq)$. By the equivariance of PH (Lemma 1.5), we have $D = \phi.\operatorname{PH}(g)$. Up to an arbitrarily small perturbation, g does not take the values g and g and g and g are element in a top dimensional stratum. The result then follows from Lemma 1.13.

The space Bar_K can thus be built as a quotient of a finite collection of (closed) simplices corresponding to the top dimensional barcode strata where some of the faces may be identified to each other and where *i*-dimensional faces may be reduced to a *j*-dimensional simplex through collapsing (i-j)-dimensional affine subspaces (corresponding to bars of length zero). The following example illustrates this.

Example 1.19. Let K represent the unit interval with vertices a,b and 1-simplex σ . The space of filters consists of two 3-dimensional strata corresponding to the induced orderings $a < b < \sigma$ and $b < a < \sigma$, which are mapped to each other via the action on K given by the involution (a,b) on its set of vertices. All faces are included in Filt_K and the two simplices are glued together along their common face corresponding to $a = b \leqslant \sigma$. Under the map PH the two 3-simplices are identified to one 3-simplex giving a unique top dimensional stratum $\mathcal{B}_{\mathrm{top}}$ in the space of barcodes Bar_K parametrising barcodes of the form $\{(x_1,\infty),(x_2,x_3)\}$ with $0 < x_1 < x_2 < x_3 < 1$. The barcode space Bar_K can then be identified with the quotient space of the closed 3-simplex where the 2-dimensional face corresponding to $0 \leqslant x_1 < x_2 = x_3 \leqslant 1$ is collapsed to the line segment $0 \leqslant x_1 = x_2 = x_3 \leqslant 1$, i.e. we identify:

$$(x_1, x_2, x_3) \sim (x_1, x_2', x_3')$$
 whenever $x_2 = x_3$ and $x_2' = x_3'$.

See Figure 1. We thus see that the fiber $\mathrm{PH}^{-1}(D)$ of a barcode $D=\{(x_1,\infty),(x_2,x_3)\}\in\mathrm{Bar}_K$ consists of two points if $x_1< x_2< x_3$, of one point if $x_1=x_2< x_3$, and of an interval (consisting of two intervals glued together) if $x_1< x_2=x_3$.

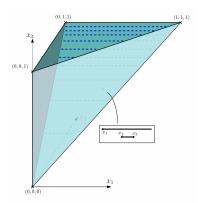


Figure 1: When K is the simplicial interval, the space Bar_K is the quotient space of the closed 3-simplex with each dotted line on the back face collapsed to one point.

2 The persistence map as a polyhedral stratified fiber bundle

Although the persistence map PH is globally not a fibration, we show in section 2.1 that it is a trivial fibration over each barcode stratum with a polyhedral complex as fiber. Using this structure in section 2.2, we derive topological properties of the fiber.

2.1 Polyhedral structure on the fiber

We strengthen Propositions 1.15 and 1.16, showing that, over each barcode stratum in the image $\operatorname{Bar}_K = \operatorname{PH}(\operatorname{Filt}_K)$, the persistence map is a trivial fiber bundle with a polyhedral complex as fiber. The intuition behind this result is that PH can be viewed as a piecewise linear projection as follows. Given a filter stratum $\mathcal{S} \subseteq \operatorname{Filt}_K$ and the barcode stratum $\mathcal{B} = \operatorname{PH}(\mathcal{S})$, the restriction $\operatorname{PH}_{|\mathcal{S}}$ can also be described as a map $\pi_{\mathcal{S}}^{\mathcal{B}} : \mathring{\Delta}^{\dim \mathcal{S}} \to \mathring{\Delta}^{\dim \mathcal{B}}$ via the coordinate charts $\mu : \mathring{\Delta}^{\dim \mathcal{S}} \stackrel{\cong}{\longrightarrow} \mathcal{S}$ and $\nu : \mathring{\Delta}^{\dim \mathcal{B}} \stackrel{\cong}{\longrightarrow} \mathcal{B}$ given by Eq. (2) and Eq. (3):

$$\overset{\mathring{\Delta}\dim \mathcal{S}}{\xrightarrow{\mu}} \overset{\mathcal{B}}{\longrightarrow} \overset{\downarrow}{\Longrightarrow} \\
\pi_{\mathcal{S}}^{\mathcal{B}} \overset{\downarrow}{\longrightarrow} \\
\mathring{\Delta}^{\dim \mathcal{B}} \xrightarrow{\nu} \overset{\mathcal{B}}{\Longrightarrow} \\$$
(5)

The map $\pi^{\mathcal{B}}_{\mathcal{S}}$ is the linear projection from $\mathbb{R}^{\dim \mathcal{S}}$ to $\mathbb{R}^{\dim \mathcal{B}}$ that records the values of a filter f which are bounded interval endpoints in the associated barcode $\mathrm{PH}(f)$. That PH can be described by this diagram follows from the following two elementary observations: (i) $\pi^{\mathcal{B}}_{\mathcal{S}}$ is $\mathrm{Aut}(\mathrm{I},\leqslant)$ -equivariant since μ and ν are equivariant, where $\mathrm{Aut}(\mathrm{I},\leqslant)$ acts on $\mathring{\Delta}^{\dim \mathcal{S}}$ and $\mathring{\Delta}^{\dim \mathcal{B}}$ coordinate-wise; and (ii) the set of bounded interval endpoints in the barcode PH is a subset of the values of f.

The fiber of such a projection map, restricted to the polyhedron $\mathring{\Delta}^{\dim \mathcal{S}} \cong \mathcal{S}$, is itself a polyhedron. In this section, we glue the polyhedra obtained in this way over the various filter strata \mathcal{S} in order to describe the whole fiber of PH over a barcode stratum as a complex of polyhedra.

Recall that a *(bounded) polyhedron* is a bounded, finite intersection of closed half-spaces in a Euclidean space. The *dimension* of a polyhedron is the dimension of its affine hull. A *face* of a polyhedron \mathcal{P} is the intersection of \mathcal{P} with a supporting hyperplane, and is itself a polyhedron. In particular, a polyhedron \mathcal{P} is a bounded convex Euclidean set. For such sets, we have the notions of relative interior and relative boundary, which offer the advantage not to depend on the ambient Euclidean space, and with a slight abuse of notations we denote them by $\mathring{\mathcal{P}}$ and $\partial \mathcal{P}$ respectively.

Definition 2.1. A polyhedral complex is a finite set Π of polyhedra in some Euclidean space \mathbb{R}^n , such that (i) if F is a face of $\mathcal{P} \in \Pi$, then $F \in \Pi$, and (ii) for all $\mathcal{P}, \mathcal{P}' \in \Pi$, the intersection $\mathcal{P} \cap \mathcal{P}'$ is either empty or is a face of both \mathcal{P} and \mathcal{P}' . By convention, the empty set is in Π . The *support* of Π is $\bigcup_{\mathcal{P} \in \Pi} \mathcal{P} \subseteq \mathbb{R}^n$. The *dimension* of a polyhedral complex is the maximal dimension of its polyhedra.

A map of polyhedral complexes, or *polyhedral map*, is a map that sends a polyhedron of the domain to a polyhedron of the co-domain surjectively, and whose restriction to each polyhedron \mathcal{P} of the domain is affine (i.e. can be extended to an affine map to the ambient space of \mathcal{P}). In the case where the polyhedra are simplicial complexes, the notion of polyhedral map coïncides with that of simplicial map, in that it is induced by a map defined on the abstract simplicial complexes. More generally, polyhedral complexes can be thought of as geometric realisations of simplicial complexes. In fact, it is a standard fact that a polyhedral complex admits a finite triangulation on the same set of vertices. For the proof of this result and more about polyhedral geometry, we refer the reader to [20].

Theorem 2.2. Let $\mathcal{B} \subseteq \operatorname{Bar}_K$ be a barcode stratum in the image of PH, and $D \in \mathcal{B}$ be any barcode. For each filter stratum $\mathcal{S} \subseteq \operatorname{Filt}_K \cap \operatorname{PH}^{-1}(\mathcal{B})$, let

$$\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D) = \overline{\mathrm{PH}^{-1}(D) \cap \mathcal{S}} \subseteq \mathbb{R}^K$$

be the (closure of the) restriction to the stratum S of the fiber of the persistence map over D. Then:

(a) Each $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$ is a polyhedron in \mathbb{R}^K of dimension $\dim \mathcal{S} - \dim \mathcal{B}$, and is affinely isomorphic to the product of $\dim \mathcal{B} + 1$ standard simplices (of various dimensions):

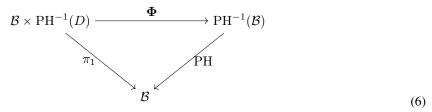
$$\Delta_0 \times \Delta_1 \times \Delta_2 \times \cdots \times \Delta_{\dim \mathcal{B}-1} \times \Delta_{\dim \mathcal{B}}$$
.

Moreover, the relative interior of $\overline{\mathrm{PH}^{-1}_{|\mathcal{S}}}(D)$ is $\mathrm{PH}^{-1}_{|\mathcal{S}}(D)$;

(b) The fiber $PH^{-1}(D)$ is the support of the polyhedral complex

$$\left\{ \overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D) \, | \, \mathcal{S} \, \text{filter stratum} \, \right\};$$

(c) There is a homeomorphism Φ giving the following commutative diagram:



where π_1 denotes the projection onto the first factor. Additionally, for any barcode $D' \in \mathcal{B}$ and filter stratum \mathcal{S} , the restricted map $\Phi(D',.)_{|\mathcal{S}}$ is an affine isomorphism between $\mathrm{PH}_{|\mathcal{S}}^{-1}(D)$ and $\mathrm{PH}_{|\mathcal{S}}^{-1}(D')$. In particular, the polyhedral structure of $\mathrm{PH}^{-1}(D)$ is the same for all barcodes in \mathcal{B} .

In assertion (a) above, the subcript in Δ_i is used as an index to order a list of standard simplices. This should not be confused with the notation Δ^i where the superscript denotes the dimension of the standard simplex. In Fig. 2 we illustrate this part of the theorem and include a sample computation for dim S – dim B.

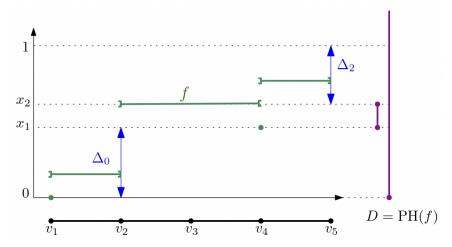


Figure 2: The filter f (green) on the complex obtained from subdividing the unit interval with five vertices yields the 2-dimensional barcode D (purple). The stratum $\mathcal S$ of f contains those filters which assign values in strictly increasing order to the vertices v_1, v_2, v_4, v_3 and v_5 and whose values over edges (v_i, v_{i+1}) are the maxima of their values over the endpoints v_i and v_{i+1} . The filters in the restricted fiber $\overline{\mathrm{PH}_{|\mathcal S}^{-1}}(D)$ must fix the equalities $f(v_1)=0$, $f(v_3)=x_2$ and $f(v_4)=x_1$, but we may vary $f(v_2)\in[0,x_1]$ and $f(v_5)\in[x_2,1]$. These two degrees of freedom correspond to the two simplices Δ_0 and Δ_2 , which are in this case of dimension 1, whereas Δ_1 is a singleton as no value of f lies in $f(x_1,x_2)$. We thus get the affine isomorphism between $\overline{\mathrm{PH}_{|\mathcal S}^{-1}}(D)$ and the product $\Delta_0\times\Delta_1\times\Delta_2$ of Theorem 2.2. In particular, we check that the dimension of the polyhedron $\overline{\mathrm{PH}_{|\mathcal S}^{-1}}(D)$ is $f(x_1,x_2)=f(x_1$

Proof. Recall from Eq. (5) that the persistence map rewrites as the projection map $\pi_{\mathcal{S}}^{\mathcal{B}}: \mathring{\Delta}^{\dim \mathcal{S}} \to \mathring{\Delta}^{\dim \mathcal{B}}$ onto the $\dim \mathcal{B}$ consecutive filter values that modify the homology groups of the sublevel sets, and that the fibers of $\pi_{\mathcal{S}}^{\mathcal{B}}$ are polyhedra. More precisely, let $(x_1, \cdots, x_{\dim \mathcal{B}}) := \nu^{-1}(D)$ be the consecutive endpoints of D, and let $1 \leq i_1 < \cdots < i_{\dim \mathcal{B}} \leq \dim \mathcal{S}$ be the projection coordinates of $\pi_{\mathcal{S}}^{\mathcal{B}}$. Then:

$$(\pi_{\mathcal{S}}^{\mathcal{B}})^{-1}(\nu^{-1}(D)) = \left\{ (x_i')_{i=1}^{\dim \mathcal{S}} \in \mathring{\Delta}^{\dim \mathcal{S}} \mid x_{i_1}' = x_1, \cdots, x_{i_{\dim \mathcal{B}}}' = x_{\dim \mathcal{B}} \right\}$$

$$= \left\{ (x_i')_{i=1}^{\dim \mathcal{S}} \in \mathbb{R}^{\dim \mathcal{S}} \mid (0 < x_1' < \cdots < x_{i_1}' = x_1) \cap \cdots \cap (x_{\dim \mathcal{B}} = x_{i_{\dim \mathcal{B}}}' < \cdots < x_{\dim \mathcal{S}}' < 1) \right\}$$

$$\cong \mathring{\Delta}^{i_1} \times \cdots \times \mathring{\Delta}^{\dim \mathcal{S} - i_{\dim \mathcal{B}}},$$

where the last homeomorphism is an affine isomorphism. Therefore the restricted fiber $\mathrm{PH}^{-1}(D) \cap \mathcal{S}$ is affinely isomorphic to the product

$$PH^{-1}(D) \cap \mathcal{S} \cong (\pi_{\mathcal{S}}^{\mathcal{B}})^{-1}(\nu^{-1}(D)) \cong \mathring{\Delta}_{0} \times \mathring{\Delta}_{1} \times \mathring{\Delta}_{2} \times \dots \times \mathring{\Delta}_{\dim \mathcal{B}-1} \times \mathring{\Delta}_{\dim \mathcal{B}}$$
(7)

of open simplices $\mathring{\Delta}_i$ whose dimensions sum up to $\dim \mathcal{S} - \dim \mathcal{B}$, where the open simplex $\mathring{\Delta}_i$ corresponds to values inbetween x_i and x_{i+1} , the i-th and i+1-th endpoint values in D (recall the convention that $x_0=0$ and $x_{\dim D+1}=1$). The linear isomorphism μ extends to a stratified linear isomorphism between the closed simplex $\Delta^{\dim \mathcal{S}}$ and $\overline{\mathcal{S}}$. Therefore, the homeomorphism in Eq. (7) extends to the closure and yields the affine homeomorphism:

$$\overline{\mathrm{PH}^{-1}(D) \cap \mathcal{S}} \cong \Delta_0 \times \Delta_1 \times \Delta_2 \times \cdots \times \Delta_{\dim \mathcal{B}-1} \times \Delta_{\dim \mathcal{B}}.$$
 (8)

This proves assertion (a). We can also deduce from the previous arguments that if $S' \subseteq \partial S$ is a boundary stratum, then

$$\overline{\mathrm{PH}_{|\mathcal{S}'}^{-1}}(D) \subseteq \partial \overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D). \tag{9}$$

To see this, let us fix $f \in \mathcal{S} \cap \mathrm{PH}^{-1}(D)$ and let f' be an arbitrary filter in $\overline{\mathrm{PH}_{|\mathcal{S}'}^{-1}}(D)$. Then the respective coordinates $\mu^{-1}(f)$ and $\mu^{-1}(f')$ are in the fiber of $\pi^{\mathcal{B}}_{\mathcal{S}}$ over $\nu^{-1}(D)$, therefore so is the straight line $[\mu^{-1}(f), \mu^{-1}(f')]$ by linearity of the projection. Since $\mu([\mu^{-1}(f), \mu^{-1}(f'))) \subseteq \mathcal{S}$, we have $\mu([\mu^{-1}(f), \mu^{-1}(f'))) \subseteq \mathrm{PH}_{|\mathcal{S}}^{-1}(D)$ and we deduce that $f' \in \overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$, and consequently $\overline{\mathrm{PH}_{|\mathcal{S}'}^{-1}}(D) \subseteq \overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$. Since $\mathcal{S}' \subseteq \partial \mathcal{S}$, the polyhedron $\overline{\mathrm{PH}_{|\mathcal{S}'}^{-1}}(D)$ in fact lies in the relative boundary of $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$.

We now show assertion (b), namely that the set of polyhedra $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$ is a polyhedral complex. So we first show that if $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$ is a polyhedron and $F \subset \overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$ is one of its face, then F is of the form $\overline{\mathrm{PH}_{|\mathcal{S}'}^{-1}}(D)$. We assume that F is a proper face, i.e. has codimension 1 in $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$. The general case follows by induction on the codimension.

The restriction of the affine homeomorphism of Eq. (8) to the face F implies that F is affinely isomorphic to the product

$$F \cong \Delta_0 \times \Delta_1 \times \Delta_2 \times \dots \times \Delta_i \times \dots \times \Delta_{\dim \mathcal{B}-1} \times \Delta_{\dim \mathcal{B}}, \tag{10}$$

where Δ_i' is a proper face of Δ_i . Note that Δ_i' is obtained by replacing one inequality between consecutive coordinates in Δ_i with an equality. This uniquely defines a stratum $\mathcal{S}'\subseteq\partial\mathcal{S}$ of codimension $\underline{1}$ in \mathcal{S} such that $\mathring{F}\subseteq\mathcal{S}'$. Since $F\subseteq\mathrm{PH}^{-1}(D)$, we have $F\subseteq\overline{\mathrm{PH}^{-1}_{|\mathcal{S}'}}(D)$. However, by Eq. (9), the polyhedron $\overline{\mathrm{PH}^{-1}_{|\mathcal{S}'}}(D)$ lies inside the relative boundary of $\overline{\mathrm{PH}^{-1}_{|\mathcal{S}}}(D)$, and in fact by convexity, inside a proper face of $\overline{\mathrm{PH}^{-1}_{|\mathcal{S}'}}(D)$. Since $F\subseteq\overline{\mathrm{PH}^{-1}_{|\mathcal{S}'}}(D)$ is itself a proper face, we deduce that $F=\overline{\mathrm{PH}^{-1}_{|\mathcal{S}'}}(D)$, as desired.

We now show that a non-empty intersection $\overline{\mathrm{PH}^{-1}_{|\mathcal{S}}}(D) \cap \overline{\mathrm{PH}^{-1}_{|\mathcal{S}'}}(D)$ of two polyhedra in the fiber is a common face of $\overline{\mathrm{PH}^{-1}_{|\mathcal{S}'}}(D)$ and $\overline{\mathrm{PH}^{-1}_{|\mathcal{S}'}}(D)$. If $\mathcal{S}=\mathcal{S}'$ there is nothing to prove. Otherwise, $\mathcal{S} \cap \mathcal{S}'=\varnothing$ so that $\mathrm{PH}^{-1}_{|\mathcal{S}'}(D) \cap \mathrm{PH}^{-1}_{|\mathcal{S}'}(D)=\varnothing$. We consider two cases:

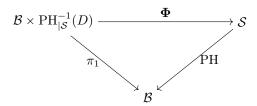
- 1. Either $\mathcal{S}' \subseteq \partial \mathcal{S}$, and then $\overline{\operatorname{PH}^{-1}_{|\mathcal{S}'}}(D)$ lies in a proper face $\overline{\operatorname{PH}^{-1}_{|\mathcal{S}''}}(D)$ of $\overline{\operatorname{PH}^{-1}_{|\mathcal{S}}}(D)$, and we must have $\mathcal{S}' \subseteq \partial \mathcal{S}''$. Thus we are done by an induction on the codimension of \mathcal{S}' in \mathcal{S} . Similarly if $\mathcal{S} \subseteq \partial \mathcal{S}'$;
- 2. Or $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$ and $\overline{\mathrm{PH}_{|\mathcal{S}'}^{-1}}(D)$ intersect only at their relative boundaries. In this case, by convexity of the two polyhedra, their intersection is the intersection $\overline{\mathrm{PH}_{|\mathcal{S}_2}^{-1}}(D) \cap \overline{\mathrm{PH}_{|\mathcal{S}_2'}^{-1}}(D)$ of some proper faces $\overline{\mathrm{PH}_{|\mathcal{S}_2}^{-1}}(D) \subseteq \overline{\mathrm{PH}_{|\mathcal{S}_2'}^{-1}}(D)$ and $\overline{\mathrm{PH}_{|\mathcal{S}_2'}^{-1}}(D) \subseteq \overline{\mathrm{PH}_{|\mathcal{S}_2'}^{-1}}(D)$. We are then left with the initial problem with polyhedra of smaller dimensions. We then conclude via an induction on the dimension of the polyhedra.

We now address the proof of assertion (c). Given a barcode $D' \in \mathcal{B}$, define $\phi_{D'}: I \to I$ to be the unique piecewise linear interpolation of the increasing map that takes the (bounded) endpoints of D' to the (bounded) endpoints of D, further fixing 0 and 1. Clearly, $\phi_{D'}$ is an orientation preserving homeomorphism. From the homeomorphism $\nu: \mathring{\Delta}^{\dim \mathcal{B}} \stackrel{\cong}{\longrightarrow} \mathcal{B}$, the intervals' endpoints in a barcode D' vary continuously with $D' \in \mathcal{B}$. Therefore $D' \mapsto \phi_{D'}$ is continuous, and in turn the map

$$\mathbf{\Phi}: (D', f) \in \mathcal{B} \times \mathrm{PH}^{-1}(D) \mapsto \phi_{D'}^{-1} \circ f \in \mathrm{PH}^{-1}(\mathcal{B})$$
(11)

is continuous. Similarly, its inverse given by $\Phi^{-1}(f')=(\mathrm{PH}(f'),\phi_{\mathrm{PH}(f')}\circ f')$ is continuous, so that Φ is a homeomorphism. Using Lemma 1.5, we have $\mathrm{PH}\circ\Phi=\pi_1$, i.e. the diagram in Eq. (6) commutes.

Let $\mathcal{S} \subseteq \mathrm{PH}^{-1}(\mathcal{B})$ be a filter stratum in the fiber. For $D' \in \mathcal{B}$, $\Phi(D',.) = \phi_{D'}^{-1} \circ _$ is the post-composition by $\phi_{D'}^{-1} \in \mathrm{Aut}(\mathrm{I}, \leqslant)$, see Eq (11), so by Lemma 1.5, the previous homeomorphism restricts to:



To finish the proof, we show that the homeomorphism $\Phi(D',.): \mathrm{PH}^{-1}(D) \cap \mathcal{S} \to \mathrm{PH}^{-1}(D') \cap \mathcal{S}$ is the restriction of an affine endomorphism of \mathbb{R}^K , for any barcode $D' \in \mathcal{B}$. Equivalently, we describe the coordinate functions $\Phi(D',.)_{\sigma}: f \mapsto \Phi(D',f)(\sigma)$ as affine forms, for each simplex $\sigma \in K$. Let $(x'_1,\cdots,x'_{\dim\mathcal{B}}):=\nu^{-1}(D')$, and let $x'_0=0$ and $x'_{\dim\mathcal{B}+1}=1$ by convention. Given $\sigma \in K$ and $f \in \mathrm{PH}^{-1}(D') \cap \mathcal{S}$, there is an index $0 \leqslant i \leqslant \dim\mathcal{B}$ such that $x'_i \leqslant f(\sigma) \leqslant x'_{i+1}$. If $f' \in \mathrm{PH}^{-1}(D') \cap \mathcal{S}$ is another filter in the fiber, we have $f'=\phi\circ f$ for some $\phi \in \mathrm{Aut}(I,\leqslant)$, and $\phi.D'=D'$ by the equivariance Lemma 1.5, so that $x'_i \leqslant f'(\sigma) \leqslant x'_{i+1}$ as well. Since $\phi_{D'}^{-1}$ is affine over $[x'_i,x'_{i+1}]$, we conclude that $\Phi(D',.)_{\sigma}$ is the restriction of an affine map, as desired.

2.2 Topology of the fiber

In this section, we collect a few results that restrict the topology of the fiber of PH over a barcode D in the image $\operatorname{Bar}_K = \operatorname{PH}(\operatorname{Filt}_K)$. We make use of the previous sections, and in particular we derive a bound for the dimension of the polyhedra in the fiber $\operatorname{PH}^{-1}(D)$.

With our first result we obtain finer control on the type of strata arising in Bar_K . Denote by

$$\operatorname{rk}(\partial_p) := \dim \operatorname{Im}(\partial_p : C_p(K, \mathbb{k}) \to C_{p-1}(K, \mathbb{k}))$$

the rank of the boundary map in the simplicial chain complex, and by $\beta_p(K) := \dim H_p(K, \mathbb{k})$ the p-th Betti of K. **Proposition 2.3.** Let $D = (D_0, \dots, D_d) \in \operatorname{Bar}_K$. Then, for any homology degree $0 \le p \le d$:

- (i) The number of infinite intervals in D_p equals $\beta_p(K)$; and
- (ii) The number of bounded intervals in D_p is smaller than or equal to $\operatorname{rk}(\partial_{p+1})$.

Proof. Item (i) follows from the fact that infinite bars in D correspond to the homology of K.

Let $D=\mathrm{PH}(f)$ be a barcode in the image, with endpoints $0< x_1< \cdots < x_{\dim D}<1$. By convention $x_0=0$ and $x_{\dim D+1}=1$. Via the Decomposition Theorem [11], there is an isomorphism between the p-th persistent homology module $\mathrm{H}_p\circ K(f)$ and $\oplus_{(b,d)\in D_p}\mathbb{I}_{[b,d)}$. Through such an isomorphism we obtain a basis of homology classes $[c_{(b,d)}]$. The representing simplicial chains $c_{(b,d)}$ are linearly independent in $C_p(K)=K(f)(1)$ and are cycles in K(f)(t) for $b\leqslant t< d$ that become boundaries in K(f)(d). In particular, the number of intervals (b,d) ending at x_i , here $i\geqslant 1$, is less or equal than $\mathrm{rk}((\partial_{p+1})_{|K(f)(x_i)})-\mathrm{rk}((\partial_{p+1})_{|K(f)(x_{i-1})})$. Summing over i yields item (ii).

Next we determine a bound for the dimension of the polyhedral complex $\mathrm{PH}^{-1}(D)$. Recall that barcode strata in Bar_K have maximal dimension $\sharp K$. Therefore by the (a) of Theorem 2.2 we have the obvious bound $\dim \mathrm{PH}^{-1}(D) \leqslant \mathrm{codim}(\mathcal{B})$, where we define the *codimension* of a barcode stratum \mathcal{B} as:

$$\operatorname{codim}(\mathcal{B}) := \sharp K - \dim(\mathcal{B}) \geqslant 0.$$

To improve this bound, we introduce the following quantity, which can be thought of as the number of missing bounded intervals in the target barcode D.

Definition 2.4. Let $\sharp K$ be the number of simplices in K, and $\sharp D$ be the number of interval endpoints in D with finite value (counted with multiplicities). The *bounded deficit* of D is the quantity:

$$\frac{\sharp K - \sharp D}{2} \geqslant 0.$$

Note that the bounded deficit is the same for all barcodes inside a given stratum.

Proposition 2.5. For any barcode $D \in Bar_K$, the dimension of the fiber of PH over D is less than or equal to the bounded deficit:

$$\dim \mathrm{PH}^{-1}(D) \leqslant \frac{\sharp K - \sharp D}{2} \leqslant \mathrm{codim}(D).$$

It follows in particular that $H_p(PH^{-1}(D)) = 0$ for $p > \frac{\sharp K - \sharp D}{2}$. We apply Proposition 2.5 to various fibers in the case where K is a triangle in Appendix A. For these fibers, the bounded deficit is almost systematically a tight upper-bound on the dimension of the fiber.

Proof. Since dim $D \leq \sharp K$ and dim $D \leq \sharp D$, we have dim $D \leq \frac{\sharp K + \sharp D}{2}$, therefore

$$\operatorname{codim}(D) = \sharp K - \dim D \geqslant \frac{\sharp K - \sharp D}{2}.$$

So we now investigate the left inequality $\dim \mathrm{PH}^{-1}(D) \leqslant \frac{\sharp K - \sharp D}{2}$.

Let \mathcal{S} be a filter stratum and let $(x_1, \cdots, x_{\dim D}) := \nu^{-1}(D) \in \mathring{\Delta}^{\dim D}$, with $x_0 := 0$ and $x_{\dim D+1} := 1$. We set $\{D\} := \{x_i\}_{i=0}^{\dim D+1}$. Let $\sigma \in K$ be a simplex. By the equivariance Lemma 1.5, a filter $f \in \mathrm{PH}^{-1}(D) \cap \mathcal{S}$ satisfy $f(\sigma) \in \{D\}$ if and only if all filters $f \in \mathrm{PH}^{-1}(D) \cap \mathcal{S}$ satisfy $f(\sigma) \in \{D\}$. There are at least $\sharp D$ such simplices, since each interval endpoint in the barcode D must correspond to at least one simplex entering the filtration.

Meanwhile, there are exactly $\dim S - \dim D$ distinct values $x = f(\sigma)$ that are not in $\{D\}$ for all filters $f \in \mathrm{PH}^{-1}(D) \cap S$. Each such value x is attained by at least 2 simplices, as otherwise $f^{-1}(x)$ would be a singleton hence would have non-zero Euler characteristic and we would have $x \in \{D\}$. We obtain

$$2 \times (\dim \mathcal{S} - \dim D) + \sharp D \leqslant \sharp K.$$

From the item (a) of Theorem 2.2, the polyhedron $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$ has dimension $\dim \mathcal{S} - \dim D$, and the above inequality yields $\dim \mathrm{PH}^{-1}(D) \leqslant \frac{\sharp K - \sharp D}{2}$.

When D is the image of an injective filter f, then $\sharp K = \dim D = \sharp D$ and hence, by Proposition 2.5, the dimension of the fiber above is zero. We thus have the following immediate consequence.

Corollary 2.6. If $f \in \text{Filt}_K$ is injective, then the fiber $\text{PH}^{-1}(\text{PH}(f))$ is a finite set.

Barcodes corresponding to injective filters are of maximal dimension. At the other extreme we have barcodes of dimension 1, i.e. barcodes D consisting simply of an infinite interval (x, ∞) , possibly with multiplicity. The constant filter with value x gives rise to such a barcode. Although the fiber $\mathrm{PH}^{-1}(D)$ does not reduce to this constant filter, we nevertheless show that it retracts to it.

Proposition 2.7. A barcode of dimension 1 in Bar_K has contractible fiber.

Proof. Let x be the unique endpoint value in D. By assumption, there exists a filter f in $PH^{-1}(D)$. Note that we must have $\min f = x$. We show that the straight line homotopy (1-t)f + tx lies in $PH^{-1}(D)$. For t < 1, the filters (1-t)f + tx and f induce the same pre-order on simplices of K, hence lie in a common stratum S. By Proposition 1.15, PH(S) = B where B is the stratum containing D, which consist in barcodes D' obtained from D by moving the unique endpoint value x to any other value x'. Hence PH((1-t)f + tx) is such a barcode, and must equal D since $x' = \min(1-t)f + tx = \min f = x$. By continuity, at t = 1, we further get that the constant filter x is in the fiber. Then $PH^{-1}(D)$ is star-shaped around x.

Remark 2.8. The dimension $\dim \mathrm{PH}^{-1}(D)$ being upper-bounded, we can ask if conversely the star of every polyhedron in the fiber has dimension $\dim \mathrm{PH}^{-1}(D)$. The example section, Appendix A, suggests that this property holds true in the case where the complex K is a manifold. In general however, two distinct filters in the fiber may have neighborhoods of distinct dimensions. Consider the following barcode D



where each endpoint value is given a different color. In Fig. 3 below we draw the simplicial complex (left), together

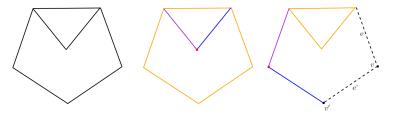


Figure 3

with two filters (middle and right) in the fiber $PH^{-1}(D)$ whose values on simplices are indicated by the colors. The filter f on the right has unspecified values on vertices and edges v, v', e, e' (colored in black), which means that whatever these values are, we can modify them as long as f(v) = f(e) and f(v') = f(e') and still get a filter whose barcode is D. Therefore, the star of f is 2-dimensional. However, the filter in the middle is alone in its neighborhood: all its values are fixed to endpoints of D and infinitesimal changes of any of these values yield filters out of $PH^{-1}(D)$.

3 The barcode category and the fiber functor

In section 3.1, we define morphisms between barcodes. This makes the image of the persistence map into a topological category, which is homotopy discrete (Theorem 3.2). We further show that morphisms of barcodes can be deformed into particularly nice morphisms which we refer to as *simplicial* morphisms (Proposition 3.4). Such morphisms can always be described as finite compositions of morphisms between codimension 1 barcodes (Proposition 3.5). In section 3.2, morphisms of barcodes are pulled-back to provide maps of fibers, that are furthermore maps of polyhedral complexes up to homotopy (Proposition 3.12).

3.1 The barcode category is homotopy discrete

We make the image Bar_K of the persistence map into a category, so that in the next section we view the fiber PH^{-1} as a functor. For this we use the action of $\operatorname{End}(I, \leq)$ on Bar_K as defined in Eq. (1) in order to define the morphisms between any two barcodes:

Definition 3.1. We denote by $\mathbf{Bar}_{\mathbf{K}}$ the **Top**-enriched *category of barcodes* with $D \in \mathrm{Bar}_K$ as objects and non-decreasing continuous maps $\phi \in \mathrm{End}(\mathrm{I}, \leq)$ such that $\phi.D = D'$ as the space of morphisms between D and D', $\mathbf{Bar}_{\mathbf{K}}(D, D')$. Two morphisms $\phi_0, \phi_1 \in \mathbf{Bar}_{\mathbf{K}}(D, D')$ are *homotopic as morphisms* if they belong to the same path connected component of $\mathbf{Bar}_{\mathbf{K}}(D, D')$.

By definition, the sets of morphisms in a **Top**-enriched category \mathcal{C} come equipped with a topology, and so taking path connected components yields the associated *homotopy category* which is denoted by $h\mathcal{C}$. The following result, whose proof is delayed to the end of the section, states that the spaces of morphisms in $\mathbf{Bar}_{\mathbf{K}}$ are made of contractible components.

Theorem 3.2. For any two barcodes D, D' in Bar_K , the space of morphisms $Bar_K(D, D')$ has finitely many path connected components each of which is contractible. In particular, the category Bar_K is homotopy discrete, i.e. $Bar_K(D, D') \simeq hBar_K(D, D')$.

In fact, when D and D' belong to the same stratum, $\mathbf{Bar}_{\mathbf{K}}(D, D')$ is contractible. Indeed, a morphism ϕ from D to D' is then simply a non-decreasing map that sends the i-th endpoint of D to the i-th endpoint of D'. Hence, given an arbitrary $\phi_0 \in \mathbf{Bar}_{\mathbf{K}}(D, D')$, the straight line homotopy $(t, \phi) \mapsto (1 - t)\phi + t\phi_0$ is a deformation retract

of $\mathbf{Bar}_{\mathbf{K}}(D, D')$ onto the point ϕ_0 . In general, however, there may be more than one homotopy class of morphisms in $\mathbf{Bar}_{\mathbf{K}}(D, D')$.

Before we embark on the proof of this, we explore some other properties of the morphism space $\mathbf{Bar_K}(D,D')$. As before, we use the coordinate charts $\nu:\mathring{\Delta}^{\dim D} \stackrel{\cong}{\longrightarrow} \mathcal{B}_D$ of Eq. (3) in order to define the consecutive endpoint values $(x_1,\cdots,x_{\dim D})=\nu^{-1}(D)$ and $(x_1',\cdots,x_{\dim D'}')=\nu^{-1}(D')$. By convention, we also set $x_0=x_0'=0$ and $x_{\dim D+1}=x_{\dim D'+1}=1$.

We denote by I(D) the simplicial complex obtained from subdividing the unit interval by the dim D endpoints $0 < x_1 < \cdots < x_{\dim D} < 1$. The morphisms from D to D' that send endpoints to endpoints piecewise linearly are of particular interest to us.

Definition 3.3. A morphism $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$ is *simplicial* if it is induced from the geometric realisation of a simplicial map from I(D) to I(D').

Proposition 3.4. For any two barcodes $D, D' \in Bar_K$, each connected component of $Bar_K(D, D')$ contains at least one simplicial map.

Proof. Let Ω be a path connected component of $\mathbf{Bar}_{\mathbf{K}}(D,D')$ and $\phi \in \Omega$. Let ψ be the piecewise linear extension of the map sending each x_i to the unique endpoint x_j' satisfying $x_j' \leq \phi(x_i) < x_{j+1}'$. The morphism ψ is then simplicial, and it is homotopic to ϕ through a straight-line homotopy.

Therefore, a morphism of barcodes is (up to homotopy) a simplicial map over the unit interval. We next show that simplicial morphisms are finite compositions of simplicial morphisms between barcodes that differ by one dimension. Note that this implies that the morphisms in $h\mathbf{Bar}_{\mathbf{K}}$ are generated by morphisms between barcodes that differ by one dimension.

Proposition 3.5. Let $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$ be a simplicial morphism. There exists a finite sequence of barcodes

$$D =: D_1, D_2, \cdots, D_k := D'$$

satisfying $0 \le \dim D_i - \dim D_{i+1} \le 1$ for $1 \le i \le k-1$, together with simplicial morphisms $\phi_i \in \mathbf{Bar}_{\mathbf{K}}(D_i, D_{i+1})$ between them, such that

$$\phi = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_2 \circ \phi_1.$$

Proof. We proceed by induction on $\dim D - \dim D'$. If $\dim D - \dim D' \in \{0,1\}$, the statement is trivial. So we assume that $\dim D - \dim D' \geqslant 2$. We first treat the case where ϕ sends the first endpoint x_1 of D to 0. Then, we may write ϕ as a composition $\phi' \circ \phi_1$ where ϕ_1 is the map that collapses the interval $[0,x_1]$ to 0 and sends $[x_1,1]$ to [0,1] linearly. Clearly then, $D_1 := \phi_1(D)$ satisfies $\dim D - \dim D_1 = 1$, and the morphisms $\phi_1 \in \mathbf{Bar}_{\mathbf{K}}(D,D_1)$ and $\phi' \in \mathbf{Bar}_{\mathbf{K}}(D_1,D')$ are simplicial. After repeating this operation as many times as necessary, we may assume that ϕ does not send any endpoint x_i to 0, for $1 \leqslant i \leqslant \dim D$. Symmetrically, we may assume that ϕ does not send any endpoint x_i to 1, for $1 \leqslant i \leqslant \dim D$.

The morphism ϕ being simplicial, there is an endpoint x'_j , here $1 \leq j \leq \dim D'$, whose pre-image by ϕ is a sequence of at least two consecutive endpoints x_i of D. We choose an endpoint $x_i \in \phi^{-1}(x_j)$ for which there exists a non-trivial interval (x_i,d) (or (b,x_i)) in D such that $\phi(d) \neq x'_j$ (or $\phi(b) \neq x'_j$). Such an endpoint must exist because x'_j is the endpoint of a non-trivial interval in D', which is the image by ϕ of an interval in D. We may assume that $x_{i+1} \in \phi^{-1}(x_j)$, as otherwise $x_{i-1} \in \phi^{-1}(x_j)$ and the rest of the proof can be conducted similarly. So $\phi(x_i) = \phi(x_{i+1}) = x'_j$, and in fact $\phi([x_i, x_{i+1}]) = x'_j$ since ϕ is non-decreasing. We may thus factor ϕ as $\phi' \circ \phi_1$, where $\phi_1 \in \operatorname{End}(I, \leq)$ is the map that collapses the interval $[x_i, x_{i+1}]$ onto x'_j , extended linearly on I. The image $D_1 := \phi_1(D)$ then satisfies $\dim D - \dim D_1 = 1$, since $\phi_1((x_i, d)) = (x'_j, \phi(d))$ is a non-trivial interval in D_1 and ϕ acts injectively on the endpoints x_k , for $k \notin \{i, i+1\}$. Finally, the morphisms $\phi_1 \in \operatorname{Bar}_{\mathbf{K}}(D, D_1)$ and $\phi' \in \operatorname{Bar}_{\mathbf{K}}(D_1, D')$ are simplicial, which concludes the proof.

We return to the study of the homotopy type of $\mathbf{Bar}_{\mathbf{K}}(D, D')$. In order to characterise when any two morphisms in $\mathbf{Bar}_{\mathbf{K}}(D, D')$ are homotopic, the following structure will be instrumental:

Definition 3.6. Let $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$. The *index* of ϕ is the collection $\Phi := \{\Phi(j)\}_{j=0}^{\dim D'+1}$, where

$$\Phi(j) := \{ 1 \leqslant i \leqslant \dim D \mid \phi(x_i) = x_j' \}.$$

Given $\phi_0, \phi_1 \in \mathbf{Bar}_{\mathbf{K}}(D, D')$, we write $\phi_0 \nearrow \phi_1$ whenever

$$\forall 0 \leq j \leq \dim D' + 1, \ \Phi_0(j) \subseteq \Phi_1(j).$$

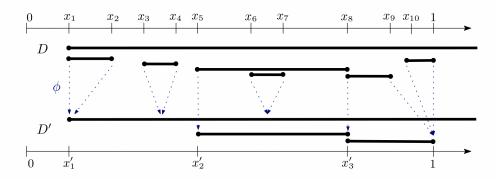


Figure 4: The map $\phi: D \to D'$ has index Φ consisting of $\Phi(0) = \emptyset$, $\Phi(1) = \{1, 2\}$, $\Phi(2) = \{5\}$, $\Phi(3) = \{8\}$ and $\Phi(4) = \{9, 10\}$.

Lemma 3.7. Let $\phi_0, \phi_1 \in \mathbf{Bar}_{\mathbf{K}}(D, D')$. Then the following are equivalent:

- (i) The two morphisms are homotopic, i.e. $\phi_0 \sim \phi_1$ as morphisms.
- (ii) For all $(b, d) \in D$ and $(b', d') \in D'$, we have $(\phi_0(b), \phi_0(d)) = (b', d')$ if and only if $(\phi_1(b), \phi_1(d)) = (b', d')$.
- (iii) The straight line interpolation $t \mapsto t\phi_0 + (1-t)\phi_1$ is a path in $\mathbf{Bar}_{\mathbf{K}}(D, D')$.
- (iv) There is some $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$ such that $\phi \nearrow \phi_0$ and $\phi \nearrow \phi_1$.

In particular, if $\phi_0 \nearrow \phi_1$ *, then* ϕ_0 *and* ϕ_1 *are homotopic.*

Proof. $[(i) \Rightarrow (ii)]$: Let ϕ_t be a path in $\mathbf{Bar_K}(D, D')$ joining ϕ_0 and ϕ_1 . Given $(b, d) \in D$ and $(b', d') \in D'$, let $\mathbf{I}_{b,d}^{b',d'} := \{t \in \mathbf{I}, \ \phi_t(b,d) = (\phi_t(b), \phi_t(d)) = (b',d')\} \subseteq \mathbf{I}$.

The sets $I_{b,d}^{b',d'}$ are closed in I. They are also open since the map $\sum_{[(b,d),(b',d')]\in D\times D'}\mathbb{1}_{I_{b,d}^{b',d'}}:I\to\mathbb{N}$ is constant and equals the number of intervals in D'. Therefore, $I_{b,d}^{b',d'}=I$ or $I_{b,d}^{b',d'}=\emptyset$.

 $[(ii) \Rightarrow (iii)]$: For $t \in I$, $\phi_t := t\phi_0 + (1-t)\phi_1$ is non-decreasing. Then, $\phi_t(b,d)$ equals a non-trivial interval $(b',d') \in D'$ if and only if $\phi_0(b,d) = \phi_1(b,d) = (b',d')$. All the other intervals $(b,d) \in D$ must then be trivialized, i.e. $\phi_0(b) = \phi_0(d)$ and $\phi_1(b) = \phi_1(d)$, so that $\phi_t(b) = \phi_t(d)$. This ensures that $\phi_t \in \mathbf{Bar}_{\mathbf{K}}(D,D')$ since for instance $\phi_0 \in \mathbf{Bar}_{\mathbf{K}}(D,D')$.

 $[(iii) \Rightarrow (i)]$: This implication is immediate. From now on, we have $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

 $[(iii) \Rightarrow (iv)] \text{: Let } \phi_t \text{ be the straight line interpolation between } \phi_0 \text{ and } \phi_1. \text{ Let } 0 \leqslant j \leqslant \dim D' + 1. \text{ Let } 1 \leqslant i \leqslant \dim D \text{ be such that } i \notin \Phi_0(j). \text{ Without loss of generality, we assume that } \phi_0(x_i) < x_j'. \text{ Since } x_j' \text{ is the endpoint of a non-trivial interval of } D' \text{ and } \phi_0(D) = D', \text{ there must exist an endpoint } x_{i'}, \text{ with } i' > i, \text{ of an interval in } D \text{ such that } \phi_0(x_{i'}) = x_j'. \text{ By the item } (ii), \text{ we also have } \phi_1(x_{i'}) = x_j'. \text{ In turn, } \phi_1(x_i) \leqslant x_j' \text{ as } \phi_1 \text{ is non-decreasing. Therefore } \phi_{\frac{1}{2}}(x_i) < x_j', \text{ and in particular } i \notin \Phi_{\frac{1}{2}}(j). \text{ We have therefore proved that } \Phi_{\frac{1}{2}}(j) \subseteq \Phi_0(j), \text{ so that } \phi_{\frac{1}{2}} \nearrow \phi_0. \text{ Similarly, } \phi_{\frac{1}{2}} \nearrow \phi_1.$

 $[(iv)\Rightarrow(i)]$: It is enough to show that if $\phi\nearrow\phi_0$, then ϕ and ϕ_0 are homotopic, as we then show in the exact same way that $\phi\sim\phi_1$, which implies $\phi_0\sim\phi_1$. Let $(b,d)\in D$ be a bounded interval (the case where $d=d'=\infty$ is dealt with similarly) such that $\phi(b,d)=(b',d')$ for some $(b',d')\in D'$. Then there are indices i< i' and j< j' such that $(b,d)=(x_i,x_{i'})$ and $(b',d')=(x'_j,x'_{j'})$. Since $\Phi(j)\subseteq\Phi_0(j)$ and $\Phi(j')\subseteq\Phi_0(j')$, we have $\phi_0(b,d)=(b',d')$ as well. Therefore, the images $\phi_0(b,d)$ of intervals (b,d) such that $\phi(b,d)\in D'$ cover all the intervals in D'. Hence, any other interval in D is trivialized by ϕ_0 , which guarantees that the assertion (ii) holds. We are done since $(ii)\Rightarrow(iii)\Rightarrow(i)$.

Remark 3.8. Two morphisms $\phi_0, \phi_1 \in \mathbf{Bar_K}(D, D')$ with the same index are homotopic. In the case where D and D' belong to strata that differ by one dimension, the converse is true as well so that in this case the index is a (complete) homotopy invariant. To see this, observe that the index Φ associated to a morphism ϕ in the case $\dim D - \dim D' = 1$ cover the endpoints of D and must consist of singletons $\Phi(j)$ except for a unique $\Phi(k)$ which is the unique pair $\{i, i+1\}$ of consecutive endpoints of D collapsed by ϕ . Therefore, if $\phi_0, \phi_1 \in \mathbf{Bar_K}(D, D')$ are homotopic, by Lemma 3.7

there is a third morphism $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$ such that $\phi \nearrow \phi_0$ and $\phi \nearrow \phi_1$. By the above restriction on the index of morphisms in $\mathbf{Bar}_{\mathbf{K}}(D, D')$, this immediately implies that ϕ_0 and ϕ_1 have the same index.

Proof of Theorem 3.2. Let $D, D' \in \operatorname{Bar}_K$. By the assertion (iv) of Lemma 3.7, if $\phi_0, \phi_1 \in \operatorname{Bar}_K(D, D')$ are two morphisms such that $\phi_0 \nearrow \phi_1$, then ϕ_0 and ϕ_1 belong to the same path connected component of $\operatorname{Bar}_K(D, D')$. Since there are finitely many possible indices, $\operatorname{Bar}_K(D, D')$ has finitely many path connected components.

Let Ω be a path connected component of $\mathbf{Bar}_{\mathbf{K}}(D, D')$. Let $\psi \in \Omega$. By the (iii) of Lemma 3.7, the map

$$(t,\phi) \in [0,1] \times \Omega \longmapsto t\psi + (1-t)\phi \in \Omega$$

is a deformation retraction of Ω onto $\{\psi\}$. Consequently, Ω is contractible.

3.2 Monodromies and polyhedral maps of fibers

We now analyse how fibers relate to each other as we cross barcode strata. We associate to each map of barcodes a map between the corresponding fibers as follows.

Definition 3.9. Let $D \in \operatorname{Bar}^{d+1}$ and $\phi \in \operatorname{End}(I, \leq)$. The *monodromy* \mathcal{L}_{ϕ} associated to ϕ is the map between fibers:

$$\mathcal{L}_{\phi}: f \in \mathrm{PH}^{-1}(D) \longmapsto \phi \circ f \in \mathrm{PH}^{-1}(\phi.D).$$

Note that the monodromy is well-defined since PH is $\operatorname{End}(I, \leq)$ -equivariant by Lemma 1.5,

$$PH(\phi \circ f) = \phi.PH(f),$$

and hence $\phi \circ f \in \mathrm{PH}^{-1}(\phi.D)$. Furthermore, given another $\phi' \in \mathrm{End}(\mathrm{I}, \leq)$ by definition

$$\mathcal{L}_{\phi'\circ\phi}=\mathcal{L}_{\phi'}\circ\mathcal{L}_{\phi},$$

and in particular, if ϕ is invertible then \mathcal{L}_{ϕ} is a homeomorphism. Furthermore, the monodromy assignment $\phi \mapsto \mathcal{L}_{\phi}$ is continuous. We thus see that monodromies turn the inverse image PH^{-1} into a functor as follows.

Definition 3.10. The *fiber functor* is a functor of **Top**-enriched categories

$$PH^{-1}: \mathbf{Bar}_{\mathbf{K}} \longrightarrow \mathbf{Top}$$

that sends a barcode D to the fiber $\mathrm{PH}^{-1}(D)$ and a morphism $\phi \in \mathbf{Bar}_{\mathbf{K}}(D,D')$ to the monodromy \mathcal{L}_{ϕ} .

Remark 3.11. PH^{-1} also descends to define a functor of homotopy categories:

$$PH^{-1}: h\mathbf{Bar}_{\mathbf{K}} \longrightarrow h\mathbf{Top}.$$

When ϕ is not simplicial (Definition 3.3), it may send a bounded interval (b,d) of D "to the middle of nowhere", i.e. $\phi(b) = \phi(d)$ may not equal an interval endpoint in D'. In turn, if this is the case, the monodromy \mathcal{L}_{ϕ} is not a polyhedral map, i.e. it is not an affine map on each polyhedron that sends polyhedra to polyhedra surjectively; see Fig. 5 below.

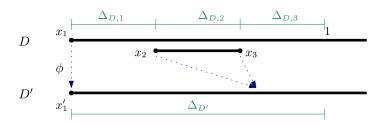


Figure 5: The barcode D has dimension 3. By Theorem 2.2, the polyhedral complex $\mathrm{PH}^{-1}(D)$ is made of polyhedra that are products $\Delta_{D,1} \times \Delta_{D,2} \times \Delta_{D,3}$ of 3 standard simplices. Each simplex $\Delta_{D,i}$ corresponds to the values of the filters in the fiber that are in-between the i-th endpoint x_i and (i+1)-th endpoint x_{i+1} of D. Likewise, a polyhedron in $\mathrm{PH}^{-1}(D')$ is simply a unique standard simplex $\Delta_{D'}$. The map ϕ collapses x_2 and x_3 strictly in-between x_1' and 1. In turn, the monodromy \mathcal{L}_{ϕ} collapses the second standard simplex $\Delta_{D,2}$ of any polyhedron in $\mathrm{PH}^{-1}(D)$ in the interior of $\Delta_{D'}$, and so is not a polyhedral map.

However, simplicial maps induce polyhedral monodromies, as stated in the following result.

Proposition 3.12. Let $D, D' \in \operatorname{Bar}_K$ and $\phi \in \operatorname{Bar}_K(D, D')$. If ϕ is simplicial, then the monodromy $\mathcal{L}_{\phi} : \operatorname{PH}^{-1}(D) \to \operatorname{PH}^{-1}(D')$ is a polyhedral map. In particular, for any $\phi \in \operatorname{Bar}_K(D, D')$, the monodromy \mathcal{L}_{ϕ} is homotopic to a polyhedral map.

Given a barcode D, we denote by $\operatorname{Aut}(I, \leq)_D$ the stabilizer of D, i.e. the group of homeomorphisms of the real line that fix the endpoints of D and hence act trivially on D. The proof of Proposition 3.12 mainly relies on the following lemma.

Lemma 3.13. Let $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$ be a simplicial morphism. Then there exists a group homomorphism

$$\phi_{\text{post}} : \text{Aut}(I, \leq)_D \longmapsto \text{Aut}(I, \leq)_{D'},$$

such that for all $\alpha \in \operatorname{Aut}(I, \leq)_D$, $\phi \circ \alpha = \phi_{\operatorname{post}}(\alpha) \circ \phi$. Similarly, there exists a group homomorphism

$$\phi_{\mathrm{pre}}: \mathrm{Aut}(I,\leqslant)_{D'} \longmapsto \mathrm{Aut}(I,\leqslant)_{D},$$

such that for all $\beta \in \operatorname{Aut}(I, \leq)_{D'}$, $\phi \circ \phi_{\operatorname{pre}}(\beta) = \beta \circ \phi$.

Proof. Let $\alpha \in \operatorname{Aut}(I, \leq)$ such that $\alpha(D) = D$. Note that, since ϕ is simplicial, for any index $0 \leq i \leq \dim D$, the following alternative holds:

- (a) Either $\phi([x_i, x_{i+1}]) = x_i'$ for some index $0 \le j \le \dim D' + 1$;
- (b) Or $\phi_{|[x_i,x_{i+1}]}$ is a linear bijection onto $[x'_j,x'_{j+1}]$ for some index $0 \le j \le \dim D'$.

According to this alternative, we define $\beta := \phi_{\text{post}}(\alpha)$ on each $\phi([x_i, x_{i+1}])$ as follows:

- (a) Either $\phi([x_i, x_{i+1}]) = x'_j$, in which case we set $\beta(x'_j) := x'_j$;
- (b) Or $\phi_{|[x_i, x_{i+1}]}$ is a linear bijection onto $[x'_j, x'_{j+1}]$, in which case we set $\beta_{|[x'_j, x'_{j+1}]} := \phi \circ \alpha \circ \phi_{|[x_i, x_{i+1}]}^{-1}$.

Note that in case (b), β is well-defined since $\alpha(D) = D$ implies that α restricts to a homeomorphism of each line segment $[x_i, x_{i+1}]$. Moreover, β is defined on the whole unit interval I since ϕ is surjective. It is then clear that we have $\beta \circ \phi \circ \alpha = \phi$ on each $[x_i, x_{i+1}]$, so that the equality holds on I as desired. Note that $\beta \in \operatorname{Aut}(I, \leq)_{D'}$ since

$$\beta(D') = \beta(\phi \circ \alpha(D)) = \phi.D = D'.$$

By construction, the association $\phi_{\mathrm{post}}: \alpha \mapsto \beta$ is a group homomorphism. Conversely, if we are rather given a map $\beta \in \mathrm{Aut}(\mathrm{I}, \leqslant)$ such that $\beta(D') = D'$, then we can construct the map $\alpha := \phi_{\mathrm{pre}}(\beta)$ satisfying $\phi \circ \alpha = \beta \circ \phi$ as follows:

- (a) Either $\phi([x_i, x_{i+1}]) = x_i'$, in which case we set $\alpha([x_i, x_{i+1}]) := \mathrm{Id}_{[x_i, x_{i+1}]}$;
- (b) Or $\phi_{|[x_i,x_{i+1}]}$ is a linear bijection onto $[x'_j,x'_{j+1}]$, in which case we set $\alpha_{|[x_i,x_{i+1}]}:=\phi_{|[x_i,x_{i+1}]}^{-1}\circ\beta\circ\phi$.

This construction also yields a group homomorphism $\phi_{\text{pre}}: \beta \mapsto \alpha$.

Proof of Proposition 3.12. The second part of the statement follows directly from Proposition 3.4, which states that any morphism of barcodes is homotopic to a simplicial map. Henceforth, we fix a simplicial morphism $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$ and show that the monodromy \mathcal{L}_{ϕ} is a polyhedral map, i.e. it is affine on each polyhedron and sends polyhedra to polyhedra surjectively.

Let $\mathcal{S} \subseteq \operatorname{Filt}_K$ be a filter stratum. We have $\overline{\operatorname{PH}_{|\mathcal{S}}^{-1}}(D) \cong \Delta_0 \times \cdots \times \Delta_{\dim D}$, where the standard simplex Δ_i corresponds to filter values that are in-between the endpoints x_i and x_{i+1} of D. Since ϕ is affine over $[x_i, x_{i+1}]$, each coordinate function $\mathcal{L}_{\phi,\sigma}: f \in \overline{\operatorname{PH}_{|\mathcal{S}}^{-1}}(D) \mapsto \mathcal{L}_{\phi}(f)(\sigma) \in \mathbb{R}$, for $\sigma \in K$, is affine as well. So the restriction of \mathcal{L}_{ϕ} to $\overline{\operatorname{PH}_{|\mathcal{S}}^{-1}}(D)$ is an affine map, as desired.

It remains to show that the image $\mathcal{L}_{\phi}(\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D))$ equals a polyhedron $\overline{\mathrm{PH}_{|\mathcal{S}'}^{-1}}(D')$. We fix a filter $f \in \mathrm{PH}_{|\mathcal{S}}^{-1}(D)$ and denote by \mathcal{S}' the stratum containing $\phi \circ f$. Note that, if $g \in \mathrm{Filt}_K$ is a filter, then:

$$g \in \mathcal{L}_{\phi}(\mathrm{PH}_{|\mathcal{S}}^{-1}(D)) \Longleftrightarrow \exists f' \in \mathrm{PH}_{|\mathcal{S}}^{-1}(D), \ g = \phi \circ f'$$

$$\iff \exists \alpha \in \mathrm{Aut}(\mathrm{I}, \leqslant)_D, \ g = \phi \circ (\alpha \circ f) \qquad \qquad \text{by the equivariance Lemma 1.5}$$

$$\iff \exists \beta \in \mathrm{Aut}(\mathrm{I}, \leqslant)_{D'}, \ g = \beta \circ (\phi \circ f) \qquad \qquad \text{by Lemma 3.13}$$

$$\iff g \in \mathrm{PH}_{\mathcal{S}'}^{-1}(D'). \qquad \qquad \text{by the equivariance Lemma 1.5}$$

Therefore, $\mathcal{L}_{\phi}(\mathrm{PH}_{|\mathcal{S}'}^{-1}(D))$ equals $\mathrm{PH}_{|\mathcal{S}'}^{-1}(D')$ and in fact $\mathcal{L}_{\phi}(\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D))$ equals $\overline{\mathrm{PH}_{|\mathcal{S}'}^{-1}}(D')$ since the image of a closed polyhedron via an affine map is again a closed polyhedron.

The following is a consequence of the Propositions 3.4, 3.5 and 3.12:

Corollary 3.14. For any $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$, the monodromy \mathcal{L}_{ϕ} is homotopic to a polyhedral map. This polyhedral map may further be chosen as a composition

$$\mathcal{L}_{\phi_k \circ \cdots \circ \phi_1} = \mathcal{L}_{\phi_k} \circ \cdots \circ \mathcal{L}_{\phi_1}$$

of monodromies \mathcal{L}_{ϕ_k} that are polyhedral maps between fibers over barcodes that differ by one dimension.

Remark 3.15. The monodromy associated to a simplicial morphism ϕ acts as a projection map on each polyhedron $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$ of the fiber. Indeed, recalling that $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$ is (isomorphic to) a product $\Delta_0 \times \cdots \times \Delta_{\dim D}$ of standard simplices, we have the commutative diagram

$$\Delta_{0} \times \cdots \times \Delta_{\dim D} \xrightarrow{\pi} \Delta_{0} \times \cdots \times \Delta_{\dim D} / \prod_{\phi(x_{i}) = \phi(x_{i+1})} \Delta_{i}$$

$$\cong \bigcup_{PH_{|S}^{-1}(D)} \underbrace{\mathcal{L}_{\phi}}$$

$$\mathcal{L}_{\phi} \xrightarrow{\mathcal{L}_{\phi}(PH_{|S}^{-1}(D))}$$

where π is the projection map. In other words, the product of simplices describing the image polyhedron $\mathcal{L}_{\phi}(\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D))$ is obtained from the product describing $\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D)$ by collapsing standard simplices Δ_i whenever ϕ collapses the i-th and i+1-th endpoint of D.

4 The space of barcodes is homotopically stratified

In this section, we take a closer look at the stratification of barcodes and show that Bar_K is homotopically stratified. This naturally leads us to introducing the entrance path category of barcodes. We observe that the entrance path category is isomorphic to the homotopy category, $h\mathbf{Bar}_K$, of barcodes. Recall from Theorem 3.2 that \mathbf{Bar}_K is homotopy discrete. Similarly, we find that the space of entrance paths between fixed barcodes is contractible.

4.1 Regularity of the stratification of barcodes

The notion of stratification used so far (Definition 1.8) is a very weak one, as it does not impose restrictions on the neighborhoods of strata. The space of filters Filt_K is in fact Whitney stratified, being a polyhedron in $\operatorname{I}^K \subseteq \mathbb{R}^K$. However, Bar_K is the quotient of Filt_K induced by PH (see Proposition 1.17), so the regularity of its stratification is less apparent. We show that the space Bar_K of barcodes in the image of the persistence map is homotopically stratified in the sense of Quinn [34]. Stratified coverings over homotopically stratified spaces are classified by the entrance path category, which we introduce and analyse in the next section in the case of barcodes.

The local neighborhoods in a homotopically stratified space X are defined in terms of paths that cross strata in decreasing order of dimension:

Definition 4.1. Let X be a stratified space. A continuous path $\gamma \in X^I$ is an *entrance path* if for any $0 \le t \le t' \le 1$, the stratum containing $\gamma(t)$ has greater or equal dimension than that containing $\gamma(t')$.

Definition 4.2. Let X be a stratified space. An entrance path γ is *elementary* if it stays in a unique stratum until the very last moment, that is if $\gamma([0,1))$ belongs to a fixed stratum. Given two strata X^i and X^j , j < i, the *homotopy link* $\operatorname{Holink}(X^i, X^j)$ is the space of elementary paths starting in X^i and ending in X^j with the compact open topology.

Definition 4.3. A stratified space X is *homotopically stratified* if it satisfies the following conditions for any pair of strata X^i and X^j , where j < i:

- 1. The inclusion $X^j \hookrightarrow X^i \cup X^j$ is *tame*, which means that there is a strong deformation retraction of a neighborhood of X^j in $X^i \cup X^j$ onto X^j such that points remain in the same stratum until the very last moment during the deformation;
- 2. The evaluation at time t = 1

$$\operatorname{ev}_1: \gamma \in \operatorname{Holink}(X^i, X^j) \longmapsto \gamma(1) \in X^j$$

is a fibration.

Note that, in the original formulation of homotopically stratified spaces [34], the strata are not necessarily topological manifolds, and so we should really refer to the spaces of Definition 4.3 as manifold stratified spaces, as done in [38] for instance. However, this distinction is irrelevant for our purposes, since the strata in $Bar_K = PH(Filt_K)$ are manifolds. **Proposition 4.4.** The filtered space Bar_K is homotopically stratified.

Proof. Let \mathcal{B} and \mathcal{B}' be two barcode strata with $\mathcal{B}' \subseteq \bar{\mathcal{B}}$. Recall that the coordinate chart ν extends to a continuous, surjective, stratum-preserving map $\nu: \Delta^{\dim \mathcal{B}} \to \bar{\mathcal{B}}$; see Eq. (4). Then, the inverse image $\nu^{-1}(\mathcal{B}')$ of \mathcal{B}' is a union of faces in $\Delta^{\dim \mathcal{B}}$. Any collection of faces has a neighborhood in $\Delta^{\dim \mathcal{B}}$ that deformation retracts back to itself such that points in the interior of the simplex are mapped to points in the interior but at the last moment. Pulling back neighborhoods and composing the deformation retraction with ν , we get the required deformation retraction of a neighborhood of \mathcal{B}' in $\mathcal{B} \cup \mathcal{B}'$ onto \mathcal{B}' . Therefore, the inclusion $\mathcal{B}' \hookrightarrow \mathcal{B} \cup \mathcal{B}'$ is tame.

To check that the evaluation map $ev_1 : \gamma \in Holink(\mathcal{B}, \mathcal{B}') \mapsto \gamma(1) \in \mathcal{B}'$ is a fibration, it is enough (see e.g. [21]) to find a section for the map:

$$(\tilde{\gamma}_h)_{h\in[0,1]}\in \operatorname{Holink}(\mathcal{B},\mathcal{B}')^{[0,1]}\longmapsto \left((\operatorname{ev}_1\circ\tilde{\gamma}_h)_{h\in[0,1]},\tilde{\gamma}_0\right)\in \mathcal{B}'^{[0,1]}\times_{\mathcal{B}'}\operatorname{Holink}(\mathcal{B},\mathcal{B}'),$$

where $\mathcal{B}'^{[0,1]} \times_{\mathcal{B}'} \operatorname{Holink}(\mathcal{B}, \mathcal{B}')$ is the fiber product

$$\mathcal{B}'^{[0,1]} \times_{\mathcal{B}'} \operatorname{Holink}(\mathcal{B}, \mathcal{B}') := \big\{ (\gamma, \tilde{\gamma}) \in \mathcal{B}'^{[0,1]} \times \operatorname{Holink}(\mathcal{B}, \mathcal{B}') \, | \, \gamma(0) = \tilde{\gamma}(1) \big\}.$$

Using the coordinate chart $\nu:\mathring{\Delta}^{\dim\mathcal{B}'}\stackrel{\cong}{\longrightarrow} \mathcal{B}'$, we may view a path $\gamma:h\in[0,1]\mapsto\gamma(h)\in\mathcal{B}'$ via its coordinates $0< x_1^\gamma(h)<\dots< x_{\dim\mathcal{B}'}^\gamma(h)<1$. Given $0\leqslant h\leqslant 1$, let $\phi_h^\gamma\in\operatorname{Aut}(I,\leqslant)$ be the map that sends the endpoint values $x_i^\gamma(0)$ to $x_i^\gamma(h)$, and is extended linearly on each line segment $[x_i^\gamma(0),x_{i+1}^\gamma(0)]$. Clearly, the association $(\gamma,h)\mapsto\phi_h^\gamma$ is continuous, and we have $\gamma(h)=\phi_h^\gamma.\gamma(0)$. Then, the map

$$(\gamma,\tilde{\gamma}) \in \mathcal{B}'^{[0,1]} \times_{\mathcal{B}'} \mathrm{Holink}(\mathcal{B},\mathcal{B}') \longmapsto (h \mapsto \phi_h^{\gamma}.\tilde{\gamma}) \in \mathrm{Holink}(\mathcal{B},\mathcal{B}')^{[0,1]}$$

is the desired section.

4.2 The entrance path category of barcodes

The entrance path category is the suitable generalisation of the fundamental groupoid for stratified spaces where ordinary paths are replaced by entrance paths between points; see [35] for the original constructions.

Definition 4.5. Let X be a stratified space. The *entrance path category* of X is $\mathbf{Ent}(X) := h\mathbf{Path}_{\leq}(X)$, where $\mathbf{Path}_{\leq}(X)$ is the topologically enriched category with X as the set of objects and spaces of entrance paths equipped with the compact open topology as morphisms. In other words, $\mathbf{Ent}(X)$ has the points of X as objects and the homotopy classes of entrance paths as morphisms.

Remark 4.6. In order to make $\mathbf{Path}_{\leq}(X)$ into a category where concatenation of paths defines a strictly associative composition of morphisms, paths of all positive lengths need to be allowed. This is analogous to replacing loop spaces by Moore loop spaces and the resulting morphism spaces are homotopy equivalent. In particular the definition of $\mathbf{Ent}(X)$ is not affected. We will ignore this subtlety in what follows.

Let ϕ in $\mathbf{Bar}_{\mathbf{K}}$ be a morphism between barcodes D and D'. Define the path:

$$\gamma_{\phi}(t) := (t\phi + (1-t)\text{Id}).D.$$

For times t < 1, $t\phi + (1-t)\mathrm{Id}$ is a homeomorphism of the unit interval and so $\gamma_{\phi}(t)$ stays in the stratum containing D. Hence, $\gamma_{\phi}(t)$ is in fact an elementary entrance path. It is then clear that the association

$$\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D') \longmapsto \gamma_{\phi} \in \mathbf{Path}_{\leq}(\mathrm{Bar}_{K})(D, D')$$

is continuous.

Proposition 4.7. The association $[\phi] \mapsto [\gamma_{\phi}]$ is functorial and induces an isomorphism of categories:

$$h\mathbf{Bar}_{\mathbf{K}} \cong \mathbf{Ent}(\mathbf{Bar}_K).$$

Before proving this, we first characterize when elementary entrance paths are homotopic, in a similar fashion to Lemma 3.7 for morphisms of barcodes. Recall that an entrance path γ from D to D' is elementary if it stays in a unique stratum until the very last moment, that is if $\gamma([0,1))$ belongs to the stratum \mathcal{B}_D . Using the coordinate chart $\nu:\mathring{\Delta}^{\dim D}\stackrel{\cong}{\longrightarrow} \mathcal{B}_D$, we may view $\gamma_{|[0,1)}$ via its coordinates $0< x_1^{\gamma}(t)< \cdots < x_{\dim D}^{\gamma}(t)< 1,\ 0\leqslant t<1$. Alternatively, for each interval $(b,d)\in D$, there is a continuously evolving interval $(b^{\gamma}(t),d^{\gamma}(t))$ starting at (b,d), $0\leqslant t<1$, and together the intervals $(b^{\gamma}(t),d^{\gamma}(t))$ form the barcode $\gamma(t)$.

Lemma 4.8. Let $\gamma_0, \gamma_1 \in \mathbf{Path}_{\leq}(\mathbf{Bar}_K)(D, D')$ be elementary entrance paths. Then the following are equivalent:

- (i) The two entrance paths are homotopic through elementary entrance paths.
- (ii) For all $(b,d) \in D$ and $(b',d') \in D'$, we have $\lim_{t\to 1^-} (b^{\gamma_0}(t),d^{\gamma_0}(t)) = (b',d')$ if and only if $\lim_{t\to 1^-} (b^{\gamma_1}(t),d^{\gamma_1}(t)) = (b',d')$.
- (iii) For all $(b,d) \in D$ and $(b',d') \in D'$, we have $\lim_{t\to 1^-} (d^{\gamma_0}(t) b^{\gamma_0}(t)) = 0$ if and only if $\lim_{t\to 1^-} (d^{\gamma_1}(t) b^{\gamma_0}(t)) = 0$.

Proof. $[(i) \Rightarrow (ii) \text{ and } (iii)]$: Let γ_h be a homotopy between γ_0 and γ_1 through entrance paths, and let $h \in [0,1]$. Using Proposition 1.4, we can partition the intervals in D into sets $D_{(a)}^{\gamma_h}$ and $D_{(b)}^{\gamma_h}$ as follows:

- (a) For each interval $(b', d') \in D'$, there is a unique $(b, d) \in D$ for which $\lim_{t \to 1^-} (b^{\gamma_h}(t), d^{\gamma_h}(t)) = (b', d')$;
- (b) For all other intervals $(b, d) \in D$, we have $\lim_{t \to 1^-} (b^{\gamma_h}(t) d^{\gamma_h}(t)) = 0$.

By continuity, the classification remains constant along the homotopy $h \mapsto \gamma_h$, i.e. $D_{(a)}^{\gamma_h}$ and $D_{(b)}^{\gamma_h}$ are the same for all $0 \le h \le 1$.

- $[(ii)\Rightarrow (iii)]$: By assumption $D_{(a)}^{\gamma_0}=D_{(a)}^{\gamma_1}$, hence $D_{(b)}^{\gamma_0}=D_{(b)}^{\gamma_1}$.
- $[(iii) \Rightarrow (ii)]$: By assumption $D_{(b)}^{\gamma_0} = D_{(b)}^{\gamma_1}$, hence $D_{(a)}^{\gamma_0} = D_{(a)}^{\gamma_1} =: D_{(a)}$. Then, γ_0 and γ_1 induce bijections from $D_{(a)}$ to the set of intervals in D'. Since γ_0 and γ_1 are entrance paths, these bijections are monotonic with respect to the endpoint values. So they are in fact the same bijections.
- [(ii) and (iii) \Rightarrow (i)]: We define a homotopy $h \mapsto \gamma_h$ between the restrictions of γ_0 and γ_1 to [0, 1) by interpolating the coordinates:

$$\forall 0\leqslant t<1,\,\forall 1\leqslant i\leqslant \dim D,\,x_i^{\gamma_h}(t):=hx_i^{\gamma_1}(t)+(1-h)x_i^{\gamma_0}(t).$$

For each interval $(b, d) \in D$, we then have

$$\forall 0 \le t < 1, (b^{\gamma_h}(t), d^{\gamma_h}(t)) = (hb^{\gamma_1}(t) + (1-h)b^{\gamma_0}(t), hd^{\gamma_1}(t) + (1-h)d^{\gamma_0}(t)).$$

Since $D_{(b)}^{\gamma_0} = D_{(b)}^{\gamma_1}$ and $D_{(a)}^{\gamma_0} = D_{(a)}^{\gamma_1}$, we have $\lim_{t \to 1^-} (b^{\gamma_h}(t), d^{\gamma_h}(t)) = (b', d')$ (resp. $\lim_{t \to 1^-} d^{\gamma_h}(t) - b^{\gamma_h}(t) = 0$) if and only if $\lim_{t \to 1^-} (b^{\gamma_0}(t), d^{\gamma_0}(t)) = (b', d')$ (resp. $\lim_{t \to 1^-} d^{\gamma_0}(t) - b^{\gamma_0}(t) = 0$). Therefore,

$$\forall h \in [0, 1], \lim_{t \to 1^{-}} \gamma_h(t) = D',$$

hence the homotopy $h \mapsto \gamma_h$ extends to a homotopy between γ_0 and γ_1 on the whole unit interval.

Proof of Proposition 4.7. The functoriality of $[\phi] \mapsto [\gamma_{\phi}]$ amounts to showing that if $\phi \in \mathbf{Bar_K}(D, D')$ and $\psi \in \mathbf{Bar_K}(D', D'')$ are two morphisms, then the entrance paths $\gamma_{\psi \circ \phi}$ and $\gamma_{\psi} \cdot \gamma_{\phi}$ are homotopic. For $t \in [0, 1]$, let ϕ_t denote the interpolated map $(1-t)\mathrm{Id} + t\phi$, so that $\gamma_{\phi}(t) = \phi_t.D$, $\gamma_{\psi}(t) = \psi_t.D'$ and $\gamma_{\psi \circ \phi}(t) = (\psi \circ \phi)_t.D$. Besides, the concatenated path $\gamma_{\psi}.\gamma_{\phi}(t)$ equals $\phi_{2t}.D$ for $t \leqslant \frac{1}{2}$ and $\psi_{2t-1}.D' = \psi_{2t-1}.(\phi.D)$ for $t \geqslant \frac{1}{2}$. A homotopy between $\gamma_{\psi \circ \phi}$ and $\gamma_{\psi}.\gamma_{\phi}$ can then be defined as:

$$H(h,t) := [h\phi_{2t} + (1-h)(\psi \circ \phi)_t].D \text{ for } t \le \frac{1}{2},$$

and

$$H(h,t) := [h\psi_{2t-1} \circ \phi + (1-h)(\psi \circ \phi)_t].D \text{ for } t \ge \frac{1}{2}.$$

Hence, we obtain a functor from $h\mathbf{Bar}_{\mathbf{K}}$ to $\mathbf{Ent}(\mathrm{Bar}_K)$, which is the identity on objects. Given barcodes D, D', we show that this functor gives a bijection $h\mathbf{Bar}_{\mathbf{K}}(D, D') \stackrel{\cong}{\longrightarrow} \mathbf{Ent}(\mathrm{Bar}_K)(D, D')$.

Let γ be an elementary entrance path from D to D'. We construct a morphism ϕ from D to D' using the classification of the maps $(b^{\gamma}(t), d^{\gamma}(t))$:

- (a) For each interval $(b',d') \in D'$, there is a unique $(b,d) \in D$ for which $\lim_{t\to 1^-} (b^{\gamma}(t),d^{\gamma}(t)) = (b',d')$. We then set $\phi(b) := b'$ and $\phi(d) := d'$;
- (b) For all other intervals $(b, d) \in D$, we have $\lim_{t\to 1^-} (b^{\gamma}(t) d^{\gamma}(t)) = 0$. We then set $\phi(b) = \phi(d)$ to be an arbitrary value such that ϕ remains non-decreasing.

We extend ϕ to a non-decreasing map of the unit interval arbitrarily. Since $\gamma(1) = D'$, we have $\phi.D = D'$ by construction. Besides, the interpolated path $\gamma_{\phi}(t) = [t\phi + (1-t)\mathrm{Id}].D$ is an elementary entrance path beween D and D', which satisfies the (i) and (ii) of Lemma 4.8 with respect to γ , hence is homotopic to γ . More generally, an arbitrary entrance path γ from D to D' is homotopic to a finite concatenation of elementary paths. In turn, γ is homotopic to an interpolated morphism γ_{ϕ} by applying the previous argument to each elementary path. Therefore, the map $[\phi] \in h\mathbf{Bar}_{\mathbf{K}}(D, D') \mapsto [\gamma_{\phi}] \in \mathbf{Ent}(\mathrm{Bar}_K)$ is surjective.

Besides, comparing the (ii) of Lemma 3.7 with the (ii) of Lemma 4.8, we see that if two morphisms between D and D' induce interpolated paths that are homotopic, then they must be homotopic. Consequently, the map $[\phi] \in h\mathbf{Bar}_K(D,D') \mapsto [\gamma_\phi] \in \mathbf{Ent}(\mathrm{Bar}_K)$ is injective.

Remark 4.9. It is well-known that coverings over a topological space X satisfying mild properties are classified by the fundamental groupoid of X. When X is stratified, it is natural to consider stratified coverings, i.e. maps restricting to coverings over each individual stratum. If X is homotopically stratified with locally simply connected and locally connected strata, the stratified coverings which are either local homeomorphisms or *branched covers* are classified by the entrance path category of X [39]. That is, functors from $\mathbf{Ent}(X)$ to \mathbf{Set} functorially give rise to such stratified coverings, in fact also to constructible cosheaves, and conversely. Although in the case of barcodes the inverse image PH^{-1} is valued in $h\mathbf{Top}$, we have a natural set valued functor:

$$\operatorname{Ent}(\operatorname{Bar}_K) \xrightarrow{\operatorname{PH}^{-1}} h\operatorname{Top} \xrightarrow{\pi_0} \operatorname{Set}.$$

We next prove an analogue of Theorem 3.2 for entrance paths.

¹See [14] for similar classifications of functors over $\mathbf{Ent}(X)$ when X is conically stratified, and [35] for some 2-categorical equivalences.

Proposition 4.10. For any two barcodes D, D' in Bar_K , the space of morphisms $\mathbf{Path}_{\leq}(Bar_K)(D, D')$ has finitely many path connected components each of which is contractible. In other words, the category $\mathbf{Path}_{\leq}(Bar_K)$ is homotopy discrete, i.e. $\mathbf{Path}_{\leq}(Bar_K)(D, D') \simeq h\mathbf{Path}_{\leq}(Bar_K)(D, D')$.

Proof. Let D and D' be two barcodes, and let $\operatorname{Holink}(D, D')$ be the space of elementary entrance paths from D to D'. For homotopically stratified metric spaces, the space of entrance paths and that of elementary entrance paths are homotopy equivalent [27, Theorem 4.9], from which we deduce that:

$$\mathbf{Path}_{\leq}(\mathrm{Bar}_K)(D,D') \simeq \mathrm{Holink}(D,D').$$

The proof of the statement then follows from Lemma 4.8. In more detail, let Ω be a path connected component in $\operatorname{Holink}(D, D')$, and let $\gamma_0 \in \Omega$. Recall that we can partition the intervals in D into sets $D_{(a)}^{\gamma_0}$ and $D_{(b)}^{\gamma_0}$ as follows:

- (a) For each interval $(b', d') \in D'$, there is a unique $(b, d) \in D$ for which $\lim_{t \to 1^-} (b^{\gamma_0}(t), d^{\gamma_0}(t)) = (b', d')$;
- (b) For all other intervals $(b, d) \in D$, we have $\lim_{t \to 1^-} (b^{\gamma_0}(t) d^{\gamma_0}(t)) = 0$.

From Lemma 4.8, any other path $\gamma \in \Omega$ satisfies $D_{(a)}^{\gamma} = D_{(a)}^{\gamma_0}$ and $D_{(b)}^{\gamma} = D_{(b)}^{\gamma_0}$. Define $r: \Omega \times [0,1] \times [0,1) \to \operatorname{Bar}_K$ by:

$$r: (\gamma, h, t) \in \Omega \times [0, 1] \times [0, 1) \longmapsto \left\{ ((1 - h)b^{\gamma}(t) + hb^{\gamma_0}(t), (1 - h)d^{\gamma}(t) + hd^{\gamma_0}(t)) \right\}_{(b, d) \in D} \in \mathcal{B}_D \subseteq \operatorname{Bar}_K.$$

We can continuously extend r at time t = 1 by $r(\gamma, h, 1) := D'$. We then get a deformation retraction

$$R: (\gamma, h) \in \Omega \times [0, 1] \longmapsto r(\gamma, h, .) \in \Omega$$

of
$$\Omega$$
 onto $\{\gamma_0\}$.

Combining Proposition 4.7 and 4.10 we can summarise our results in this section with the following.

Corollary 4.11. For any two barcodes $D, D' \in Bar_K$ we have

$$\mathbf{Path}_{\leq}(\mathrm{Bar}_K)(D,D') \simeq h\mathbf{Path}_{\leq}(\mathrm{Bar}_K)(D,D') = \mathbf{Ent}(\mathrm{Bar}_K)(D,D') \cong h\mathbf{Bar}_K(D,D') \simeq \mathbf{Bar}_K(D,D'),$$

and hence the natural weak equivalences of categories

$$\mathbf{Path}_{\leqslant}(\mathrm{Bar}_K) \overset{\simeq}{\longrightarrow} h\mathbf{Path}_{\leqslant}(\mathrm{Bar}_K) = \mathbf{Ent}(\mathrm{Bar}_K) \overset{\cong}{\longleftarrow} h\mathbf{Bar}_K \overset{\simeq}{\longleftarrow} \mathbf{Bar}_K.$$

5 Variations of the fiber problem

We adapt our analysis to two further situations of interest, namely when we remove the constraint that filters and barcodes take value in the unit interval, and when we restrict PH to the subspace of filters determined by their values on vertices. Finally, we point out that the action of the symmetries of K on the filters restricts to the fibers.

5.1 The case of unbounded filters and barcodes

In the previous sections, the values of filters and the interval endpoints of barcodes were constrained to lie in the unit interval. Here we briefly outline how our analysis can be adapted when we consider the unbounded case and replace the interval I by the real line \mathbb{R} . We denote by $\operatorname{Filt}(\mathbb{R})_K$ the filter functions with unrestricted real values and by $\operatorname{Bar}(\mathbb{R})$ the space of finite barcodes with unrestricted endpoints. As before, persistent homology defines a map

$$\mathrm{PH}:\mathrm{Filt}(\mathbb{R})_K\longrightarrow\mathrm{Bar}(\mathbb{R}).$$

Let $\operatorname{Aut}(\mathbb{R},\leqslant)$ be the group of continuous automorphisms of the ordered real line that are the identity outside a compact set. Similarly let $\operatorname{End}(\mathbb{R},\leqslant)$ be the monoid of continuous order preserving maps of the real line that are the identity outside a compact set. Both spaces then act on the extended spaces of filters and barcodes. The proof of Lemma 1.5 generalises to show that the persistence map above is equivariant with respect to these extended actions. The actions are continuous as in Proposition 1.7 when we equip $\operatorname{Aut}(\mathbb{R},\leqslant)$ and $\operatorname{End}(\mathbb{R},\leqslant)$ with the L^∞ topology. Using the action of $\operatorname{Aut}(\mathbb{R},\leqslant)$ one can construct stratifications of filter and barcode spaces such that the $\operatorname{Aut}(\mathbb{R},\leqslant)$ -orbits are the strata and the analogues of Propositions 1.9 and 1.12 hold.

Indeed, most of the results and their proofs can easily be adapted with the caveat that fibers no longer have to be compact. Thus, item (a) of Theorem 2.2 needs to be reinterpreted: The polyhedron in the fiber are not necessarily products of closed standard simplices but instead we have

$$\overline{\mathrm{PH}_{|\mathcal{S}}^{-1}}(D) \cong \Delta_1 \times \cdots \times \Delta_{\dim \mathcal{B}-1} \times \Delta'_{\dim \mathcal{B}},$$

where the last term may be a simplex missing its last face, i.e. $\Delta'_{\dim \mathcal{B}}$ may be the closed *i*-simplex Δ^i or of the form:

$$\Delta_{\infty}^{i} := \{0 \leqslant x_1 \leqslant \cdots \leqslant x_i < \infty\}.$$

Modulo this subtlety, Theorems 2.2 and Proposition 3.12 hold also in the unbounded case we consider here, and PH is again a stratified fiber bundle whose fibers are (possibly unbounded) polyhedra. The proofs of these results in the unbounded situation do not present additional difficulties.

Next we provide a necessary and sufficient criterion for the simplicial complex K that ensures that all the fibers of PH are bounded.

Definition 5.1. A subset $L \subseteq K$ is k-removable, or simply removable, if $K \setminus L$ is a subcomplex of K and the inclusion $K \setminus L \hookrightarrow K$ induces an isomorphism on (standard) homology with k-coefficients. K is said to be k-essential, or simply essential, if it has no removable subsets.

For instance, any pair (σ, σ') where σ' has σ as its only co-face provides an example of a removable subset for any $\mathbb R$. This is an elementary collapse familiar from simple homotopy theory. More elaborate examples include the wedge product $K \vee A$ of two simplicial complexes K and K, where K is $\mathbb R$ -acyclic, i.e. K has trivial reduced (ordinary) homology with $\mathbb R$ coefficients. Then $K = K \setminus \{0\}$ is removable. A rich source of such K are the classifying spaces of perfect groups, or the classifying spaces of finite groups when $\mathbb R$ is of characteristic zero.

Proposition 5.2. The fibers of the persistence map PH are all compact if and only if the complex K is essential.

Proof. If K is not essential, it has a removable subset $L \subseteq K$. Given a partial filter $f: K \setminus L \to \mathbb{R}$ and a real value x with $x \geqslant \max_{\sigma \in K \setminus L} f(\sigma)$, f can be extended to a filter f_x on all of K by assigning the common value x to all the simplices in L. Thus the fiber of $D = \mathrm{PH}(f_x)$ contains the open half line $\{f_x \mid x \in [\max_{\sigma \in K \setminus L} f(\sigma), \infty)\}$ and is hence not compact.

Conversely, if PH has a non-compact fiber over some barcode D, it means there exists an $f \in PH^{-1}(D)$ attaining values higher than the largest (bounded) endpoint $\max(D)$ of D. Therefore the set L of simplices on which f takes value larger than $\max(D)$ is removable.

Next we will exhibit a family of simplicial complexes that are essential. We say that K is a *triangulated (oriented)* manifold if its geometric realisation |K| is homeomorphic to a closed (orientable) manifold. Note that this manifold will necessarily be compact since K is finite.

Proposition 5.3. Let K be a triangulated manifold. If either the field of coefficients \mathbb{k} is of characteristic 2 or K is oriented, then K is essential.

Proof. Without loss of generality, we may assume that K is connected and of dimension d. Then any filter f attains its maximum value on a top dimensional simplex, since all lower-dimensional simplices have co-faces. By our assumptions, we have $H_d(K) \cong \mathbb{R}$ and a generator of this top dimensional homology class is the sum of all top dimensional simplices of K (with appropriate signs). At the level of barcodes, this means that there is an infinite interval in homological degree d starting at $\max_{\sigma \in K} f(\sigma)$. Therefore the fiber of the persistence map over any barcode D is bounded, and since it is closed by continuity of PH, it is also compact and hence essential by the previous result.

The converse of Proposition 5.3 is false as can be seen from the following simple counterexample.

Example 5.4. Let K be the wedge product of two triangles. So K has five vertices and six 1-simplices. Its first homology group is of rank 2 but any subcomplex will have at most rank 1. Thus K is essential, but K is not a manifold.

The point-set topology is a little delicate when working with the unbounded real line \mathbb{R} instead of the compact interval I. This is part of the reason why we chose to work with I for the main part of our paper. For example, Proposition 1.17 cannot be adapted to the unbounded situation: If the fibers of PH are not compact then the bottleneck topology and the quotient topology induced by PH do not necessarily agree on the image $\operatorname{Bar}_K(\mathbb{R})$ as Example 5.5 below shows.

Example 5.5. Let us consider again the Example 1.19 of the complex K representing the unit interval with vertices a,b and 1-simplex σ . As a set $\mathrm{Bar}_K(\mathbb{R})$ can be identified (as in Example 1.19) with the 3-simplex $\Delta_3' = \{(x_1,x_2,x_3), -\infty < x_1 \le x_2 \le x_3 < \infty\}$ with two missing faces, where in addition the 2-dimensional face corresponding to $-\infty < x_1 < x_2 = x_3 < \infty$ is collapsed to the line segment $-\infty < x_1 = x_2 = x_3 < \infty$. Consider the set

$$U := \{(x_1, x_2, x_3) \mid x_3 - x_2 < e^{-x_2}\} \subseteq \text{Bar}_K(\mathbb{R})$$

of barcodes whose unique bounded bar (x_2, x_3) has length less than e^{-x_2} . We then have

$$\mathrm{PH}^{-1}(U) := \left\{ f \in \mathrm{Filt}_K(\mathbb{R}) \, | \, f(\sigma) - \max(f(a), f(b)) < e^{-\max(f(a), f(b))} \right\} \subseteq \mathrm{Filt}_K(\mathbb{R}) \subseteq \mathbb{R}^3,$$

which is open in $\mathrm{Filt}_K(\mathbb{R})$ for the usual topology induced by the $\|.\|_{\infty}$ -metric. Therefore U is an open set in the quotient topology. However it does not contain any bottleneck ball, hence is not an open set in the bottleneck topology.

Furthermore, the choice of topology on $\operatorname{End}(\mathbb{R},\leqslant)$ and $\operatorname{Aut}(\mathbb{R},\leqslant)$ matters in the unbounded situation: Replacing the L^{∞} topology by the compact open topology results in the actions not being (sequentially) continuous as can be seen in the following example.

Example 5.6. Consider the sequence of barcodes D_n containing a single interval $(n,n+2^{-n})$. The sequence D_n converges to the empty diagram D_{\emptyset} in the bottleneck topology. In addition, let $\phi_n:\mathbb{R}\to\mathbb{R}$ be the map such that $\phi_n(n)=n, \phi_n(n+2^{-n})=n+1, \phi_n|_{[n,n+2^{-n}]}$ and $\phi_n|_{[n+2^{-n},n+2]}$ are linear, and outside [n,n+2] ϕ_n is the identity. Then the sequence ϕ_n converges to the identity map of the real line in the compact open topology. If the action were continuous in both variables, the sequence $\phi_n.D_n$ would converge to the empty diagram $\mathrm{Id}.D_{\emptyset}=D_{\emptyset}$. However, each of the barcodes $\phi_n.D_n$ contains a unique interval (n,n+1), and the sequence does therefore not converge in the bottleneck topology.

However, in our analysis, we have never needed to make full use of the continuity of the action of $\operatorname{End}(I,\leqslant)$. Instead, it is enough to ensure that the action is continuous w.r.t. the choice of ϕ . Namely, fixing $D\in\operatorname{Bar}$, the map $\phi\in\operatorname{End}(I,\leqslant)\mapsto\phi.D\in\operatorname{Bar}$ is continuous. In the current unbounded situation, it can also be proven that the map $\phi\in\operatorname{End}(\mathbb{R},\leqslant)\mapsto\phi.D\in\operatorname{Bar}(\mathbb{R})$ is continuous, where we consider the compact open topology on $\operatorname{End}(\mathbb{R},\leqslant)$. This is precisely what is needed to carry the analysis through in a similar fashion.

Finally, if we do not impose that maps in $\operatorname{End}(\mathbb{R}, \leq)$ and $\operatorname{Aut}(\mathbb{R}, \leq)$ equal the identity outside a compact set, then the analysis breaks down in the L^{∞} topology. For instance, straight line interpolations on which our results rely, would not always give continuous paths.

5.2 The case of lower star filters

The lower star filtration form an interesting subspace of the space of all filters on K and one might want to restrict one's attention to these as for example in [15]. We summarise briefly how our analysis can be adapted and compared to this case.

Let K be a finite simplicial complex with vertex set V. A *lower star filter* on K is a filter $f \in Filt_K$ such that for any simplex $\sigma \in K$:

$$f(\sigma) = \max_{v \in \sigma} f(v).$$

Being determined by their values on vertices, such filters offer many advantages in practice. Any function $f:V\to I$ can be extended uniquely to a lower star filter. Hence, the subspace $\mathrm{Low}_K\subseteq\mathrm{Filt}_K$ of lower star filters is canonically isomorphic to I^V . We denote its image under the persistence map PH by $\mathrm{Bar}_K^{\mathrm{Low}}$.

The actions of $\operatorname{Aut}(I,\leqslant)$ and $\operatorname{End}(I,\leqslant)$ by post-composition restrict to I^V and, by the equivariance of PH, also to $\operatorname{Bar}_K^{\operatorname{Low}}$. As the strata are given by $\operatorname{Aut}(I,\leqslant)$ -orbits, we see that both Low_K and $\operatorname{Bar}_K^{\operatorname{Low}}$ are sub-stratified spaces, each consisting of a subcollection of full strata from Filt_K and Bar_K respectively. Thus PH restricts to a strongly stratified map

$$\mathrm{PH}_{\mid \mathrm{Low}} : \mathrm{Low}_K \longrightarrow \mathrm{Bar}_K^{\mathrm{Low}}$$

and hence satisfies similar properties as PH. In particular, the fiber $\mathrm{PH}_{\mathrm{Low}}^{-1}(D)$ has again the structure of a polyhedral complex and the analogue of Theorem 2.2 holds.

The space $\operatorname{Bar}^{\operatorname{Low}}_K$ also gives rise to a subcategory $\operatorname{Bar}_K^{\operatorname{Low}}$ of Bar_K . We note that this is a full subcategory. Thus Theorem 3.2 and Proposition 3.12 also hold for this subcategory. In particular, $\operatorname{Bar}_K^{\operatorname{Low}}$ is homotopy discrete. As before, we can associate to morphisms in $\operatorname{Bar}_K^{\operatorname{Low}}$ monodromies between fibers, turning the inverse image into a functor

$$\mathrm{PH}^{-1}_{|\mathrm{Low}}:\mathbf{Bar_K}^{\mathrm{Low}}\longrightarrow\mathbf{Top}.$$

Up to homotopy, the monodromies between fibers are again polyhedral maps.

Remark 5.7. Vice-versa, we may also consider the space of filters on K as a subspace of the space of lower star filters on its barycentric subdivision \hat{K} . Recall that the vertices of \hat{K} are the simplices of K. Thus $f \in \operatorname{Filt}_K$ uniquely gives rise to $\hat{f} \in \operatorname{Low}_{\hat{K}}$ via

$$\hat{f}(\hat{\sigma}) := f(\sigma),$$

where $\hat{\sigma}$ is the vertex of \hat{K} corresponding to the simplex σ in K. This way, we get a nested sequence of spaces:

$$Low_K \subset Filt_K \subset Low_{\hat{K}}$$
.

It is a straightforward exercise to show that $PH(f) = PH(\hat{f})$. Thus we also have a nested sequence of barcode spaces:

$$\operatorname{Bar}_K^{\operatorname{Low}} \subset \operatorname{Bar}_K \subset \operatorname{Bar}_{\hat{K}}^{\operatorname{Low}}.$$

All these inclusions are also $\operatorname{End}(I, \leq)$ -equivariant and PH defines an equivariant map between these nested sequences. Thus, by similar arguments as before, PH and the inclusions are compatible with the stratifications in the strongest sense giving rise to a sequence of full, homotopy discrete subcategories

$$\mathbf{Bar_K}^{\mathrm{Low}} \subset \mathbf{Bar_K} \subset \mathbf{Bar_{\hat{\mathbf{K}}}}^{\mathrm{Low}}$$

It would be interesting to analyse how the fibers of PH, or more generally the fiber functors on these three categories are related.

5.3 Symmetries restricted to fibers

In this brief section we examine how symmetries of the simplicial complex restrict to the fibers of the persistence map. For simplicity we return to filters and barcodes in the unit interval I, but the analysis can be carried out in the unbounded situation or when restricting to lower star filters in the same way.

Let $\mathbb{G}(K)$ be the group of isomorphisms of the simplicial complex K. Then $\mathbb{G}(K)$ can be identified as the subgroup of the group of symmetries $\mathrm{Sym}(K_0)$ of the vertices K_0 which consists of all those s that map a subset $\sigma \in K$ to a subset $s(\sigma) \in K$. Pre-composition with the inverse induces a left action of $\mathbb{G}(K)$ on the space Filt_K of filters via

$$s.f := f \circ s^{-1}.$$

Thus f and s.f take the same values and furthermore, if $S \subseteq \operatorname{Filt}_K$ is a filter stratum then s.S is another stratum of the same dimension, $\dim s.S = \dim S$, and s maps S to s(S) via an affine isomorphism.

Proposition 5.8. For all $f \in \text{Filt}_K$ and all $s \in \mathbb{G}(K)$ we have

$$PH(f) = PH(s.f).$$

Equivalently, the action of $\mathbb{G}(K)$ on Filt_K restricts to the fiber $\mathrm{PH}^{-1}(D)$ for every $D \in \mathrm{Bar}_K$. Furthermore, $\mathbb{G}(K)$ acts through maps of polyhedra on $\mathrm{PH}^{-1}(D)$.

Proof. The symmetry s maps the sublevel-set filtration of f isomorphically to that of s.f. Thus the associated persistence modules are isomorphic and so the two resulting barcodes in Bar_K are the same. Hence, f and s.f are in the same fiber. Since s defines an affine isomorphism from a stratum s to the stratum s. s and preserves fibers, it restricts to an affine isomorphism from $\operatorname{PH}^{-1}(D) \cap s$ to $\operatorname{PH}^{-1}(D) \cap s(s)$ for any barcode s Bar_K .

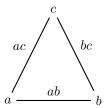
Proposition 5.9. Given two barcodes D, D' and a morphism $\phi \in \mathbf{Bar}_{\mathbf{K}}(D, D')$, the monodromy $\mathcal{L}_{\phi} : \mathrm{PH}^{-1}(D) \to \mathrm{PH}^{-1}(D')$ is $\mathbb{G}(K)$ -equivariant.

Proof. Let $f \in PH^{-1}(D)$, $s \in \mathbb{G}(K)$ and $\sigma \in K$. Then the statement of the proposition follows from

$$\mathcal{L}_{\phi}(s.f)(\sigma) = (\phi \circ s.f)(\sigma) = \phi(f(s^{-1}(\sigma))) = \mathcal{L}_{\phi}(f)(s^{-1}(\sigma)) = s.\mathcal{L}_{\phi}(f)(\sigma).$$

A A detailed example: the fiber of the persistence map over the triangle

In this section we illustrate our theory developed so far on an example. The simplest non-contractible simplicial complex K is a triangle. We denote its vertices by a, b, c and its edges by ab, ac, bc.



In this simple case, a filter is any map $f: K \to I$ such that its value on an edge is greater (or equal) than the value on the endpoints of this edge. By the elder rule and since K is connected, $\min(f(a), f(b), f(c))$ is the left endpoint of the unique infinite interval in $\mathrm{PH}_0(f)$. For similar reasons, $\mathrm{PH}_1(f)$ contains a unique unbounded interval with left endpoint given by $\max(f(ab), f(bc), f(ac))$.

In addition, the example of the triangle has interesting symmetries. The symmetries of the triangle $\mathbb{G}(K) = \mathbf{D}_3$ is the dihedral group which can also be identified with the symmetric group Σ_3 , the set of bijections of the set $\{a,b,c\}$ of vertices of K.

Our goal in this section is to determine all the barcode strata and the corresponding fibers, see Figure 6; describe the action of $\mathbb{G}(K) \simeq \mathbf{D}_3$ on the fibers, and compute the monodromies between the non-discrete fibers.

Summary of the results:

- 1. In section A.1 we compute all 34 barcode strata in the image $Bar_K = PH(Filt_K)$;
- 2. In section A.2 we compute the fibers of PH over the distinct barcode strata. We find only five strata with fibers that are not discrete. By Theorem 2.2, these fibers are polyhedral complexes, and we exhibit the polyhedra in \mathbb{R}^K making up the complexes. The results in section 5.3 guarantee that $\mathbb{G}(K)$ acts on each fiber, and we describe this action on the fibers for these five barcode strata;
- 3. Finally, by the results in section 3.2, the closure containment relations between barcode strata yield monodromies between fibers, which up to homotopy are polyhedral maps. We describe the monodromies between non-discrete fibers in section A.3.

This simple example of the triangle shows that the fibers of the persistence map can be topologically distinct from each other. Furthermore, the topology of the fibers can also be more complex than that of the underlying simplicial complex K, especially for low dimensional strata in the space of barcodes. Similar observations hold when restricting the fibers to the subspace Low_K of lower star filters, as we detail in section A.4. We contrast this with the case studied in [15] where K is a triangulation of the interval [0,1] and lower star filters are considered. In that case the fibers of the persistence map are all disjoint unions of contractible sets. In particular, our example of the triangle shows that the fiber of the persistence map does not have to be a union of contractible sets.

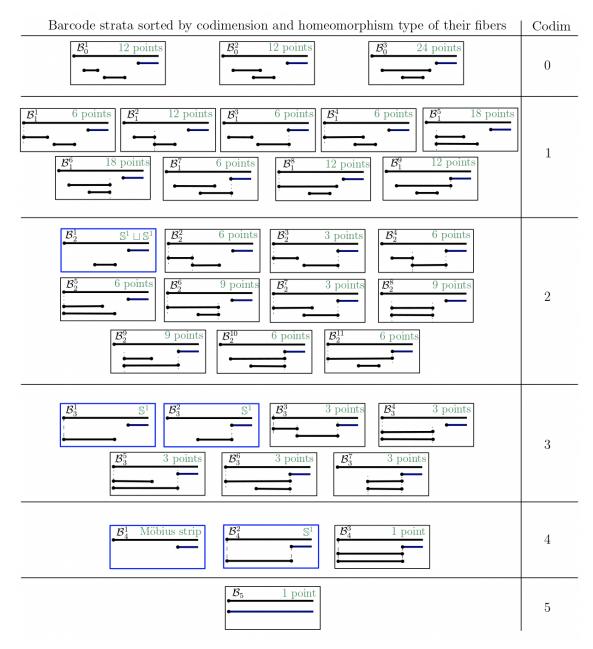


Figure 6: Each stratum is represented by one of its barcodes, where the blue interval is the one corresponding to the degree 1 homology of the triangle and the others correspond to degree 0 homology. Strata come with a label, for instance \mathcal{B}_0^2 , whose subscript gives the codimension and the superscript allows to enumerate strata of a given codimension. The homeomorphism type of the fiber of PH over each stratum is given in green. For instance, the fiber PH⁻¹(\mathcal{B}_1^2) is finite and consists of 12 distinct filters. The blue boxes highlight the five strata with non-discrete fibers.

A.1 Computation of Bar_K and its strata

For notational convenience, we replace the unit interval with a bigger interval, I := [-10, 10]. This allows considering barcodes with only integer valued endpoints. In addition, we restrict ourselves to strata of barcodes with endpoints strictly in (-10, 10), since they completely determine strata (and their fibers) where the endpoints -10 and 10 are allowed.

The top dimensional filter strata of the space of filters Filt_K correspond to injective filters and can equivalently be thought of as orderings of the simplices in the triangle, where an edge must appear after its vertices. This allows us to count these top dimensional strata. Namely, considering the case where the vertices of the triangle appear before all the edges, and separately the case where one edge appears before the last vertex, we get the following count of possible orderings and hence

$$3! \times 3! + 3 \times 2 \times 1 \times 1 \times 2! = 36 + 12 = 48$$
 top dimensional filter strata.

The top dimensional barcode strata in the image $\mathrm{Bar}_K = \mathrm{PH}(\mathrm{Filt}_K)$, by Proposition 1.18, are given by the image of the top dimensional filter strata. We argue that there are precisely three: From section 5.3, the persistence map is $\mathbb{G}(K)$ -equivariant, so we may restrict ourselves to those filter strata, viewed as orderings, for which the vertex a is first, f(a)=0, followed by b, f(b)=1. The following simplex must either be ab or c. In the first case, f(ab)=2, all filters satisfying this yield the same barcode $[\{(0,\infty),(1,2),(3,4)\},\{(5,\infty)\}]$. We denote by \mathcal{B}^1_0 the corresponding codimension 0 barcode stratum. In the second case, f(c)=2, we get two other barcodes depending on whether ab is the next simplex in the ordering or not, $[\{(0,\infty),(1,3),(2,4)\},\{(5,\infty)\}]$ and $[\{(0,\infty),(1,4),(2,3)\},\{(5,\infty)\}]$. We denote the corresponding barcode strata by \mathcal{B}^2_0 and \mathcal{B}^3_0 .

The list of all barcode strata in Bar_K , by Proposition 1.18, can be derived from the three top dimensional barcode strata \mathcal{B}_0^1 , \mathcal{B}_0^2 and \mathcal{B}_0^3 by collapsing their interval endpoints. We draw all the 34 resulting strata in Figure 6, sorted by codimension, and give them labels that are used in the rest of the section.

A.2 Computation of fibers and the action of $\mathbb{G}(K)$

The fibers over the various barcode strata in the image of the persistence map, whose computations are detailed in this section, are summarised in Figure 6 (in green in the top right corner of each box).

Strata with discrete fibers. Most of the barcode strata have finite fibers. For instance, the unique lowest-dimensional stratum is the one labelled by \mathcal{B}_5 : there only the two essential homological features are present, and appear at the same time. This implies that all the simplices of the triangle must appear at a given time, and therefore the fiber of PH over \mathcal{B}_5 consists of a unique constant function. This agrees with the prediction of Proposition 2.7 that the fiber is contractible.

The barcode strata with maximal number of bounded intervals, that is 2 such intervals, have zero bounded deficit (Def. 1.2). There are 28 such strata. By Proposition 2.5, their fibers are discrete. For instance, a simple counting argument gives discrete fibers for the top dimensional barcode strata \mathcal{B}_0^1 , \mathcal{B}_0^2 and \mathcal{B}_0^3 , with 12, 12, and 24 points in their fibers respectively. More generally, Proposition 2.5 upper-bounds the dimension of the fibers over arbitrary strata by their bounded deficit. In this example, note that the bounded deficit in fact equals the dimension of the fiber in all cases, except in the degenerate case of the stratum \mathcal{B}_5 . In the remainder of this section we compute explicitly the fibers over the five barcode strata that have fibers of dimension greater than 0.

Stratum \mathcal{B}_2^1 . We take a representative barcode $D := [\{(b_1, \infty), (b_2, d_2)\}, \{(b_3, \infty)\}] \in \text{Bar}^2$, where $b_1 < b_2 < d_2 < b_3$, of the codimension 2 stratum \mathcal{B}_2^1 .



Without loss of generality, $(b_1, b_2, d_2, b_3) = (0, 1, 2, 3)$. We use the identification of a filter f with the vector $(f(a), f(b), f(c), f(ab), f(ac), f(bc)) \in I^6$. If f yields barcode D, then:

- There is a vertex v (resp. an edge e) at which f attains its minimum (resp. maximum) value, equal to 0 (resp. 3).
- There is another vertex v' and edge e' such that f(v'), f(e') = 1, 2.
- The remaining vertex and **incident** edge have same value $0 \le t \le 3$.

Let us assume that the vertex a should create the first connected component, while the vertex c should create the second one. Then the vertex b must appear at the same time as an edge connecting b to either a or c. If b appears at the same time as ab, and if we wish that bc is the edge closing the loop, we get a 1-simplex $\{(0,t,1,t,2,3)\}_{0\leqslant t\leqslant 3}$ in the fiber, which we break into three 1-simplices that lie in (the closures of) different filter strata: $\{(0,t,1,t,2,3)\}_{0\leqslant t\leqslant 1}$,

 $\{(0,t,1,t,2,3)\}_{1\leqslant t\leqslant 2}$ and $\{(0,t,1,t,2,3)\}_{2\leqslant t\leqslant 3}$. If rather the edge ac closes the loop of the triangle, we get the two sets $\{(0,t,1,t,3,2)\}_{1\leqslant t\leqslant 2}$ and $\{(0,t,1,t,3,2)\}_{0\leqslant t\leqslant 1}$ in the fiber. Now, let us assume that b appears at the same time as bc, which imposes that b appears after time 1. Then if ab closes the loop of the triangle, we get the two sets $\{(0,t,1,3,2,t)\}_{1\leqslant t\leqslant 2}$ and $\{(0,t,1,3,2,t)\}_{2\leqslant t\leqslant 3}$ in the fiber, while if we wish that the edge ac closes this loop, we get the unique set $\{(0,t,1,2,3,t)\}_{1\leqslant t\leqslant 2}$ in the fiber. All in all, we have gathered 8 embeddings of the standard 1-simplex in I^6 . By symmetry, if we vary the choice of two vertices that create the first two connected components, we get 6 analogous collections of 8 embeddings of the standard 1-simplex described above, which together cover the fiber. This provides a description of the fiber in terms of a graph, whose incidence structure is explicited in Fig. 7. In particular, the fiber of PH over D is homeomorphic to $\mathbb{S}^1 \sqcup \mathbb{S}^1$.

We further depitct the action of $\mathbb{G}(K)$ on $\mathrm{PH}^{-1}(D)$ in Fig. 7. We consider the cyclic map $(a,b,c)\mapsto (b,c,a)$ and the elementary transposition $(a,b,c)\mapsto (b,a,c)$ as generators. We see that the cyclic map preserves the two connected components of the fiber $\mathbb{S}^1\sqcup\mathbb{S}^1$, whereas the elementary transposition swaps them. More generally, even permutations $g\in\mathbb{G}(K)$ preserve the connected components of the fiber, while odd permutations exchange them.

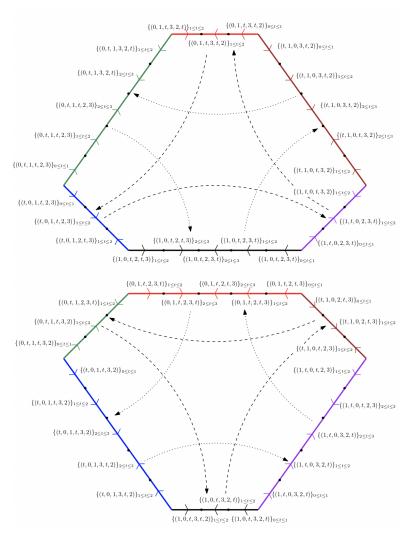


Figure 7: The fiber of the barcode in stratum \mathcal{B}_2^1 . We observe two connected components. Each edge represents an embedding of the standard 1-simplex in the fiber, oriented with an arrow toward increasing values of the parameter t. The six colors correspond to the six possible choices of two vertices responsible for the appearance of the first two connected components. If a and c (resp. a and b, b and a, b and c, c and a, c and b) create these components, we color the edge in green (resp. red, black, blue, purple and brown). The cyclic symmetry $(a, b, c) \mapsto (b, c, a)$ acts on the fiber by rotating each connected component by an angle of $\frac{2\pi}{3}$, as depicted by the dotted arrows. The transposition $(a, b, c) \mapsto (b, a, c)$ acts by swapping the connected components by reflecting along the horizontal axis.

Stratum \mathcal{B}_3^2 . Fixing two endpoints via $d_2 = b_3$ in the stratum \mathcal{B}_2^1 , we get the stratum \mathcal{B}_3^2 represented below.



As a representative of this stratum, we consider the barcode $D = [\{(0,\infty),(1,2)\},\{(2,\infty)\}] \in \operatorname{Bar}^2$. The fiber of D can be obtained similarly as in the previous case, that is by providing a cover of the fiber by embeddings of the 1-simplex in I^6 . Imposing that vertices a and then c should be responsible for the appearance of the first two components, we get the sets $\{(0,t,1,t,2,2)\}_{0\leqslant t\leqslant 1},\{(0,t,1,t,2,2)\}_{1\leqslant t\leqslant 2}$ and $\{(0,t,1,2,2,t)\}_{1\leqslant t\leqslant 2}$ in the fiber. By symmetry in the choice of these two vertices, we get a total of 6×3 embeddings of the standard 1-simplex that together cover the fiber. This decomposition describes the fiber as a graph, which is described in Fig 8 and is homeomorphic to \mathbb{S}^1 . The action of the symmetries on the circle is described in the figure as well.

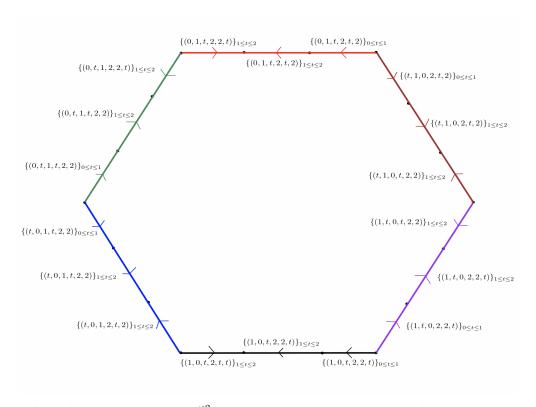


Figure 8: The fiber of the barcode in stratum \mathcal{B}_3^2 . Each edge represents an embedding of the standard 1-simplex in the fiber, oriented toward increasing values of the parameter t. The six colors correspond to the six possible choices of two vertices responsible for the appearance of the first two connected components in the barcode. The coloring convention is the same as in case \mathcal{B}_2^1 , Fig 7. The cyclic permutation of the triangle acts as a rotation by an angle of $\frac{2\pi}{3}$ oriented counter-clockwise, while the elementary transposition acts as the horizontal symmetry of the hexagon.

Stratum \mathcal{B}_3^1 . A representative of the stratum \mathcal{B}_3^1 is $D := [\{(0, \infty), (0, 1)\}, \{(2, \infty)\}] \in \text{Bar}^2$.



We describe the fiber with an explicit cover by embeddings of the 1-simplex in I^6 . Imposing that vertices a and c should be responsible for the appearance of the first two components, we get the following embeddings:

• $\{(0,t,0,1,2,t)\}_{0\leqslant t\leqslant 1}, \{(0,t,0,2,1,t)\}_{0\leqslant t\leqslant 1}, \{(0,t,0,t,1,2)\}_{0\leqslant t\leqslant 1}, \{(0,t,0,t,2,1)\}_{0\leqslant t\leqslant 1}, \{(0,t,0,t,1,2)\}_{1\leqslant t\leqslant 2}$ and $\{(0,t,0,2,1,t)\}_{1\leqslant t\leqslant 2}.$

By symmetry in the choice of these two vertices, we get a total of 3×6 embeddings of the standard 1-simplex that together cover the fiber. By inspecting adjacency relations, this decomposition describes the fiber as a graph isomorphic to the graph of the fiber over \mathcal{B}_3^2 . In particular, the fiber is homeomorphic to \mathbb{S}^1 .

Stratum \mathcal{B}_4^1 . Let $D = [\{(b_1, \infty)\}, \{(b_2, \infty)\}] \in \text{Bar}^2$ (where $b_1 < b_2$) be a barcode in the stratum \mathcal{B}_4^1 .



We may set $b_1=0$ and $b_2=1$ for simplicity. If a filter $f:K\to I$ yields the barcode D, then there are two pairs $(v_i,e_i)_{i=1,2}$ of vertices and edges such that $0\leqslant f(e_i)=f(v_i)=:t_i\leqslant 1$ and the remaining vertex v_0 and edge e_0 realize the minimum 0 and maximum 1 of f respectively.

For a fixed choice of $(v_i, e_i)_{i=0,1,2}$ as above, the two parameters $t_1 \leqslant t_2$ describe a 2-simplex. This 2-simplex corresponds to one of the top dimensional polyhedron in the fiber of PH over D, which we know is a polyhedral complex from Theorem 2.2. To count and describe these 2-simplices, fix v_0 , say $v_0 = a$. Then we distinguish between the two cases: (i) where e_0 contains v_0 and (ii) where e_0 does not contain v_0 . In case (i), if we choose $e_0 = ac$, then b and ab simultaneously appear at time t_1 and finally c and bc appear at time t_2 . The resulting simplex in the fiber is denoted by $S := \{(0, t_1, t_2, t_1, 1, t_2)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}$. In case (ii), there is only one choice for e_0 , i.e. $e_0 = bc$. The other vertex-edge pairs have to be the pairs (b, ab) and (c, ac). If we decide that the pair (b, ab) appears before (c, ac), we obtain the simplex $\bar{S} := \{(0, t_1, t_2, t_1, t_2, 1)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}$. All the other simplices in the fiber may be derived from the action of $\mathbb{G}(K)$ on S and \bar{S} . More precisely, letting $\tau: (a, b, c) \mapsto (b, a, c)$ be the elementary transposition and $c: (a, b, c) \mapsto (b, c, a)$ the cyclic permutation, we obtain all the 2-simplices in the fiber:

- (i) $S = \{(0, t_1, t_2, t_1, 1, t_2)\}, \ \tau.S = \{(t_1, 0, t_2, t_1, t_2, 1)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}, \ c.S = \{(t_2, 0, t_1, 1, t_2, t_1)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}, \ c^2.S = \{(t_1, t_2, 0, t_2, t_1, 1)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}, \ \tau c.S = \{(0, t_2, t_1, 1, t_1, t_2)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}, \ \tau c^2.S = \{(t_2, t_1, 0, t_2, 1, t_1)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1};$
- (ii) $\bar{S} = \{(0, t_1, t_2, t_1, t_2, 1)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}, \quad \tau.\bar{S} = \{(t_1, 0, t_2, t_1, 1, t_2)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}, \quad c.\bar{S} = \{(t_2, 0, t_1, t_2, 1, t_1)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}, \quad c.\bar{S} = \{(t_1, t_2, 0, 1, t_1, t_2)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}, \quad \tau c.\bar{S} = \{(0, t_2, t_1, t_2, t_1, 1)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}, \quad \tau c.\bar{S} = \{(t_2, t_1, 0, 1, t_2, t_1)\}_{0 \leqslant t_1 \leqslant t_2 \leqslant 1}.$

All the 12 sets are embeddings of the 2-simplex Δ^2 in I^6 , where we represent the simplex Δ^2 in \mathbb{R}^2 conveniently for our purpose as in Fig 9.

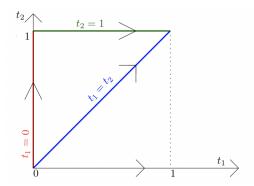


Figure 9: An embedding of the standard simplex Δ^2 , with faces in red, green and blue.

The 2-simplices in the orbit of \bar{S} meet with 3 distinct other 2-simplices at its 3 faces (obtained by setting $t_1=0,\,t_2=1$ or $t_1=t_2$). The 2-simplices in the orbit of S only meet with two other simplices. Glued together, these simplices form a Möbius band embedded in \mathbb{R}^6 , as explicited in Figure 10. Therefore the fiber of the persistence map over the barcodes in the stratum \mathcal{B}^4_1 is isomorphic, as a simplicial complex, to the Möbius band.

We again consider how the symmetry group Σ_3 of the triangle acts on the fiber. Note that the action on the fiber must preserve the orientations and colors described in Fig. 10. The elementary transposition simply rotates the Möbius band by an angle of π . The action of the cyclic permutation is slightly more involved as it reverses and translates the Möbius band.

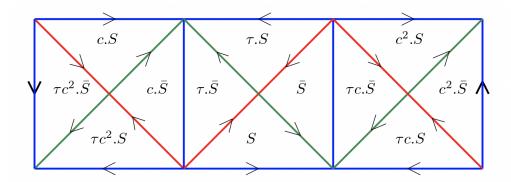


Figure 10: The 2-simplices in the fiber of the stratum \mathcal{B}_4^1 glued together in a Möbius strip. Blue (resp. red and green) edges correspond to setting $t_1=t_2$ (resp. $t_1=0$ and $t_2=1$) in their co-faces. The left and right extreme blue oriented edges are identified. The action of the transposition $(a,b,c)\mapsto (b,a,c)$ on the fiber can be described as a rotation of the Möbius strip by an angle of π . The action of the cyclic permutation $(a,b,c)\mapsto (b,c,a)$ on the fiber can be described as the composition of (i) the symmetry of the Möbius strip around its middle horizontal line, followed by (ii) a unit translation on the left of each simplex.

Stratum \mathcal{B}_4^2 . The last stratum of barcodes whose fiber we explicitly compute has representative $D := [\{(0,\infty)\},\{(0,1),(1,\infty)\}]$:



In this case, the fiber is the union of the segments:

$$\bullet \ \{(0,t,0,t,1,1)\}_{0\leqslant t\leqslant 1}, \quad \{(0,t,0,1,1,t)\}_{0\leqslant t\leqslant 1}, \quad \ \{(0,0,t,1,1,t)\}_{0\leqslant t\leqslant 1}, \quad \ \{(0,0,t,1,1,t)\}_{0\leqslant t\leqslant 1}, \quad \ \{(0,0,t,1,t,1)\}_{0\leqslant t\leqslant 1},$$

which assemble into a regular hexagon, and therefore the fiber of PH over the stratum \mathcal{B}_4^2 is homeomorphic to \mathbb{S}^1 .

A.3 Computation of monodromies between fibers

We describe the monodromies between non-discrete fibers. We focus on pairs of barcode strata that differ by one dimension, since a monodromy between an arbitrary pair of strata is a composition of such elementary monodromies by Corollary 3.14.

Monodromy from \mathcal{B}_2^1 to \mathcal{B}_3^1 and \mathcal{B}_3^2 . From the previous section, the fibers over \mathcal{B}_3^1 and \mathcal{B}_3^2 are isomorphic cyclic graphs. We only describe monodromies from $\mathrm{PH}^{-1}(\mathcal{B}_2^1)$ to $\mathrm{PH}^{-1}(\mathcal{B}_3^2)$ since monodromies from $\mathrm{PH}^{-1}(\mathcal{B}_2^1)$ to $\mathrm{PH}^{-1}(\mathcal{B}_3^1)$ are identical. The stratum \mathcal{B}_2^1 contains the stratum \mathcal{B}_3^2 in its closure. From Lemma 1.13, this ensures that the set of morphisms $\mathrm{Bar}_{\mathbf{K}}(D_2,D_3^2)$ is non-empty for any representatives $(D_2,D_3^2)\in\mathcal{B}_2^1\times\mathcal{B}_3^2$. For clarity, we take as representatives the barcodes from the previous section, that is $D_2=[\{(0,\infty),(1,2)\},\{(3,\infty)\}]$ and $D_3^2=[\{(0,\infty),(1,2)\},\{(2,\infty)\}]$.

From section 3.2, the number of different homotopy classes of monodromies from $PH^{-1}(D_2)$ to $PH^{-1}(D_3^2)$ is upper-bounded by the cardinality of $\pi_0(\mathbf{Bar_K}(D_2,D_3^2))$. Moreover, the homotopy type of an element $\phi \in \mathbf{Bar_K}(D_2,D_3^2)$ is completely characterized by its index by Remark 3.8. In the current situation, any map ϕ from D_2 to D_3^2 must collapse the third and fourth endpoint of D_2 . This means that up to homotopy, there is a unique monodromy map from $PH^{-1}(D_2)$ to $PH^{-1}(D_3^2)$ to describe. It is then natural to choose ϕ to be any linear extension of the map:

$$\phi: \{0, 1, 2, 3\} \mapsto \{0, 1, 2, 2\}.$$

The resulting monodromy \mathcal{L}_{ϕ} , from Proposition 3.12, is a simplicial map between fibers. To visualize \mathcal{L}_{ϕ} , it is convenient to see how the fiber $\mathrm{PH}^{-1}(D_2)$ is progressively deformed into a subset of $\mathrm{PH}^{-1}(D_3^2)$ under the path $t \mapsto t\mathcal{L}_{\phi} + (1-t)\mathrm{Id}$. As depicted in Fig. 11, this transformation has the effect to collapse some edges and to identify some edges and nodes.

In the same figure, we see that the monodromy is a surjection onto the fiber $PH^{-1}(D_3^2)$. By Proposition 5.9, the equivariance of the monodromy w.r.t. the $\mathbb{G}(K)=\Sigma_3$ action on the fibers predicts that the image, through the monodromy, of the action on $PH^{-1}(D_2)$ must equal the action on $PH^{-1}(D_3^2)$. Let us for instance consider the cyclic permutation $(a,b,c)\mapsto (b,c,a)$ of the triangle, which acts on the two hexagons constituting $PH^{-1}(D_2)$ by rotation of an angle of $\frac{2\pi}{3}$. The monodromy identifies the two hexagons in the fiber, hence the induced action of g on $PH^{-1}(D_3^2)$ is the rotation by the same angle, which agrees with the direct computation of the action of g on $PH^{-1}(D_3^2)$ in Fig. 8. The same observation can be made about the elementary transposition $\tau:(a,b,c)\mapsto (b,a,c)$ of the triangle.

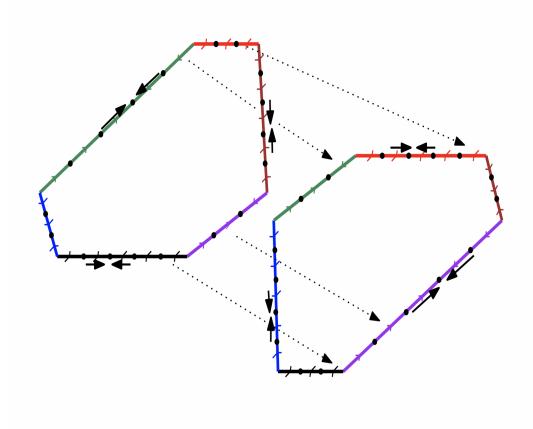


Figure 11: Starting with the two irregular hexagons describing the fiber of the barcode $[\{(0,\infty),(1,2)\},\{(3,\infty)\}]$ of stratum \mathcal{B}_2^1 (see Fig. 7), the arrows describe the process of continuously tracking the fiber as the interval $(3,\infty)$ gets closer to the interval $(2,\infty)$, thus ending to the barcode $[\{(0,\infty),(1,2)\},\{(2,\infty)\}]$ of stratum \mathcal{B}_3^2 . Plain arrows show edges of the fiber that are collapsed during this process. Meanwhile, the two hexagons merge into the regular hexagon depicted in Fig. 8. This merging happens by identifying edges of the two components, of the same color and orientation, following the dotted arrows.

Monodromy from \mathcal{B}_3^1 and \mathcal{B}_3^2 to \mathcal{B}_4^1 . The stratum \mathcal{B}_3^2 contains the stratum \mathcal{B}_4^1 in its closure. We let $D_3^2 := [\{(0,\infty),(1,2)\},\{(2,\infty)\}]$ be the representative barcode of the stratum \mathcal{B}_3^2 , and $D_4^1 := [\{(0,\infty)\},\{(1,\infty)\}]$ be that of \mathcal{B}_4^1 , both as in the previous section. The set of morphisms $\mathbf{Bar}_{\mathbf{K}}(D_3^2,D_4^1)$ is non-empty, and in fact contains a unique homotopy class by Remark 3.8, since all maps $\phi \in \mathbf{Bar}_{\mathbf{K}}(D_3^2,D_4^1)$ must collapse the interval (1,2). It is then natural to choose ϕ to be any linear extension of the map:

$$\phi: \{0, 1, 2\} \mapsto \{0, 1, 1\}.$$

From Proposition 3.12, the resulting monodromy $\mathcal{L}_{\phi}: \mathrm{PH}^{-1}(D_3^2) \to \mathrm{PH}^{-1}(D_4^1)$ is a map of polyhedral complexes. Recall that the fiber $\mathrm{PH}^{-1}(D_3^2)$ is a polyhedral complex described in Fig. 8. The monodromy map \mathcal{L}_{ϕ} collapses 12 out of the 18 edges in $\mathrm{PH}^{-1}(D_3^2)$. The remaining 6 edges form a regular hexagon which is mapped onto the green circle of the Möbius strip describing the fiber of \mathcal{B}_4^1 , see Fig 10. Likewise, the monodromy from the fiber of (a representative of) the stratum \mathcal{B}_3^1 to the fiber over \mathcal{B}_4^1 is unique up to homotopy, and collapses 12 out of the 18 edges in the fiber over \mathcal{B}_3^1 , sending the remaining 6 edges onto the red circle of the Möbius strip describing the fiber of \mathcal{B}_4^1 .

Monodromy from \mathcal{B}_3^1 and \mathcal{B}_3^2 to \mathcal{B}_4^2 . Recall that the fiber over \mathcal{B}_3^2 is a cyclic graph with 18 edges depicted in Figure 8, while the fiber over \mathcal{B}_4^2 is a regular hexagon. There is again a unique (up to homotopy) monodromy \mathcal{L}_{ϕ} between these fibers. The simplicial map \mathcal{L}_{ϕ} collapses 12 out of the 18 edges in $\mathrm{PH}^{-1}(\mathcal{D}_3^2)$. The remaining 6 edges (one for each color in Figure 8) form the regular hexagon $\mathrm{PH}^{-1}(\mathcal{B}_4^2)$. In particular, \mathcal{L}_{ϕ} is a fibration. The monodromy from $\mathrm{PH}^{-1}(\mathcal{D}_3^1)$ to $\mathrm{PH}^{-1}(\mathcal{B}_4^2)$ can be described in the same way.

A.4 Lower star filters

If we consider the restriction of PH to the subspace $Low_K \subseteq Filt_K$ of lower star filters, there are only 2 barcode strata in the image: \mathcal{B}^1_4 and \mathcal{B}_5 . The reason for this is that edges enter the sublevel set filtration of a lower star filter at the same time as one of their vertices, hence there are no bounded intervals in the resulting barcode. The fibers of the restricted persistence map over the two strata can be derived from the general case. Namely, the fiber over \mathcal{B}_5 is the constant filter, while the fiber over \mathcal{B}_4^1 is a hexagon which embeds into the Möbius strip as the green zig-zag in Fig. 10.

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