Avicenna's Philosophy of Mathematics

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I hereby declare that this dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except where specifically indicated in the text. I further state that no substantial part of my dissertation is the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution. This dissertation does not exceed the 80,000-word limit. The second chapter, with slight modifications, has been published in *Dialogue: Canadian Philosophical Review*. The first and thrid chapters, again with some revisions, are accepted for publication, respectively, in *Oriens* and *Archiv für Geschichte der Philosophie*. The final chapter is under review for publication in another journal.

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ABSTRACT. I discuss four different aspects of Avicenna's philosophical views on mathematics, as scattered across his various works. I first explore the negative aspect of his ontology of mathematics, which concerns the question of what mathematical objects (i.e., numbers and geometrical shapes) are not. Avicenna argues that mathematical objects are not independent immaterial substances. They cannot be fully separated from matter. He rejects what is now called mathematical Platonism. However, his understanding of Plato's view about the nature of mathematical objects differs from both Plato's actual view and the view that Aristotle attributes to Plato. Second, I explore the positive aspect of Avicenna's ontology of mathematics, which is developed in response to the question of what mathematical objects are. He considers mathematical objects to be specific properties of material objects actually existing in the extramental world. Mathematical objects can be separated, in mind, from all the specific kinds of matter to which they are actually attached in the extramental word. Nonetheless, inasmuch as they are subject to mathematical study, they cannot be separated from materiality itself. Even in mind they should be considered as properties of material entities. Third, I scrutinize Avicenna's understanding of mathematical infinity. Like Aristotle, he rejects the infinity of numbers and magnitudes. But he does so by providing arguments that are much more sophisticated than their Aristotelian ancestors. By analyzing the structure of his Mapping Argument against the actuality of infinity, I show that his understanding of the notion of infinity is much more modern than we might expect. Finally, I engage with Avicenna's views on the epistemology of mathematics. He endorses concept empiricism and judgment rationalism regarding mathematics. He believes that we cannot grasp any mathematical concepts unless we first have had some specific perceptual experiences. It is only through the ineliminable and irreplaceable operation of the faculties of estimation and imagination upon some sensible data that we can grasp mathematical concepts. By contrast, after grasping the required mathematical concepts, independently from all other faculties, the intellect alone can prove mathematical theorems. Other faculties, and in particular the cogitative faculty, can assist the intellect in this regard; but the participation of such faculties is merely facilitative and by no means necessary.

To Samaneh Ehsaninezhad

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Contents

Introduction

Abū ʿAlī Ibn Sīnā (ca. 970-1037)—to whom I refer by his Latinized name 'Avicenna'—was the most significant philosopher in the Islamic world. His immense influence on the later philosophical traditions, in both the Islamic world and the Latin west, can hardly be exaggerated. Moreover, as a polymath, he wrote on a wide range of (what we now categorize as) *scientific* topics including medicine, pharmacology, mineralogy, astronomy, and mathematics. Avicenna's familiarity with some of these sciences clearly had a great impact on how his philosophy was formed. In particular, the traces of his knowledge of medicine and mathematics are easily discernible in such central areas of his philosophy as epistemology and logic. Thus a comprehensive understanding of Avicenna's philosophy is impossible unless we have a reliable picture of his views *about* these two sciences. Roughly speaking, we can say that his epistemology combines empiricist and rational elements, with medicine and mathematics respectively serving as paradigm sciences: medicine is often invoked when Avicenna illustrates the role of sense experience in science, while mathematics would seem to be a non-empirical science.[1] While Avicenna's medicine and the role of medical examples in his empiricism have been studied to some extent, Avicenna's views on philosophy of mathematics have been largely neglected. While there are some impressive studies on Avicenna's mathematical works,[2] they mainly concern technical aspects of Avicenna's treatment of mathematics and have nothing to do with his philosophical views *about* it. To the best of my knowledge there is only one short book and a few (fewer than five) articles about Avicenna's philosophy of mathematics, which by no

[1] It is well known that Avicenna, following the Aristotelian tradition, considers some sciences, like music and astronomy, to be branches of the mathematical sciences. But, having the modern conception of *mathematics* in mind, I will focus only on geometry and arithmetic. More precisely, I will focus on what Avicenna (2005, chap. I.3, p. 17, l. 10) calls 'pure mathematics.' So, when I speak of Avicenna's philosophy of mathematics, I speak of his philosophical views concerning three dimensional Euclidean geometry and the arithmetic of natural numbers.

[2] See, among others, Al-Daffa and Stroyls (1984), Rashed (1984), Djebbar (1999), and Luther (2004).

means provide a systematic and comprehensive picture of his philosophical views about mathematics.[1] This is the gap that is intended to be filled by this PhD project.[2]

When we think about philosophy of mathematics, we are faced with two major questions. One pertains to the *ontology* of mathematical entities, while the other is related to the *epistemology* of mathematical concepts and propositions. Thus in this dissertation I deal with some ontological and some epistemological issues related to mathematics. In the following two chapters I discuss Avicenna's views on the nature and existence of what are today called mathematical objects (e.g. numbers and geometrical shapes). In chapter one I discuss Avicenna's criticisms of the views he finds implausible on the nature of mathematical objects. In particular, I give a detailed analysis of his arguments against what is known today as mathematical Platonism. As we will see, his empiricist epistemology plays a crucial role in his rejection of the existence of mathematical objects as independent immaterial substances. In chapter two, I turn to Avicenna's own alternative theory about the nature of mathematical entities. Another ontological question with which I engage in this dissertation is the existence and qualifications of mathematical infinities (e.g., infinite magnitudes and infinite sets of numbers). As will be clarified, Avicenna's account of the ontology of mathematics has some significant consequences for his theory of infinity. The subtleties of this theory are clarified in chapter three.

Having clarified Avicenna's position on the ontology of mathematics, I will be in a position to explain his views on the formation of mathematical concepts (e.g., the concept TWO or the concept TRIANGLE) and the assent to the truth of mathematical propositions (e.g. that 2+2=4 or that the sum of interior angles of a triangle is equal to two right angles). Chapter four is dedicated to Avicenna's positions on these issues.

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[1] Tahiri (2016) discusses Avicenna's epistemology of mathematics and its connection to his general theory of knowledge. Rashed (2016) discusses Avicenna's views regarding some ontological issues about mathematics. Ardeshir (2008) provides a more inclusive picture of Avicenna's ontology and epistemology of mathematics. For a general survey of philosophy of mathematics in the Arabic tradition see Rashed (2008).

[2] McGinnis (2007b, p. 185, n. 41) confirms that "accounts of Avicenna's philosophy of mathematics are few."

In each of the four chapters of the dissertation the contents of the previous chapters are presupposed. Nonetheless, these chapters are designed to be readable as independent papers. The second chapter, with slight modifications, has been published in *Dialogue: Canadian Philosophical Review*.[1] The first and third chapters, again with some modifications, are accepted for publication, respectively, in *Oriens* and *Archiv für Geschichte der Philosophie*.[2] The final chapter is under review for publication in another journal.[3] These chapters are heavily interrelated with each other and should be considered as different pieces of the jigsaw puzzle that is Avicenna's philosophy of mathematics. Admittedly, though, there remain significant issues pertaining to Avicenna's philosophy of mathematics that, due to restrictions in space, I could not address in this dissertation. For instance, Avicenna's understanding of the nature of mathematical proofs and his account of mathematical continuum are postponed to future studies. But, hopefully, this will not prevent us from achieving an overall picture of the most essential elements of Avicenna's philosophy of mathematics.

[1] Zarepour (2016).

[2] Zarepour (n.d.-a, n.d.-b).

[3] All parts of this project have been undertaken during my PhD studies and no part of it has already been submitted, or is being submitted, for any other qualification.

1. Against Mathematical Platonism

In this chapter I investigate Avicenna's criticisms of the separateness of mathematical objects and their principleness for natural things. These two theses form the core of Plato's view of mathematics; i.e., mathematical Platonism*. Surprisingly, Avicenna does not consider his arguments against these theses as attacks on Plato. This is because his understanding of Plato's philosophy of mathematics differs from both Plato's original view and what Aristotle attributes to Plato.*

1.1. Introduction

Avicenna's view about the nature of mathematical objects has two distinct aspects. Its negative aspect was developed in response to the question of what mathematical objects are not. Its positive aspect, on the other hand, clarifies what mathematical objects are. The negative aspect of Avicenna's ontology of mathematics, which is interwoven with his rejection of the theory of Platonic forms, is the focus of this chapter.[1] Avicenna criticizes and rejects three Platonic ideas, which can be formulated as the following theses:

> **Separateness of Forms (SF):** Forms (*ṣuwar*) are independent immaterial substances, fully separate (*mufāriq*) from matter and material objects.
>
> **Separateness of Mathematical Objects (SM):** Mathematical objects are independent immaterial substances, fully separate from matter and material objects.
>
> **Principleness of Mathematical Objects (PM):** Mathematical objects are the principles (*mabādi'*) of natural things. Mathematical objects have some sort of primacy over natural forms which makes the latter dependent on (or grounded in or caused by) the former.

These theses originate in ancient Greek philosophy. Plato in some way endorses all of them. (SF) forms one of the bedrocks of his famous theory of forms. He explicitly holds (SM) and can also plausibly be understood as being committed to (PM) in a specific sense. These two latter theses form the foundation of Plato's stance regarding the ontology of mathematics. As I will shortly explain, some other ancient Greek philosophers defended one or two of these theses. By contrast, Aristotle finds none of them plausible; and as the prominent representative of the Aristotelian tradition of his day, Avicenna follows the same approach. He strongly criticizes the theory of Platonic forms and argues that mathematical objects are neither independent immaterial substances nor the principles of material existents.

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[1] The positive aspect of Avicenna's views about the nature of mathematical objects will be discussed in the next chapter.

Avicenna's understanding of the ancient Greek views about mathematics, as is to be expected, was formed mainly through Aristotle's *Metaphysics* and its commentaries. In particular, his understanding of Plato's theory of mathematical objects is evidently based on what Aristotle attributes to Plato. Avicenna, like Aristotle, believes that mathematical objects for Plato are *intermediates* between immaterial Platonic forms and physical objects. However, despite this apparent similarity, Avicenna seems to be completely ignorant of the meaning of the *intermediateness* of mathematical objects for Plato as it is described in Aristotle's *Metaphysics*. Indeed, Avicenna's description of the so-called *theory of intermediates* is so different from Aristotle's version of this theory that Avicenna's direct reading of any (even relatively) *reliable* translation of the related passages of Aristotle's *Metaphysics* (in which this theory is introduced) is highly improbable. Moreover, it is not clear whether or not Plato's original view is completely in accord with any of these two versions of the theory of intermediates which are attributed to him by Aristotle and Avicenna. Therefore, we are dealing with three distinct things: (1) Plato's original view, (2) the version of the theory of intermediates that Aristotle attributes to Plato, and (3) the version of the theory of intermediates that Avicenna attributes to Plato. Interestingly, Avicenna does not criticize (3). Therefore, it is only from an external point of view that Avicenna's arguments against (SM) and (PM) can be considered as a rejection of Plato's view on mathematical objects. Avicenna does not object to what he himself considers to be Plato's original ontology of mathematics. Although Avicenna was evidently aware that Plato embraces (SF), he believes neither that mathematical objects are forms for Plato nor that Plato endorses (SM) and (PM). This indicates that Avicenna himself did not consider his rejection of the aforementioned theses to be a rejection of Plato's view about mathematical objects—which should warn us that Avicenna might not have been sympathetic to the title of this chapter!

In the next section, I briefly sketch Plato's theory of mathematical objects and the version of the theory of intermediates that Aristotle attributes to him. From section 1.3 to the end of this chapter, I discuss the negative aspects of Avicenna's ontology of mathematics, relying on what he puts forward in chapters 2 and 3 of the seventh book of *The Metaphysics of the*

Healing.[1] Avicenna's understanding of ancient Greek views about the ontology of mathematics and his categorization of these views into three major groups (and some subsidiary ones) is explained in section 1.3. This categorization is based more or less on the possible positions one might hold regarding the above theses. By investigating Avicenna's interpretation of Plato's view, I clarify how Avicenna's version of the theory of intermediates differs from that of Aristotle. In the same section, I deal with certain arguments that Avicenna attributes to the proponents of the aforementioned theses. In section 1.4, I briefly examine Avicenna's criticisms of the proposed arguments for (SM) and (PM). Sections 1.5 and 1.6 are devoted to Avicenna's own positive arguments against these two theses. I close this chapter in section 1.7 with some concluding remarks.

1.2. Plato's Theory of Mathematical Objects

Plato's thoughts about mathematics are presented mainly in *Meno*, *Republic* and *Letter VII*.[2] There are also some brief discussions of issues pertinent to the philosophy of mathematics scattered in his other works.[3] Unfortunately, there is no consensus on his exact view about the nature of mathematical objects. Nonetheless, it seems beyond question that for Plato mathematical objects, inasmuch as they are subject matters of mathematical studies, are independent immaterial objects. In other words, he undoubtedly espouses (SM). His main

[1] Avicenna (2005, Chapter VII.2-3). Chapter VII.2 of *The Metaphysics of the Healing* has been discussed by Marmura (2006), Porro (2011), and Uluç (2012). The main foci of these papers are, however, the general features of Avicenna's criticism of the Platonic theory of forms, rather than his particular arguments against the mathematical ontologies which he found implausible. Unfortunately, chapter VII.3 has been largely neglected in modern Avicenna scholarship.

[2] The authenticity of *Letter VII* has been disputed by some Plato scholars. See Burnyeat and Fred (2015) for a recent detailed discussion on the spuriousness of this letter.

[3] For a detailed list of the passages in which Plato discusses mathematical issues, see the appendices of Wedberg (1955).

argument for endorsing the immateriality of mathematical objects can be best portrayed as follows:

The Platonic Argument for (SM):

(P1) Mathematical theorems are true of some independent existents.

(P2) Mathematical theorems are not true of material objects in the sensible world.

Therefore:

(P3) Existents of which the theorems of mathematics are true—i.e., mathematical objects—are not material objects in the sensible world. They are independent immaterial objects.

Although Plato himself nowhere explicitly mentions this argument, many scholars have interpreted him as having something very similar to this argument in mind.[1] Indeed, the attribution of this argument to Plato can be traced back to Aristotle. He says:

> **TEXT # 1.1.** But those who make number separable assume that it exists and is separable because the axioms would not be true of sensible things, while the statements of mathematics are true and delight the soul; and similarly with the magnitudes of mathematics.[2]

But how can we defend the attribution of such an argument to Plato? The combination of three considerations implies that Plato holds premise (P1). First, there is no pre-modern philosopher who takes a skeptical position towards mathematics and rejects the truth of mathematical theorems. In the same way, Plato takes it for granted that the theorems of mathematics are genuinely true. Second, a theorem cannot be true unless there exist some

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[1] See, among others, Burnyeat (1987), Shapiro (2000, Chapter 3), Bostock (2009, Chapter 1, 2012), and Panza and Sereni (2013).

[2] *Metaphysics* (1090b4-1722). All translations of Aristotle's passages are taken from *The Revised Oxford Translation of The Complete Works of Aristotle* (1984).

things of which it is true. No theorem can be true of things that do not exist at all.[1] If the objects to which a theorem refers do not exist, then that theorem does not express any facts and, consequently, cannot be true. This can be considered as a corollary of the correspondence theory of truth, to which he is committed.[2] That all truths, including all mathematical ones, are about existing things is beyond dispute also for Aristotle, who attacks Plato's ontology of mathematics. Aristotle clearly holds that the subject of the discussion for mathematical objects 'will be not whether they exist but *how* they exist.'[3] The third point we should consider with respect to (P1) is the ontological independence of mathematical objects. The truth of mathematical theorems does not seem to be dependent upon our minds. Even if there were no humans, mathematical theorems would still be true. This shows that they are not true of constructions dependent upon our minds; they are not mental objects whose existence is intertwined with the existence of human minds. The main sign of Plato's commitment to the objectivity and independence of mathematics is his non-empiricist epistemology of mathematics and his faith in—what is now called—the *a priority* of mathematics.[4] The combination of these three points together shows why (P1) is true for Plato.[5]

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[1] For the sake of simplicity, we can put aside the case of negative existentials. There are many true positive mathematical statements.

[2] See, for example, *Cratylus* (385b2) and *Sophist* (263b).

[3] *Metaphysics* (1076a37); my emphasis.

[4] Plato's epistemology of mathematics appears in *Meno*. Although some scholars, e.g., Bostock (2012), believe that the so-called *theory of recollection* which was at the center of his discussions in *Meno* is abandoned in Plato's later works, there is no doubt that he never abandoned the *a priority* of mathematics and its ontological independence from our mind and perceptual experiences.

[5] Some people who attribute a fictionalist account of mathematics to Aristotle reject (P1). They interpret mathematical objects as fictional entities whose existence depends upon our minds and our cognitive mechanisms. Given this interpretation, mathematical objects are merely mental/representational constructions. See, for example, Lear (1982), Hussey (1991), and Corkum (2012) for different fictionalist interpretations of Aristotle's ontology of mathematics. However, such an attribution to Plato is rare and in strong conflict with the mainstream understanding of his ontology of mathematics. Franklin (2012) defends

The second premise of the Platonic Argument for (SM) is justified by various pieces of evidence he provides for the perfection of mathematical objects and the imperfection of the sensible things existing in the physical world. For example, in *Republic* Plato argues that the main concern of geometers is not the figures they draw on paper. These sensible objects are just auxiliary tools to facilitate the geometers' thinking of the ideal objects of which the drawn figures are merely images:

> **TEXT # 1.2.** Then you also know that, although they [i.e., geometers] use visible figures and make claims about them, their thought isn't directed to them but to those other things that they are like. They make their claims for the sake of the square itself and the diagonal itself, not the diagonal they draw, and similarly with the others. These figures that they make and draw, of which shadows and reflections in water are images, they now in turn use as images, in seeking to see those others themselves that one cannot see except by means of thought.[1]

Plato also distinguishes between number as the subject matter of arithmetic and number as a property of physical objects. Numbers in the latter sense—what we may call physical numbers—are the numbers of groups of physical objects which are not necessarily similar to each other in all respects. By contrast, numbers in the former sense—what we may call arithmetical numbers—are pluralities of perfect indivisible units that are equal to one another in every respect; units that are completely indiscernible from one another.[2] Obviously, such perfect units do not exist in the physical world. They must, therefore, be ideal

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this uncommon view. I do not touch on this alternative reading of Plato, and confine myself to the explanation of the mainstream view.

[1] *Republic* (510d-511a). All English translations of Plato's passages are taken from *Plato: Complete Works* (1997). There are remarkable points in common between TEXT # 1.2 and a text in *The Demonstration* part of *The Healing* (1956, Chapter II.10, p. 186, ll. 15-18). See my discussion of TEXT # 2.7 in the next chapter.

[2] It obviously provokes some questions about how two things can be both indiscernible and distinct from each other at the same time. This remains one of the important problems of contemporary philosophy of mathematics, but discussing it would take us far afield. For Plato and Aristotle's stances on the problem of the identity of indiscernibles, see Eslick (1959, 1960) and Garner (2018).

immaterial objects.[1] As a result, arithmetical numbers are not physical numbers.[2] They cannot be numbers of collections of armies and cows. In the Platonic dialogues, arithmetic is explicitly disassociated from *logistic* (or *computation*), which is the art or skill of counting physical objects.[3] Physical numbers, Plato believes, are objects of logistic, rather than arithmetic. The fundamentals of this view are presented by Socrates in *Philebus*:

> **TEXT # 1.3.** There are those [i.e., ordinary people] who compute sums of quite unequal units, such as two armies or two herds of cattle, regardless whether they are tiny or huge. But then there are the others [i.e., mathematicians] who would not follow their example, unless it were guaranteed that none of those infinitely many units differed in the least from any of the others.[4]

The mathematical theorem '1+1=2' is not true of any couple of objects in the physical world. Mathematically speaking, the two '1's of this theorem are referring to objects that are equal to one another in all respects. But there are no such two things in the sensible world. Arithmetical units reflect nothing but pure unity. There is, however, no such thing in the material world.[5] This indicates that the theorems of mathematics, either geometrical or

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[1] Shapiro (2000, p. 58) mentions that 'there is no consensus on Plato's opinions concerning the nature of number. One interpretation has it that Plato took numbers to be ratios of geometric magnitudes.' This would mean that for Plato arithmetic is a branch of geometry and his understanding of numbers is very similar to what Euclid puts forward in the tenth book of *Elements*. But even if so, since geometrical objects—inasmuch as they are subject matters of mathematics—do not exist in the sensible world, numbers which are, according to this interpretation, derivations of geometric magnitudes do not exist in the sensible world either. So this uncertainty about Plato's view about numbers has no impact on the above line of reasoning.

[2] According to the *literalist* interpretation of Aristotle's philosophy of mathematics, as Shapiro (2000, p. 68) remarks, 'numbers are numbers of collections of ordinary objects. Aristotle's numbers are Plato's physical numbers.' If so, Avicenna's account of numbers is tantamount to that of Aristotle. See secion 2.4 of the next chapter.

[3] See *Gorgias* (451a-451c) for the distinction between logistic (or computation) and arithmetic.

[4] *Philebus* (56d-56e). See also *Phaedo* (72a-77e), *Theaetetus* (195e-196a), and *Republic* (525d-526a).

[5] To be precise, this example does not presuppose that one is a number. It merely says that two, inasmuch as it is an object for arithmetic, is combined of two units that are perfectly similar to each other in all respects. Plato's conception of numbers is exactly what Euclid (1908, bk. VII, def. 2, p. 277) puts forward: 'a number is a

arithmetical, cannot be true of sensible things. Ergo, (P2) is true for Plato. In sum, on the one hand, mathematical objects are independent existents. On the other, they are not material objects. Therefore, they must be independent immaterial objects. This, many scholars believe, is how Plato arrives at his belief in (SM).[1]

I now turn to Plato's treatment of (PM). This thesis can quickly be derived if we accept that mathematical objects are Platonic forms. A focal element of Plato's theory of forms is that they are principles of natural things and they cause sensible objects to have the properties they actually have.[2] Therefore, if mathematical objects are Platonic forms, then they can in one way or another be considered as (at least some of) the principles of natural things.[3] However, (SM) on its own does not imply that mathematical objects are Platonic forms. It merely yields that for Plato mathematical objects, like forms, are immaterial independent objects. So (PM) cannot be established in this way unless we provide independent evidence that Plato sees mathematical objects as Platonic forms. Regardless of whether such evidence can be found, there are other observations which can convince us that Plato accepts (PM). For example, there is a detailed discussion in *Timaeus* on the pivotal role of mathematical objects in the construction of the various dimensions of the cosmos. On the one hand, he

multitude composed of units.' According to this definition, sine 1 is not a multitude, it cannot be counted as a number. See Pritchard (1995, Chapter 5). Like Plato and Aristotle, Avicenna does not consider one as a number ('*adad*). He believes that 'the smallest number is two' (1985, p. 545). It is worth noting, however, that although one is not a number, it can be an object of arithmetical studies. There are some arithmetical facts about one including the fact which can be expressed by the sentence '1+1=2'. So *one* can be (and is indeed) one of the objects of arithmetic. But this does not imply that it is a number.

[1] For an opposite view see Franklin (2012) who rejects the argument under discussion and attributes a more sophisticated ontology of mathematics to Plato.

[2] *Phaedo* (100bff). The idea of the principleness of the Platonic forms for natural things can be formulated as an independent thesis worth adding to the three theses set out above. I refrain from doing so, however, since Avicenna did not discuss this idea independently. When he demonstrates that Platonic forms do not exist at all, he has *a fortiori* rejected their being principles of natural things.

[3] See *Metaphysics* (1090a) in which Aristotle links the existence of numbers as Platonic forms to their principleness for other things.

propounds that the soul of the world is constructed from a series of numbers.[1] On the other hand, he argues that the body of the world is constructed by certain geometrical objects.[2] This should persuade us that Plato admits (PM).

(PM) describes a generalized version of a well-known Pythagorean doctrine according to which numbers are the principles and causes of the things that are.[3] There is no doubt that some of the views discussed in the Platonic dialogues developed under the influence of the Pythagorean teachings.[4] The principleness of mathematical objects over natural things is one one of those teachings that Plato borrowed from the Pythagoreans and gave the flavor of his own philosophy. It is worth noting, however, that although Plato and the Pythagoreans share the idea of the principleness of mathematical objects, they disagree about the nature of mathematical objects. Contrary to Plato, the Pythagoreans do not accept (SM). They do not treat numbers as entities fully separate from matter.

In his late period, Plato likely became even more explicitly committed to Pythagorean views, at least if we accept as reliable Aristotle's reports about Plato's 'unwritten' doctrines. Aristotle describes Plato as believing in many Pythagorean ideas.

> **TEXT # 1.4.** He [i.e., Plato] agreed with the Pythagoreans in saying that the One is substance and not a predicate of something else; and in saying that the numbers are the causes of the substance of other things.[5]

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[1] *Timaeus* (35-36b).

[2] *Timaeus* (53c-56c). See also Aristotle's *On The Soul* (404b7-404b26).

[3] See Zhmud (1989).

[4] The footprint of the Pythagorean teachings is easily traceable in, among others, *Gorgias*, *Phaedo*, *Republic*, and *Timaeus*. For the influence of the Pythagoreans on Plato, see Dillon (1996, pp. 1–11), Kahn (2001, Chapter IV), and Riedweg (2005, pp. 116–118).

[5] *Metaphysics* (987b23-987b25). See also fragments F 28 R3 and F 203 R3 extracted from Alexander's *Commentarius in Metaphysica* in which Aristotle explicitly says that for Plato, like the Pythagoreans, numbers are the first principles of all existing things.

The above discussion shows that Plato endorsed (SM) and (PM). However, it does not yet reveal whether Plato considered mathematical objects to be forms. There is evidence that he takes at least some mathematical objects to be forms. For example, Plato usually uses the phrase 'the X itself' to refer to the immaterial form of X, and there are places—e.g., TEXT # 1.2 quoted above—in which he employs this locution to talk about the subject matters of geometry. This might indicate that Plato treated geometrical objects as forms. Plato also seems to have treated some numbers as forms.[1] Moreover, it is said in some parts of the Aristotelian corpus that Plato identified (at least some) numbers with forms.[2] One might therefore conclude from these points that for Plato mathematical objects are forms. Nonetheless, this claim cannot be considered established until any evidence to the contrary is rebutted. Perhaps the most important such evidence is Aristotle's attribution of the theory of intermediates to Plato. Aristotle says:

> **TEXT # 1.5.** Besides sensible things and Forms he [i.e., Plato] says there are the objects of mathematics, which occupy an intermediate position, differing from sensible things in being eternal and unchangeable, from Forms in that there are many alike, while the Form itself is in each case unique.[3]

Here, Aristotle claims that for Plato mathematical objects are neither physical objects nor forms, though they have some similarities with both of these two kinds of entities. Mathematical objects, like forms, are immaterial, eternal, unchangeable, and independent. However, contrary to forms, they are not unique universals; they are rather independent immaterial particulars. In this sense, they are similar to physical objects in the sensible world. Put otherwise, on the one hand mathematical theorems are not true of imperfect physical objects. As a result, mathematical objects cannot be identified with physical objects. They must, Plato believes, be some sort of perfect immaterial objects. On the other hand,

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[1] As mentioned by Annas (1975, p. 150) and Pritchard (1995, p. 33), in *Phaedo* (101 & 103-105) numbers are treated as forms.

[2] *On The Soul* (404b23-404b26). See also *Metaphysics* (1090a).

[3] *Metaphysics* (987b14-987b18). See also *Metaphysics* (1028b18-1028b21) in which Aristotle refers to Plato by name and claims that for Plato mathematical objects are distinct from both forms and physical objects.

many mathematical theorems express facts about distinct objects of the same kind. For example, the theorem ‘2+2=4’ expresses a fact about twice two. In other words, these two ‘2’s in this theorem refer to two distinct objects. Nonetheless, there is only a unique Platonic form for twoness. Therefore, these ‘2’s must be distinct from the Platonic form of twoness.[1] In the same way, when we say, for example, that through any two points there is exactly one line, we are talking about two distinct points. They cannot therefore be identified with the unique Platonic form of point. In sum, mathematical objects, though detached from matter and materiality, cannot be considered to be Platonic forms. Mathematical objects are many. By contrast, forms are one. Ergo, mathematical objects are not forms.[2] Many scholars believe that, from Aristotle’s point of view, Plato’s main reason for positing intermediates is to respond to this specific problem about mathematical objects.[3] Julia Annas refers to this problem as ‘The Uniqueness Problem’ and describes it along with its Platonic solution as follows:

> A Form has to be unique of its kind, whereas mathematical statements seem to refer to a plurality of entities, and these cannot be identified either with Forms or with physical objects. Hence intermediates are posited to be the objects of such statements.[4]

It is widely accepted that Aristotle, both in his exposition and in his criticism of Plato’s ontology of mathematics, presumes that mathematical objects for Plato are the intermediates. Admittedly, some of Aristotle’s reports can be read as saying that Plato

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[1] It is worth emphasizing that the claim that numbers, inasmuch as they are objects of arithmetic, are not Platonic forms does not imply that numbers have no Platonic forms. Conversely, having a Platonic form (i.e., being an instantiation of a Platonic form) is not the same as being a Platonic form. There are some passages, e.g. *Metaphysics* (1080a13-b16), in which Aristotle seems to be claiming that for Plato, numbers, as objects of arithmetic, are fully separate entities which have their own ideal forms, though they are not themselves forms.

[2] In the opening paragraph of the prologue of his commentary on the first book of Euclid’s *Elements*, Proclus (1970, Prologue: Part One, sec. 3, p. 3) follows the same line of argument to establish the intermediary status of mathematical objects.

[3] See, among others, Wilson (1904), Annas (1975), Bostock (2012), and Arsen (2012).

[4] Annas (1975, p. 151).

identified (at least some) mathematical objects with forms.[1] There are, however, many stronger and more explicit textual witnesses showing that Aristotle thought that Plato's mathematical objects are the intermediates.[2] If so, the following table accurately portrays Aristotle's understanding of the Platonic ontology:

forms	separate	unique	
intermediates	separate	plural	mathematical objects
sensible particulars	non-separate	plural	

Table 1

A question that might immediately be raised is whether Plato actually believed the theory of intermediates. There is no explicit discussion of this theory in the Platonic dialogues. But the identification of mathematical objects with the intermediates seems to be implicitly indicated in some passages.[3] As might be expected, there is no agreement on the valid construal of these passages. Some scholars believe that there is no evidence at all for the claim that Plato rendered mathematical objects as the intermediates.[4] Some others argue that Plato did posit the intermediates in the dialogues.[5] A third group of scholars believe that although the theory of intermediates cannot be found in the dialogues, we are justified in thinking that Plato developed this theory in his later unwritten doctrines.[6]

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[1] See, for example, *On The Soul* (404b23-404b26) and *Metaphysics* (1090a).

[2] See Annas (1975) and Bostock (2012) for these witnesses.

[3] According to Annas (1975), only the following passages 'put forward a line of argument which can be seriously treated as an attempt to establish the existence of intermediates': the notoriously famous passage of the Divided Line in *Republic* (509d-511a), *Republic* (523c-526b), and *Philebus* (56c-59d, 61d-62b).

[4] See, among others, Wilson (1904), Cherniss (1945), Lear (1982), and Moravcsik (2000).

[5] See, among others, Wedberg (1955, pp. 99–111), Adam (1963, pp. 156–163), Burnyeat (1987), and Yang (2005).

[6] See, for example, Bostock (2012). Arsen (2012) argues that given the evidence that the intermediates are necessary for Plato's philosophy of mathematics, we are *justified* in believing that he posited them, even if he has never actually done this.

Since the focus of this chapter is not Plato's philosophy of mathematics, I go no deeper into the exegetical investigations concerning Plato's actual view about the intermediates. However, regardless of whether Plato actually endorsed everything that Aristotle attributed to him, it is indisputable both that Plato accepted the triple of theses (SF), (SM) and (PM), and that Aristotle acknowledged Plato's endorsement of them. Therefore, even if Aristotle was wrong in attributing the theory of intermediates to Plato, he was right in thinking that for Plato mathematical objects are independent immaterial objects. It is particularly worth highlighting that Aristotle understood the intermediates to be independent immaterial objects. As we will shortly see, Avicenna seems to have a completely different understanding of these objects. Moreover, although his knowledge of Plato's philosophy of mathematics is apparently formed through the reports of Aristotle and his commentators, Avicenna does not seem to believe that Plato was committed to (SM) and (PM). So before discussing Avicenna's arguments against these theses, we need to investigate how he understood the previous philosophers' attitudes towards them.

1.3. Avicenna's Historiography of Philosophy (of Mathematics)

Avicenna's discussion of the theory of Platonic forms and mathematical Platonism in chapter 2 and 3 of the seventh book of *The Metaphysics of the Healing* to some extent reflects Aristotle's discussions of these views.[1] This does not mean, however, that Avicenna follows Aristotle in all details. Chapter VII.2 of *The Metaphysics of the Healing* commences with a prologue on how previous philosophers have arrived at the idea of the forms as completely separated from matter and how they dealt with the thesis mentioned in the opening section of this chapter. In a sense, he sketches the history of philosophical ideas about forms and their relations with mathematical objects. The first level of his analysis includes a general claim about the development of philosophical sciences over time:

[1] Aristotle's discussions of these views are mainly presented in *Metaphysics* (I.5-6 and I.8-9). Some other relevant discussions are scattered in *Metaphysics* (XIII and XIV).

> **TEXT # 1.6.** Every art (*ṣināʿa*) has a genesis wherein it is raw and unripe, except that after a while it matures and after some more time, it develops and is perfected. For this reason, philosophy in the early period of the Greeks' occupation with it was rhetorical. It then became mixed with error and dialectical argument. Of its divisions, it was the natural which first attracted the masses (*al-jumhūr*). Then they began to give attention to the mathematical [division], then to the metaphysical. They were involved in transitions from one part [of philosophy] to another that were not sound.[1]

Interestingly, Avicenna's understanding of the development of the philosophical sciences mirrors his understanding of the abstraction mechanism. The different stages of the maturation of the sciences are in a sense parallel to the different degrees of abstraction. In the abstraction mechanism, we start by collecting sense data from physical things through perception and we finally arrive at universal concepts fully abstracted from matter and materiality. In other words, at the beginning of this process we are dealing with the objects of the natural sciences and at the end we have the objects that ought to be studied by metaphysics. Avicenna believes that the early philosophers were attracted to studying the less abstract things (i.e., the subject matters of the natural sciences), while the later philosophers paid more attention to the more abstract things (i.e., the subject matters of metaphysical sciences). So it seems that Avicenna finds a parallel between a psychological cognitive mechanism and a historical fact. He seems to believe that since the less abstract objects are more easily apprehensible, the earlier philosophers were more quickly attracted by them than by purely abstract things. Given this picture of the history of the sciences, any transition from a lower level of maturation of the sciences to a higher one seems to correspond to a transition from a lower degree of abstraction to a higher one. The idea of the

[1] Avicenna (2005, Chapter VII.2, sec. 2). Unless otherwise specified, all translations of the passages from *The Metaphysics of the Healing* are Marmura's. Phrases in single square brackets are added by him. Italic Arabic transliterations and phrases in double square brackets are mine. I prefer to translate all occurrences of the words '*ʿaql*', '*mufāriq*', and '*maʿnā*', respectively, to 'intellect', 'separate', and 'meaning'. So, in what follows, I have changed to these English terms all other translations that Marmura has occasionally chosen for these Arabic terms. For reasons I will discuss in chapter four, in some contexts 'connotational attribute' is also a plausible translation for '*maʿnā*'.

intelligible immaterial forms that stand on their own emerged, Avicenna believes, because of a deficiency in one of these transitions:

> **TEXT # 1.7.** When they first made the transition from what is apprehended by the senses (*maḥsūs*) to what is apprehended by the intellect (*maʿqūl*), they became confused. [One] group thought that the division necessitates the existence of two things in each thing—as, for example, two humans in the meaning of humanity: a corruptible, sensible human; and an intellectually apprehended, separate (*mufāriq*), eternal, and changeless human. For each of the two they assigned an existence. They termed the separate existence "exemplary existence" (*wujūd mithālī*), and for each of the natural things they made a separate form that is intellectually apprehended, being the [very] one that the intellect receives, since the intelligible is something that does not undergo corruption, whereas every sensible of these [natural things] is corruptible. They [further] rendered the sciences and demonstrative proofs move in the direction of [the incorruptible intelligibles], these being the ones they treat. [1]

Avicenna here provides a compact presentation of the theory of exemplary forms. However, the structure of the argument he is attributing to the proponents of this idea is not clear. The most compelling reconstruction of the argument might be something on these lines: What is apprehended by the intellect is not corruptible. By contrast, what is apprehended by the senses is corruptible. Therefore, they are not identical. But these apprehensions can be of the same thing, e.g., humanity. Consequently, there must exist two things in each thing. One of them is the corruptible object of sense perception. The other is the incorruptible object of intellection. This argument, which I call (SF-1), is purported to establish the existence of the separate things (*mufāriqāt*). Although the example Avicenna gives is about a universal form (i.e., humanity), the argument itself does not determine whether or not the separate things are shareable universals. In fact, it is only in an extreme version of this view, Avicenna explains, that the separate things under discussion are considered as shareable forms:

[1] Avicenna (2005, Chapter VII.2, sec. 3).

> **TEXT # 1.8.** It was known that Plato and his teacher, Socrates, went into excess in upholding this view, saying that there belongs to humanity one (*wāḥid*) existing meaning (*ma'nā*) in which individuals participate and which continues to exist with their ceasing to exist. This [they held] is not the sensible, multiple (*al-mutakaththir*) and corruptible meaning and is therefore the intelligible separate meaning.[1]

This passage assures us that Avicenna was well aware that Socrates and Plato accepted (SF). Therefore, in this respect there is no difference between Avicenna's understanding of Plato's philosophy and that of Aristotle. The passage seems to suggest two interrelated arguments which can be proposed for (SF). The first argument can be reconstructed as follows: Since all individuals are of one meaning (i.e., humanity), that very one thing must be found in all individuals. But the sensible meaning (*al-ma'nā al-maḥsūs*) is multiple. A sensible material thing cannot be found simultaneously in many things. Therefore, the universal meaning that all individuals share (i.e., the form of humanity) is the intelligible meaning which is fully separate from matter. I call this argument (SF-2). The second argument is apparently based on the permanent existence of (at least some) forms. Humanity exists forever.[2] But nothing material can exist forever. Every human is corruptible and ceases to exist at some point in time. Therefore, humanity itself is something fully separate from matter. I call this argument (SF-3). Admittedly, if we consider this passage isolated from all other things Avicenna says in the chapters under discussion, one might find it hard to believe that the passage is pointing to these two arguments. But Avicenna's criticism of these two arguments could serve as a hint about how to demystify this passage. I will return to this issue in the next section.

[1] Avicenna (2005, Chapter VII.2, sec. 4).

[2] In the phrase 'there belongs to humanity one existing meaning in which individuals participate and which continues to exist with their ceasing to exist,' Avicenna seems to be referring to the Aristotelian doctrine of the eternity of species which he himself endorses. Avicenna believes that the successive instantiation of the species in time makes them eternal. However, the eternity of a species does not necessitate the eternal existence of any of its instantiations, or so Avicenna believes (2005, Chapter VII.2, sec. 20). For Avicenna's approach towards the particular case of the eternity of humanity and some of its philosophical consequences, see Marmura (1960).

There is another noteworthy point in the above passages. Following Aristotle, Avicenna is generally very critical of Platonic approach to philosophical problems. The conjunction of the last three passages tacitly implies that Avicenna considers the theory of Platonic forms as an *unripe* and *immature* idea.[1] He blatantly rejects this theory and offers an alternative to it.[2] Nevertheless, since Avicenna's main concern in the chapter under discussion is primarily the nature of mathematical objects, before casting doubts on the theory of Platonic forms, he illustrates his predecessors' views about mathematical objects by categorizing them into three major groups.

> **TEXT # 1.9.** Another group did not perceive a separate existence for this [[universal]] form, but only for its principles. They rendered the mathematical entities that are separate in definition (*bi-l-ḥudūd*) as deserving to be separate in existence (*bi-l-wujūd*). They made those natural forms that are not separate in definition not separate in essence (*bi-l-dhāt*). They [further] made the natural forms to be generated only through the connection of these mathematical forms with matter—as, for example, concavity. For it is a mathematical meaning: once it connects with matter, it becomes "snub-nose-ness" and becomes a natural meaning. It is then for concavity inasmuch as it is mathematical to separate [in existence], even though inasmuch as it is natural it is not for it to separate.[3]

Avicenna extends his description of these philosophers' views by clarifying that:

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[1] In the epilogue of *The Sophistics* (1958, Chapter II.6, pp. 114-115) Avicenna makes the general claim that philosophy in the time of Plato had not matured. See Gutas (2014, pp. 25–29) for a translation of this epilogue.

[2] In addition to the criticisms appearing in chapter VII.2 of *The Metaphysics of the Healing*, which I briefly discuss in the next section, Avicenna criticizes the theory of Platonic forms in other places; e.g., Avicenna (Avicenna, 1952d, Chapter VII, pp. 39-45, 2005, Chapter V.2). For two other discussions of Avicenna's criticism of the theory of Platonic forms as appearing in chapter VII.2 of *The Metaphysics of the Healing* see Marmura (2006) and Porro (2011). For Avicenna's own theory of universals see Marmura (1979, 1992) and Bahlul (2009).

[3] Avicenna (2005, Chapter VII.2, sec. 5). The example of 'snub-nose-ness' has apparently been borrowed from Aristotle's *Metaphysics* (1025b26-1026a5).

TEXT # 1.10. They rendered the principles of natural things mathematical objects, rendered them the things that are in truth intelligibles, and rendered them in truth the separate entities [in existence]. They stated that if they strip (*jarradū*) the corporeal states from matter, then only magnitudes, shapes, and numbers remain [[…]]. There results from [all] this [the notion] that everything that is not quantitative is attached to matter (*muta'allaq bi-l-mādda*). [But] the principles of what is attached to matter is [itself] not attached to matter. Hence, mathematical things [[which are not attached to matter]] become the principles and the things that are, in truth, the intelligibles, everything else not being an intelligible.[1]

These two passages show that this group of philosophers—to whom Avicenna is referring—rejects the independent immaterial existence of forms. They instead believe both that mathematical objects are independent immaterial existents and that they are the principles of natural things. In other words, they deny (SF) and accept (SM) and (PM). Although Avicenna does not mention his name, this position was seemingly advocated by Speusippus and his followers.[2] The argument for (SM) that Avicenna seems to attribute to this group of philosophers can be articulated as follows: Mathematical objects are separate from matter in definition and in mind. Therefore, they should be separate from matter in existence. On the other hand, natural things fail to be separate from matter in existence, since they are not separate from matter in definition. In other words, these philosophers seem to assume separateness in definition and in mind as both necessary and sufficient for separateness in existence. I call this argument (SM-1).

The argument these philosophers have for (PM), as elaborated by Avicenna, is grounded on (SM). They first presuppose that the principles of material things can only be things fully

[1] Avicenna (2005, Chapter VII.2, sec. 7). TEXT # 1.10 does not appear immediately after TEXT # 1.9. In between these two passages, Avicenna engages with Plato's view about mathematical objects to which I will turn below. However, as Marmura has clarified (2006, p. 357), these two passages are referring to the ideas of the same group of philosophers.

[2] See Dancy (2016) for more information about Speusippus' philosophy. For the outlines of his philosophy of mathematics see Mueller (1986).

separate from matter. Coupling this premise with (SM), according to which mathematical objects are fully separate from matter, these philosophers conclude that mathematical objects are the principles of natural things, or so Avicenna tells. As we will see in the next section, Avicenna finds this argument, which I call (PM-1), wanting. Now it is time to scrutinize his comprehension of Plato's view about mathematical objects. Avicenna says:

> **TEXT # 1.11.** As for Plato, most of his inclination was [toward the view] that it is the forms that are separate. Regarding mathematical [entities], for him these were [[intermediate]] meanings between forms and natural things. For even though they are separate in definition, it is not permissible, according to him, that there should be a dimension (*bu'd*) that does not subsist in matter. [This, he argued, is] because [the dimension] is either finite or infinite. If [it is] infinite and this is a consequence of its being simply a nature, then every dimension would be infinite. If this is a consequence of being denuded from matter, then matter becomes the thing that furnishes restriction and form. Both ways are impossible. Indeed (*bal*), the existence of an infinite dimension is impossible. If it is finite, then its restriction within a limited bound and measured shape is due only to an affection occurring to it externally, not to its very nature. [[But]] form would be affected only by its [[connection to]] matter. It would thus be [both] separate and not separate. This is impossible. Hence, [mathematical things] must be intermediate (*mutawassiṭa*).[1]

At first glance, it might seem reasonable to think that Avicenna's claim that mathematical objects for Plato 'were [[intermediate]] meanings between forms and natural things' is extracted directly from Aristotle. As we saw, TEXT # 1.5 attributes the same position to Plato.[2] However, there is a significant divergence between Aristotle's understanding of the

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[1] Avicenna (2005, Chapter VII.2, sec. 6). Marmura has translated the first occurrence of the term '*bu'd*' in this passage as 'spatial dimension'. I removed the adjective 'spatial' which seems to be superfluous. Moreover, to be in consonance with modern English scholarship of ancient Greek philosophy, I prefer to employ the term 'intermediate', rather than 'intermediary' which Marmura uses, as the translation of the Arabic term '*mutawassiṭa*'.

[2] Bertolacci (2006, p. 23) has pointed to the connection between TEXT # 1.5 (from Aristotle's *Metaphysics*) and TEXT # 1.11 (from *The Metaphysics of the Healing*).

intermediacy of mathematical objects and that of Avicenna. It was expounded in the previous section that, according to Aristotle's narrative, the intermediates are objects fully separate from matter; and this is the main resemblance between the Platonic forms and the intermediates. From Aristotle's point of view, proposing an intermediary status for mathematical objects is Plato's way out from the plight of the Uniqueness Problem. On the one hand, mathematical objects are so perfect and exact that they cannot be identified with imperfect sensible things. Therefore, they are immaterial existents. On the other hand, mathematicians often engage with more than one instance of each type of mathematical object. This implies that there must exist many (perhaps infinitely many) instances of each kind of mathematical object. Accordingly, mathematical objects cannot be identified with the Platonic forms that are unique. In this respect, they are like sensible particulars. In other words, in Aristotle's version of the theory of intermediates, these entities have both the immateriality of the Platonic forms and the plurality of material objects. By contrast, Avicenna seems to reject that mathematical objects for Plato are fully separate from matter. In the above passage, he first confirms that mathematical objects for Plato are the intermediates. So Avicenna agrees with Aristotle in this respect. But, to explain why mathematical objects should have the intermediary status, Avicenna sets forth an argument which aims to establish that although mathematical objects are separate in definition, they have no independent separate existence in the extramental world. He then concludes that 'they must be intermediate.' This shows that, contrary to Aristotle's understanding of the intermediacy of mathematical objects which preserves their separateness from matter, Avicenna's interpretation of this idea undermines the immateriality of mathematical objects. Thus, surprisingly, Avicenna does not consider Plato as believing in (SM). Consequently, the version of the theory of intermediates that Aristotle attributes to Plato clashes with that of Avicenna. The latter rejects (SM), while the former affirms it.

It is worth remarking that Avicenna never criticized the version of the theory of intermediates that he attributed to Plato. Although Avicenna refutes (SF), (SM), and (PM), he himself does not consider the rejection of these theses as a criticism of Plato's ontology of mathematics. This is because Avicenna thinks that Plato's ontology of mathematics on its own implies none of these theses. According to Avicenna, mathematical objects for Plato are not forms. Therefore, Avicenna's rejection of (SF) is irrelevant to his position on Plato's ontology of mathematics. Moreover, Avicenna's version of the theory of intermediates is compatible with the rejection of (SM). Finally, Avicenna does not mention anything about Plato's view on (PM). So we are justified in concluding that he does not see the rejection of these theses as a criticism of Plato's ontology of mathematics.

Now it is time to discuss the structure of the argument Avicenna proposes against (SM) in the above passage. This argument aims to show that geometrical dimensions (and geometrical shapes in general) have no independent immaterial existence. So it does not seem to be applicable, at least in the current formulation, to the case of numbers.[1] The argument goes as follows: Suppose that there is a dimension (or a geometrical shape in general) fully separate from matter. It must be either infinite or finite. If it is infinite, then its infiniteness is either because of the nature of being a dimension or because of its separation from matter. It cannot however be the former case. Otherwise, every dimension—which by definition has the nature of being a dimension—must be infinite. Nor can the infiniteness of that immaterial dimension be because of its separation from matter. Otherwise, what restricts things and gives them their shapes must be their matter. More precisely, the restricting element of form must be matter. We however know that it is the other way around. Form is the restricting element of matter.[2] Therefore, this path is blocked too. The immaterial dimension under discussion cannot be infinite.[3]

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[1] In fact, in his discussion of the positive aspect of the ontology of mathematics, Avicenna (1952c, Chapter I.2, pp. 13-14, 2005, Chapter I.3, sec. 17-19, pp. 18-19) argues that numbers, contrary to geometrical shapes, can indeed be separated from matter and materiality. However, if we consider them as objects fully separate from matter and materiality, they would not be receptive to increase and decrease; and, accordingly, they cannot be the subject of mathematical studies. They should therefore be studied by metaphysics. See the next chapter for Avicenna's view regarding the different ontological status of numbers and geometrical shapes.

[2] For the relation between matter and form in Avicenna's philosophy, see Hyman (1977), Lizzini (2004), Richardson (2012), and Shihadeh (2014).

[3] At the end of this part of argument, Avicenna adds that 'indeed (*bal*), the existence of an infinite dimension is impossible.' One might be inclined to read this phrase as meaning that whether or not a dimension is separate from matter, it cannot be infinite for independent reasons. However, the success and validity of almost all other

Now consider the second horn, in which the immaterial dimension is finite. Since the dimension is finite it has limits and a determined shape. But these limits and shape are not due to the very nature of being a dimension. Otherwise, all dimensions would have the same shape. Thus, the shape must be imposed by an external element. In other words, the form of this dimension is affected by an external element to receive the given shape it has. But a form can be affected only through its connection to its matter. This means that it is only because of its connection to matter that the dimension's form can be affected to receive a certain shape. Therefore, to have a certain shape, the dimension under discussion must be attached to matter. But we supposed at the beginning that it is fully separate from matter. Ergo, contradiction. The dimension cannot be finite either. This entails that there is no dimension fully separate from matter. This is because it can be neither infinite nor finite; and, as a result, it is not a dimension at all.

Reconsider the structure of the above passage. Avicenna first attributes the theory of intermediates to Plato. Then he argues that although mathematical objects are separate from matter in definition, they are not independent separate substances; they are not separate in existence. He puts forward an argument to justify this claim and concludes, at the end of the passage, that mathematical objects must be intermediates. So it seems that Avicenna's understanding of Plato's reasons for the intermediacy of mathematical objects can best be explained as follows: On the one hand, mathematical objects cannot be identified with sensible objects because, contrary to the latter, the former are separate from matter in definition. On the other hand, mathematical objects cannot be identified with ideal Platonic forms because, contrary to the latter, the former are not separate from matter in existence. So the association of mathematical objects with matter is weaker than that of sensible objects and stronger than that of Platonic forms. In this sense, they are intermediates between

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arguments which Avicenna proposes against the actual infiniteness of dimensions (e.g. the mapping argument, the ladder argument, and the collimation argument) depend on the materiality of the dimensions under discussion. These arguments are not automatically applicable to the case of immaterial dimensions. So it is not clear if there are independent reasons because of which all dimensions are finite, regardless of their being material or immaterial. See McGinnis (2010b) and the third chapter of this dissertation for Avicenna's arguments against the actuality of infinity.

sensible objects and Platonic forms. This indicates that, contrary to Aristotle who considers the theory of intermediates as a response to the Uniqueness Problem, Avicenna sees this theory as a compromise for combining the separability of mathematical objects in definition (and in mind) with their inseparability in extramental existence. He simply overlooks the Uniqueness Problem and says nothing about the similarity between mathematical objects and sensible particulars with respect to their plurality. Once again, we can see how different Avicenna's understanding of the theory of intermediates is from that of Aristotle. While Aristotle's discussion of intermediates concentrates on the notion of the *plurality* of mathematical objects, Avicenna's main concern is their *separability* (in definition and in existence).

If this account is accurate, then Avicenna's understanding of the intermediacy of mathematical objects is very similar to his own positive view about the nature of mathematical objects. In his discussions of the division of the sciences, Avicenna argues that mathematics has an intermediate status between natural sciences and metaphysics in the sense that it studies objects that lie between the two groups of objects that are studied by metaphysics and natural sciences. Objects that are separate from matter neither in definition, nor in mind, nor in the extramental world must be studied by natural sciences; and objects that are separate from matter both in mind and in extramental reality must be studied by metaphysics. Mathematical objects lie between these two categories of objects because although they are separate from matter in definition and (in a very specific sense) in mind, they do not exist extramentally as independent immaterial objects. So their association with matter is weaker than that of the former group of objects and stronger than that of the latter.[1]

[1] Avicenna has discussed the division of the sciences in several parts of his oeuvre; e.g., Ch. 2 of Bk. I of *Isagoge* (1952c), Chs. 1-3 of Bk. I of *The Metaphysics of the Healing* (2005), and Chs. 1-2 of *The Metaphysics of 'Alā'ī Encyclopedia* (1952d). See Marmura (1980) and Gutas (2003) for two modern commentaries on the Avicennan classification of the sciences. Addressing the subtleties of Avicenna's view about the intermediary status of mathematics between natural sciences and metaphysics is beyond the scope of this chapter. See the next chapter for a discussion of this issue.

Avicenna rejects the existence of ideal Platonic forms. Therefore, *a fortiori* he denies that mathematical objects are intermediate between Platonic forms and physical objects. Nonetheless, Avicenna agrees with *his* Plato that mathematical objects have an intermediary status in terms of separability from matter. It is not unexpected, therefore, that Avicenna does not criticize the theory of intermediates. It is true that, from his point of view, the immature idea of Platonic forms should be substituted with an abstractionist conception of forms. But this change does not have any immediate consequence for the status of mathematical objects with respect to separability from matter. The following table summarizes Avicenna's understanding of the Platonic ontology:

forms	separate in definition and mind	separate in existence	objects should be studied by metaphysics
intermediates	separate in definition and mind	non-separate in existence	objects should be studied by mathematics (i.e., mathematical objects)
sensible particulars	non-separate in definition and mind	non-separate in existence	objects should be studied by natural sciences

Table 2

The idea of the aforementioned tripartite division of the theoretical sciences according to the ontological status of their objects goes back to Aristotle.[1] He believes that mathematics lies between metaphysics and physics in the sense that mathematical objects have an intermediary status between the objects that are studied by the two other sciences. So both Plato and Aristotle endorse a three-level ontology of objects and put mathematical objects on the intermediate level. Nonetheless, this does not suffice to show that they both consider

[1] Aristotle proposed this tripartite division of the theoretical sciences in *Metaphysics* (bk. VI, sec. 1). Cleary (1994) has discussed various aspects of Aristotle's view regarding the classification of the theoretical sciences.

mathematical objects to have the same ontological nature. Plato's ontology, as reported by Aristotle, comprises ideal forms, intermediates, and sensible particulars. Mathematical objects for Plato are counted as intermediates in the sense that, on the one hand, they are independent immaterial entities (like ideal forms), and, on the other, they are not unique and have many instances (like sensible particulars). By contrast, Aristotle does not think that mathematical objects are independent immaterial existents. Although they are somehow *separable* from matter in mind, they cannot independently exist in the extramental world as *separate* entities. They are intermediates between the objects of metaphysics and physics in the sense that, on the one hand, they are attached to the matter in the extramental world (like the objects of physics), and, on the other hand, they can be thought of as separate entities in mind (like the objects of metaphysics). But it is undoubted that they have no independent immaterial existence. So Aristotle's tripartite ontology is not equivalent or reducible to that of Plato. Nonetheless, Avicenna seems to confuse the Aristotelian sense of the intermediateness of mathematical objects with its counterpart Platonic notion. Accordingly, what Avicenna attributes to Plato regarding the nature of mathematical objects is highly compatible with his own view, which (as we will see in the next chapter) is basically a reconstruction of the original Aristotelian position. Some ancient commentators (e.g., Iamblichus and Proclus) have tried to show that Aristotle's tripartite division of the sciences is inspired by the Platonic tripartite ontology.[1] But I do not know of any commentators who go the other way around and understand Plato's tripartite ontology through the lens of the Aristotelian division of the sciences. Yet this is exactly what Avicenna does. These observations—especially that Avicenna seems to be completely ignorant of Aristotle's discussion of the Uniqueness Problem—may reinforce the hypothesis that Avicenna's understanding of Plato's philosophy of mathematics was not formed through a first-hand reading of an even relatively reliable translation of Aristotle's works. Unfortunately,

[1] See Merlan (1975, Chapter III).

however, I could not identify any pre-Avicennan commentary on Aristotle's *Metaphysics* which suggests this interpretation of the theory of intermediates.[1]

Now we can resume our discussion of Avicenna's brief historiography of the philosophy of mathematics. The Pythagoreans are the third group of philosophers Avicenna addresses:

> **TEXT # 1.12.** Some people [however] made these [[i.e., numbers and magnitudes]] principles but did not make them separate. These are the followers of Pythagoras. They composed everything from unity and duality. [2]

This passage shows that, according to Avicenna, the Pythagoreans reject (SM) but embrace (PM).[3] So there is no disagreement between Avicenna and Aristotle in this regard. Avicenna also mentions two other groups of philosophers for whom the principles consist of 'the excessive (*al-zā'id*), the deficient (*al-nāqiṣ*), and the equal (*al-musāwī*).' While the first group, says Avicenna, makes 'the equal take the place of hyle,' the second group makes it 'take the place of form.'[4] He does not clarify whether or not the excessive, the deficient, and the equal should be counted as mathematical objects. Nor he does say anything about the separateness of them. If these things could be counted as mathematical *objects*, then these two latter groups of philosophers should be considered as branches either of the Pythagoreans or of the followers of Speusippus (depending on whether or not those three things are separate for these philosophers). If, on the other hand, these three things are not mathematical

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[1] Bertolacci (2006, pp. 23–24) has previously shown, based on other textual evidence, that Avicenna's description of the intermediate status of mathematical objects is not extracted from Ẓarīf's Arabic translation of Aristotle's *Metaphysics* I. By contrast to Avicenna's text that precisely conveys Aristotle's point that mathematical objects are intermediates between forms and particular physical objects, Ẓarīf's text, as quoted by Averroes, obscurely suggests that mathematical species are intermediates between two realities, without any further clarification about the characteristics of these realities.

[2] Avicenna (2005, Chapter VII.2, sec. 8).

[3] Aristotle attributes the same position to the Pythagoreans. See *Metaphysics* (985b23-986a13 & 1090b4-1722).

[4] Avicenna (2005, Chapter VII.2, sec. 9). Aristotle has discussed the views of this group of philosophers in *Metaphysics* (bk. XIV, sec. 1).

objects, then their views about the principles have nothing to do with mathematical objects, and Avicenna's criticism of their views has no straightforward connection to his criticism of problematic views about mathematics (to which this chapterr is devoted). So it seems reasonable to conclude that Avicenna has categorized his predecessors' views regarding the status of mathematical objects into three major groups:

(1) Plato accepts (SF) but does not believe in either (SM) or (PM).
(2) Speusippus and his followers reject (SF) but endorse (SM) and (PM).[1]
(3) The Pythagoreans do not believe in either (SF) or (SM) but they defend (PM).

After introducing these three major groups of ideas, Avicenna discusses how the defenders of (PM) 'branched out regarding the matter of compositing the whole [[of a natural thing]] from mathematical entities.'[2] He accordingly divides these groups of philosophers into subgroups. Due to constraints of space, I do not discuss the details of this subcategorization. Instead, it is now time to look at Avicenna's criticism of the arguments he attributes to the proponents of (SM) and (PM).

1.4. Avicenna's Criticism of Arguments for (SM) and (PM)

Shortly after discussing the views of the ancient philosophers regarding the triad of (SF), (SM), and (PM), Avicenna investigates 'the bases of the causes of error (*asbāb al-ghalaṭ*), in all the things wherein these people have gone astray.'[3] Relying on his notorious doctrine of the indifference of essences (*māhīyāt*), Avicenna elaborates five misunderstandings which he counts as the *causes* of these people's false belief in these theses.[4] These causes have been studied in some excellent recent works.[5] There is however one important thing which, I

[1] As I pointed out above, Avicenna does not mention Speusippus by name.

[2] Avicenna (2005, Chapter VII.2, sec. 10).

[3] Avicenna (2005, Chapter VII.2, sec. 15).

[4] Avicenna (2005, Chapter VII.2, sec. 15-22).

[5] See particularly Marmura (2006) and Porro (2011).

think, is not illustrated and emphasized explicitly enough: the link between these causes and (at least) some of the aforementioned arguments which Avicenna attributes to the proponent of these theses. Indeed, in his discussion of (at least some of) these causes, Avicenna seems to have had in mind some arguments which he previously attributed to the ancient friends of the triad. However, going through the details of Avicenna's criticism of the arguments proposed for (SF)—i.e., the arguments (SF-1), (SF-2), and (SF-3) which I discussed in the previous section—would takes us far from our main focus on Avicenna's ontology of mathematics. Accordingly, in the sequel, I confine myself to scrutinizing Avicenna's criticisms of (SM) and (PM). In this section, I will show how two of the mentioned *causes of the false beliefs* make the arguments (SM-1) and (PM-1) fallacious.[1]

1.4.1. (SM-1) is Fallacious

As I clarified in my analysis of TEXT # 1.9 and TEXT # 1.10, Avicenna seems to attribute an argument for (SM) to the advocates of this thesis. This argument, which I have called (SM-1), can be formalized as follows:

(1) Mathematical objects are separate in definition.

(2) Everything that is separate in definition, is separate in existence.

Therefore:

(3) Mathematical objects are separate in existence.

In his discussion of the first cause of the false beliefs, Avicenna explicitly refutes this argument:

> **TEXT # 1.13.** [[It is mistaken to think that]] if a thing is abstracted (*jurrida*) such that the consideration of another thing is not connected with it, then it is separate from it in existence. It is as though, if attention is paid to the thing alone—[a thing] that has

[1] Hereafter, for brevity, by a *cause* (*sabab*) *of the false beliefs* I mean a misunderstanding which causes an erroneous belief in (at least) one of the theses under discussion. This is in accordance with how Avicenna uses the Arabic term '*sabab*' in this context.

> an associate—in a manner that gives no attention to its associate, [this] would render it not adjoining its associate. In short, if it is considered without the condition (*lā bi-sharṭ*) of [its] conjunction with another, then it is believed that it is considered with the condition that there is no conjunction [with another], so that [according to this view] it was only suitable to be examined because it was not conjoined, but separate [[...]]. It is not difficult for us to direct attention through perception (*bi-l-idrāk*) or some other [[cognitive]] state to one of the two things whose character (*sha'n*) is not to separate from its companion in subsistence (*bi-l-qiwām*)—even though it separates from it in definition, meaning, and reality (*ḥaqīqa*), since its reality is not entered in the reality of the other. For conjunction necessitates connectedness [[in existence]], not permeation (*mudākhila*) in meanings [[and definitions]].[1]

At first glance, it might seem that here Avicenna is simply rejecting the second premise of the argument (SM-1) on the basis that conceivability in mind (*dhihn*) does not entail possibility in the extramental realm. Avicenna believes that mathematical objects are, in a specific sense, separate from matter in definition and mind. Nonetheless, he denies that mathematical objects can actually be separated from matter in the extramental realm. Therefore, it might be thought that Avicenna denies that conceivability in mind entails possibility in the extramental world. Consequently, although the separability of mathematical objects from matter is conceivable in mind, it is impossible for them to exist as independent separate entities in the extramental world. If so, even if the first premise of (SM-2) is true, the second premise is false and the argument is accordingly unsound. I think, however, that Avicenna here follows a more delicate line of argument which can be sketched as follows: He illustrates that the notion of separability in definition and mind can be understood in two different senses, one of which is weaker than and does not entail the other. Mathematical objects, Avicenna believes, are separate in definition and mind only in the weaker sense of this notion. This means that the first premise of (SM-1) is true only if we endorse the weaker sense of separability. By contrast, the second premise of the argument

[1] Avicenna (2005, Chapter VII.2, sec. 16-17). Marmura has translated the Arabic term '*sha'n*' as 'role'; I have replaced it with 'character'.

is true only if the stronger sense of separability is endorsed. But since the weaker sense does not imply the stronger, those two premises cannot be true at the same time. The moral is that the argument is not sound. But what are those two senses of separability? Let me explain.

To show that the notion of 'separability in definition and mind' can be interpreted in two different ways, Avicenna appeals to his famous distinction between 'X without the condition of (*lā bi-sharṭ*) Y' and 'X with the condition of not (*bi-sharṭ lā*) Y'.[1] Although the latter entails the former, the other way around does not necessarily hold, since the latter notion expresses a stronger condition. Based on this distinction, Avicenna explains in the above passage that separability in definition or mind can be construed in two different senses:

> (a) Something is separate in definition (or in mind) if and only if it can be considered (or conceived) without the condition of materiality (i.e., without the condition of being attached to matter).
> (b) Something is separate in definition (or in mind) if and only if it can be considered (or conceived) with the condition of immateriality (i.e., with the condition of being detached from matter).

Separability in definition or mind in the sense of (a)—which is, as the first sentence of the passage suggests, the very notion of *abstraction* (*tajrīd*)—follows from separability in the sense of (b), but not vice versa. Avicenna believes that mathematical objects can be considered as separate things in the former sense. Mathematical objects can be conceived in mind without considering their materiality in the extramental world; i.e., without the

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[1] This distinction is grounded in the distinction between simple (or plain) negation and negation by retraction (or privation) (*'udūl*) which Avicenna, inspired by Al-Fārābī, discusses in his logical works; e.g. his commentary on *De Interpretatione* (1970, Chapter II.1-2) and the logic parts of *The Salvation* (1985, pp. 26–29) and *The Pointers and Reminders* (1983, Chapter III.7). For a meticulous study of Al-Fārābī's view about retracted (or privative) judgments, see Thom (2008). Hodges (2012) and Kaukua (n.d.) analyze Avicenna's view on the two kinds of negation and investigate the historical background of this idea and its development from Aristotle to Al-Fārābī.

condition that they are actually attached to some physical objects. As a result, if we endorse (a) as the standard interpretation of separability in definition and mind, then the first premise of (SM-1) is true.[1] Moreover, this premise cannot be true if we understand separability in the sense of (b). The mere fact that we can disregard the association of mathematical objects with matter in the extramental world does not guarantee that we can conceive them as immaterial objects, independently existing in that realm. Avicenna denies the separability of mathematical objects in the sense of (b). This shows that the premises of (SM-1) are mutually inconsistent, and this is because the second premise of (SM-1) can be true only if we understand separability in the sense of (b). If it is conceivable that X is immaterial (i.e., if it is possible to consider X with the condition that it is immaterial), then—by assuming that possibility can be driven from conceivability—we can conclude that it is possible for X to be an independent immaterial existent in the extramental world. But the mere conceivability of X without the condition that it is material is too weak a piece of evidence to establish the possibility of X's existence as an independent immaterial thing in the extramental world, or so Avicenna seems to think. This reveals that, from Avicenna's point of view, the two premises of (SM-1) cannot be true at the same time. The second premise can be true only if separability is understood in the sense of (b). Quite to the contrary, the first premise is true in the sense of (a) but false in the sense of (b). So Avicenna's

[1] It is worth noting that Avicenna has a very specific and subtle understanding of the separability of mathematical objects in the sense of (a). Although he agrees that mathematical objects can be separated, in the sense of (a), from all specific kinds of matters to which they are actually attached in the extramental word, he denies that mathematical objects can be separated, even in the sense of (a), from the materiality itself. Indeed he distinguishes separation from all specific kinds of matter (e.g., gold, wood, etc.) from separation from materiality itself. In his discussion of the fifth cause of the false beliefs, he explicitly mentions that the definitions of 'mathematical [objects] absolutely do not dispense with matter, even though they can do without some kind of matter' (2005, Chapter VII.2, sec. 21). In the next chapter I will discuss in detail this distinction and its consequences for Avicenna's philosophy of mathematics.

objection to (SM-1) is not based on simply denying that possibility can be drawn from conceivability.[1] Now we can go on to examine Avicenna's criticism of (PM-1).

1.4.2. (PM-1) is Fallacious

The arguments that Avicenna attributes, in the last lines of TEXT # 1.10, to the advocates of (PM) can be formalized as follows:

(1) Mathematical objects are separate. In other words, (SM) is true.

(2) The principles of the natural things (which are attached to matter) cannot be themselves attached to matter. In other words, the principles of the natural things are separate.

Therefore:

(3) Mathematical objects are the principles of the natural things.

This argument is unsound; but not merely because of its reliance on (SM) which for Avicenna is implausible. In his discussion of the fifth cause of the false beliefs, Avicenna says:

> **TEXT # 1.14.** [[It is mistaken to think that]] if material things are caused, then their causes are necessarily any of the things than can separate. For it is not the case that if material things are caused and mathematical things are separate, then it follows necessarily that the mathematical things are their causes. Rather, [their causes] may be other substances that are not among the nine categories.[2]

[1] To be precise, I do not claim that here Avicenna has endorsed that conceivability entails possibility. Rather, I am arguing that Avicenna's criticism of (SM-1) is more sophisticated than simply being grounded in the claim that separability in definition and mind does not entail separability in the extramental world. For a recent study on Avicenna's view about the relation between conceivability and possibility see Kukkonen (2014).

[2] Avicenna (2005, Chapter VII.2, sec. 21).

Here Avicenna is arguing that from the claim that the cause of a material thing should be separate it does not follow that *every* separate thing can be the cause of that natural thing. Therefore, even if we accept (SM) and affirm that mathematical objects are actually separate from matter, we still need some further justification to arrive at the conclusion that mathematical objects, rather than other separate things, are the causes of the natural things. Avicenna seems to believe that the friends of (PM) do not provide such justification, and it is exactly for this reason that they cannot establish (PM) on the basis of (SM). Put otherwise, even if both of the premises of (PM-1) are true, they are not sufficient to imply the conclusion of the argument. Ergo, (PM-1) is invalid.[1]

1.5. Avicenna's Argument against (SM)

In the previous section, I investigated Avicenna's criticism of the arguments for (SM) and (PM). In what follows, I focus on his own arguments against these theses as they are portrayed in the beginning of the third chapter of book VII of *The Metaphysics of the Healing*. I start, in this section, by analyzing Avicenna's argument against (SM); then I turn in the following section to his argument against (PM). His argument against (SM) is presented in three steps. In the first step, Avicenna argues that, whatever our view of the existence of separate immaterial mathematical objects, we should concede that there are sensible mathematical objects. In the second step, he shows that even if there are immaterial mathematical objects fully separated from matter, they must have the same nature as sensible mathematical objects. If so, the essence and definition of mathematical objects should be indifferent with respect to materiality, or so Avicenna argues in the third step. This conclusion, Avicenna believes, contradicts (SM) according to which mathematical objects are

[1] As I said, since I have restricted my focus to philosophy of mathematics, I do not discuss the link between the other causes of the false beliefs and the arguments Avicenna has attributed to the proponents of (SF) in the chapter under discussion. But just for the record, the second and fourth causes put forward objections against the arguments (SF-2) and (SF-3) respectively. See Avicenna (2005, Chapter VII.2, sec. 18 & 20).

independent immaterial substances. Now, let me go through the details of these three steps of the argument. Avicenna says:

> **TEXT # 1.15.** We say: If among mathematical things there is a mathematical object separate from the sensible mathematical object (*al-taʿlīmī al-maḥsūs*) at all, then in the sensible thing either there would be no mathematical object or there would be [a mathematical object]. If in the sensible thing there is no mathematical object, then it necessarily follows that there is no quadrilateral, circular, or numbered (*maʿdūd*) sensible thing. If none of [these things] is sensible, then what way is there to establish their existence [or], indeed, [even] to imagine them? For the principle of their being imagined is likewise [derived] from sensible existence—so much so that, if we suppose, through our estimative faculty, an individual who has apprehended none of [these] by the senses, we will judge that he does not imagine, nor, indeed, intellectually apprehend any of them. However, we have established the existence of many of them in what is sensible.[1]

Here Avicenna tries to establish the existence of sensible mathematical objects which are attached to matter.[2] To justify this claim, he offers two parallel reasons, one ontological and the other epistemological. On the ontological side, he says that sensible mathematical objects exist for the simple reason that quadrilateral, circular, and numbered sensible things exist. One might leap to the view that he is begging the question here; but he is not, or so it seems to me. Mathematical objects for Avicenna are certain specific properties of physical objects existing in the sensible world. Therefore, the existence of mathematical objects in the sensible world can be concluded from the existence of those specific properties in the

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[1] Avicenna (2005, Chapter VII.3, sec. 1). Marmura has translated '*maʿdūd*' as 'enumerable'.

[2] In the last chapter I will clarify that what Avicenna means by the existence of sensible mathematical objects is indeed the existence of sensible objects in which mathematical objects exist as non-sensible connotational attributes. According to Avicenna, mathematical objects are not themselves sensible. So it might be better to avoid using the misleading phrase 'sensible mathematical object'. Nonetheless, the distinction between mathematical objects that are themselves sensible and non-sensible mathematical objects that exist in sensible objects does not play any crucial role in Avicenna's argument in this section. Thus using that phrase seems to be unproblematic; and I do so for the sake of brevity.

sensible world. Now, Avicenna claims that the existence of these properties in the sensible world is implied by the existence of sensible things which have those properties. For example, the existence of the property of circularity—which is an object of mathematical study—in the sensible world is implied by the existence of sensible circular things. This position will be better understood if we consider its differences from, for example, the nominalist view. Although a nominalist confirms that there are some sensible circular things, she denies that the property of circularity itself exists.

A more important feature of Avicenna's argument in the above passage is the epistemological evidence he provides for the existence of mathematical objects in the sensible world. From the fact that we can imagine mathematical objects and that we can intellectually apprehend them, Avicenna concludes that there must exist sensible mathematical objects. He thinks that the imaginability and, more generally, apprehensibility of mathematical objects is grounded in their sensible existence which can be perceived through our senses. To justify this claim he puts forward a thought experiment. Consider a person who has never had any perceptual experiences of sensible mathematical objects.[1] Such an individual could probably never exist, but if she does, then it seems intuitively obvious that she would never have an imagination or intellectual apprehension of mathematical objects, or so Avicenna believes.[2] The lack of sense perception of the mathematical objects existing in the sensible world will

[1] What enables us to consider such a person in this weird situation is, as the passage shows, the faculty of estimation (*wahm*). See Black (1993) and Hall (2006) for the different crucial roles this cognitive faculty plays in Avicennan epistemology. The functions and applications of thought experiments in Avicenna's philosophy have been studied by Kukkonen (2014).

[2] Interestingly, this thought experiment can be considered as a brief version of the Flying Man argument restricted to the context of mathematics. In a sense, Avicenna is here asking us (1) to conceive someone who has no causal interaction with sensible mathematical objects and (2) to think whether or not this individual has any knowledge of mathematical objects. This hypothetical individual is intended to be less restricted than the original Flying Man; and compared to the original version of the Flying Man thought experiment, the thought experiment of the above passage is designed to arrive at a more restricted conclusion which is exclusively about mathematics. For studies on the Flying Man argument see, among others, Marmura (1986) and Alwishah (2013).

result in the lack of imagination and apprehension of mathematical objects. Thus here Avicenna is apparently embracing a version of concept empiricism about mathematics.[1] He believes that grasping mathematical concepts (e.g., the concept of circle) is impossible unless we have some perceptual experiences of sensible mathematical objects (e.g., sensible circularity). But, no doubt, we all have some mathematical concepts, meaning not only that there are some sensible mathematical objects, but also that we have already perceived (at least some of) them.[2] So far, then, the existence of sensible mathematical objects is established. But this does not, on its own, refute (SM). There might exist two different kinds of mathematical object, one of a material essence and the other essentially separate from matter. Avicenna takes a further step and argues that this cannot be the case:

> **TEXT # 1.16.** If [on the other hand] the nature of mathematical objects may also exist in sensible things, then there would be a consideration (*iʿtibār*) of that nature (*ṭabīʿa*) in itself. Its essence, then, either would correspond in definition and meaning with the separate [mathematical entity] or would differ from it. If it were different from it, then the intelligible mathematical objects would be things other than those we imagine and intellectually apprehend. We would [then] require a new proof to establish the existence [of the former] and, after this, [to] engage in examining the state of their separateness. Thus, what they have done in rendering [mathematical objects] eternal so as to dispute with establishing their existence, and in preoccupying themselves

[1] This does not mean, however, that Avicenna is committed to empiricism in all other contexts. Nor does it mean that Avicenna is empiricist about mathematical judgments and propositions. This merely shows that, according to Avicenna, having mathematical concepts without having the relevant perceptual experiences of some sensible mathematical objects is impossible. Some scholars, e.g. Gutas (2012), believe that Avicenna's general theory of knowledge is strongly committed to an inclusive version of empiricism about all instances of conceptual and propositional knowledge in all contexts. I will address some aspects of this interesting view in the last chapter.

[2] Mathematical concepts for Avicenna are formed based on what we obtain from our perceptual experiences and through the abstraction mechanism. See the last chapter for the details of this process.

with giving priority to the task of showing their separateness, represents an unreliable course of action.[1]

In TEXT # 1.15 Avicenna proves that there are sensible mathematical objects and that our concepts of mathematical objects are formed based on the perceptual experiences we have of those sensible mathematical objects. Now we are dealing with the question of whether or not sensible and separate mathematical objects have the same nature (the same definition or the same meaning). If they have different natures, then we have no way, Avicenna claims, to know anything about the separate mathematical objects. On the one hand, as the thought experiment in the previous passage shows, the only way we can imagine or intellectually apprehend mathematical objects is by having some relevant sense perceptions. On the other hand, the objects of these sense perceptions are sensible mathematical objects. So we have no knowledge of other mathematical objects (if any) whose natures are different from the nature of sensible mathematical objects. Therefore, even if we accept the existence of separate mathematical objects, they must have the same nature as their sensible counterparts. Otherwise, we can by no means establish the existence of such entities.[2] Engaging in any fruitful discussion on the separability of these entities is *a fortiori* impossible.

Here Avicenna is putting forward a very compelling epistemological objection to (SM) which has no parallel in Aristotle's corpus.[3] This objection is still considered, in the contemporary

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[1] Avicenna (2005, Chapter VII.3, sec. 2).

[2] To be strictly precise, Avicenna does not explicitly say that we have no way to establish the existence of separate mathematical entities. He merely says that if we believe that the nature of separate mathematical objects differs from that of sensible mathematical objects, then we need some new proofs for establishing the existence of separate mathematical entities. Nonetheless, since he does not discuss any alternative proof (while he usually considers many alternative maneuvers on behalf of his opponents), it seems plausible to understand the passage as claiming that if separate and sensible mathematical objects have different natures, then we have no way of knowing that the separate mathematical entities exist.

[3] To the best of my knowledge, the only epistemological complaint Aristotle expresses against the ideal Platonic forms is that they are epistemologically dispensable, since they make no contribution to our knowledge of physical objects. See *Metaphysics* (991a9-991a31).

philosophy of mathematics, as one of the most powerful challenges to mathematical Platonism.[1] If we believe that mathematical objects are non-sensible entities which exist only in a fully immaterial realm, then there seems to be no way to know about the existence and properties of these objects. If the advocates of (SM) insist that the nature (or essence) of separate mathematical objects is totally different from that of sensible mathematical objects, then our cognitive link between the sensible world and the realm in which those separate mathematical entities exist will be cut off.[2] Thus, a necessary condition for defending the existence of separate mathematical objects (and, accordingly, (SM)) is accepting that separate and sensible mathematical objects have identical natures (essences and definitions). However, this latter view, Avicenna says, has some consequences which diverge from the original position of the defenders of (SM). So the third step of the argument against (SM) goes as follows:

> **TEXT # 1.17.** If [the nature of the mathematical object in sensible things] corresponds with and shares [the] definition of [the separate mathematical object], then either those [mathematical objects] in sensible things have come to be by reason of their nature alone—but, then, how would that which has its definition become separate?—or else this is something that occurs to them through some cause (they being exposed to this, their definitions not preventing this [from] occurring to them). Thus, it would be the prerogative of these material things to become separate. But this is contrary to what they hold and [to that] upon which they built the basis of their view.[3]

If separate and sensible mathematical objects have the same nature and definition, then what does cause the latter group of objects to be material and sensible? They are not so just because of their nature and definition. They have the same nature and definition as separate mathematical objects. But materiality and sensibility cannot be just because of a nature

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[1] This objection has come to the attention of contemporary philosophers of mathematics via Benacerraf's seminal paper (1973).

[2] Again, it is incautious to generalize hastily this view to contexts other than mathematics.

[3] Avicenna (2005, Chapter VII.3, sec. 3).

which is shared with some non-sensible separate things. Therefore, sensible mathematical objects are not sensible by nature. In the same way, the separateness and immateriality of separate mathematical objects (if they exist) cannot be merely due to their nature and definition which is shared with some sensible and material entities. Coupling these two facts, we should conclude that the nature and definition of mathematical objects is itself indifferent with respect to materiality and separability. This is the effect of an external cause (i.e., not their nature on its own) which makes mathematical objects separate or sensible. So sensible and separate mathematical objects can somehow transform into each other. Put otherwise, neither materiality nor immateriality is an essential element of being a mathematical object. But this contradicts (SM) which counts immateriality as an inevitable feature of mathematical objects. Ergo, (SM) is false.

To see better how the above complex argument works against (SM) we can summarize it as follows: (1) There are some sensible mathematical objects. (2) If there are separate mathematical objects, they should have the same nature and definition as their sensible counterparts. Therefore, (3) there is no mathematical object that is separate and immaterial by nature. So (SM) is false. But one might be inclined to endorse a weaker interpretation of (SM) according to which although immateriality is not an essential element of being a mathematical object and we have both separate and sensible mathematical objects, separate mathematical objects are prior to and more fundamental than the sensible ones in the sense that the former are the principles for the latter. Avicenna's argument against (PM), which I discuss in the next section, rejects this possibility. So we can safely conclude that (SM) is false.

Before closing this section, I would like to highlight a striking point about Avicenna's argument against (SM). As we saw in section two, the core idea behind the Platonic Argument for (SM) is that mathematical propositions cannot be true of sensible objects in the material world because they are not perfect and exact. But since (at least some) mathematical propositions are true, then they should be true of some perfect immaterial entities. By contrast, Avicenna does not touch on the issue of (im)perfectness and (in)exactness in his discussion of sensible and separate mathematical objects. He says that these two groups of objects are of the same nature and definition, without mentioning anything about their differences in terms of perfectness. This is because he believes that sensible mathematical objects can, at least in principle, be perfect and exact. That is why he thinks that separate mathematical entities and their sensible counterparts are of the same definition (*ḥadd*). An imperfect mathematical object cannot have the same definition as a perfect one. Their definitions can at best be very close to each other, but not exactly the same. So if someone, like Avicenna, believes that sensible and separate mathematical objects have exactly the same nature and definition, she should affirm that there are (or, at least can be) perfect mathematical objects in the sensible world. Coupling this with the other evidence we have for Avicenna's belief in the existence of perfect mathematical objects in the sensible world,[1] it is plausible to think that Avicenna's reaction to the Platonic Argument for (SM)—if he was actually dealing with that—would be to reject the argument by refuting its second premise. Mathematical theorems for Avicenna can, in principle, be true of sensible objects. This shows that in the context of Avicenna's philosophy we have a resource for replying to the Platonic Argument for (SM) which we lack, as many scholars believe, in the context of Aristotle's philosophy.[2]

1.6. Avicenna's Argument against (PM)

It is now time to investigate Avicenna's argument against (PM). As I showed in sections two and three, (PM) is independent from (SM). One might endorse (PM) without being committed to (SM). Avicenna refers to a group of philosophers (probably Speusippus and his followers) who defend both (SM) and (PM), and to the Pythagoreans who reject the former

[1] I will later return to the problem of the perfectness of mathematical objects in secions 2.5 and 4.4.

[2] David Bostock (2012) expresses this shortcoming of Aristotle's philosophy in this way: 'As I see it, Plato's basic argument is very simple: mathematics concerns objects that are perfect or ideal in a way that no perceptible object is; but the statements of mathematics are true; therefore such objects must exist. What is wrong with this reasoning? Aristotle *never* explicitly addresses this question, though surely he should have done.' Emphasis in original.

but endorse the latter. Therefore, Speusippus and his followers who believe in the separateness of mathematical entities would understand (PM) as follows:

> (PM*) Separate mathematical objects are the principles of natural things.

Plato—the original one, not Avicenna's Plato—had endorsed both (SM) and (PM), and had the same understanding of the principleness of mathematical objects. By contrast, the Pythagoreans, who believed that mathematical objects are not separate entities, had a different rendition of (PM) in mind:

> (PM**) Material (or sensible) mathematical objects are the principles of natural things.

Although Avicenna rejects both of these interpretations of (PM), I dedicate my discussion to his argument against (PM*), which is the Platonic version of (PM) and directly relevant to the focus of the chapter.[1] Consider a separate mathematical object. If there is any natural thing for which the separate mathematical object is a principle, it should be, in the first place, its sensible counterpart. If a separate mathematical is not a principle of its sensible counterpart, then it is untenable to think that this separate entity can be a principle of any other natural thing. Avicenna rejects (PM*) by showing that a separate mathematical object cannot be a principle even for its own sensible counterpart. The argument goes as follows:

> **TEXT # 1.18.** This matter [[of the sensible object]], which is conjoined with accidents (*ʿawāriḍ*), either will need the separate [entities] or will not. If it needs separate [entities], it would then only need the separate [things] other than itself [[because of]] their natures, and hence the separate [things, in turn,] would need other [[separate things and we have an infinite regress]]. If it needed the separate [things] only because of what has accidentally occurred to it—so that, if it were not for that occurrence, it would not at all have needed the separate [things] and there would be

[1] The argument (PM-1) which Avicenna attributes to the advocates of (PM) is in fact an argument for (PM*), rather than (PM**).

> no necessity at all for the existence of the separate things—then an accidental occurrence to the thing would necessitate of the existence something prior to it and which has no need of it. It will [also] render the separate things in need of [matter] in order that they would necessarily have existence. [1]

If separate mathematical entities are the principles of sensible mathematical entities then it is legitimate to ask why the latter need the former. Consider a separate mathematical object (e.g., a separate triangle) and its sensible counterpart (e.g., a sensible triangle[2]). It is either because of the essence and nature of the sensible object that it needs the separate object, or because of some of its accidents. But Avicenna has already shown that the separate and sensible mathematical objects have the same nature. Therefore, if it is because of the essence of the sensible object that it needs the separate object, then the separate object (which is of the same nature as the sensible object) itself needs another separate thing, and this regress goes on infinitely.[3] If, on the other hand, the sensible object needs the separate object because of one of its accidents, then we are in a vicious circle. If that accident did not exist, the sensible object would not need the separate object. Therefore, the existence of the separate thing in some sense depends on the existence of the accident.[4] So the accident is dependent on the sensible object, the sensible object is dependent on the separate object, and the separate object is dependent on the accident. Contradiction. Therefore, the sensible object needs the separate object neither because of its essence nor because of its accidents. So, apparently, there is nothing because of which the sensible mathematical entities need the separate entities. Moreover, if we accept that, for whatever reason, the sensible object with

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[1] Avicenna (2005, Chapter VII.3, sec. 4).

[2] Since mathematical objects for Avicenna are the *properties* of physical objects, it is more accurate to say *triangularity* instead of *triangle*. However, for the sake of simplicity I stick to the usual terminology we use in discussing mathematical objects.

[3] In *Metaphysics* (1076b39-1702) Aristotle proposes a similar objection based on the absurdity of the aforementioned infinite regress.

[4] Avicenna seems to assume that if the sensible object does not need the separate object, then there is no need for the existence of the separate object at all. See the discussion of the next passage.

all of its accidents needs and is caused by the separate object, then a more controversial issue arises:

> **TEXT # 1.19.** If … the existence of separate [things] necessitates the existence [of matter] with this accident, then why is it that the accident is necessitated in [what is] other than [the separate things and] not in themselves, when the nature [of the separate and non-separate thing] agree? If [[the sensible objects]] do not need the separate [things], the separate [things] would not be in any manner whatsoever their causes, nor first principles. It follows necessarily that these separate [things] would be deficient (*nāqiṣ*). For some actions and powers—[things] that do not exist in separate [things]—have attached to this thing that is connected with matter [i.e., the sensible mathematical object].[1]

The separate and sensible mathematical objects have the same nature. So if the former causes the latter to be material and to have some specific accidents, it should do the same thing to itself. If X and Y have the same essence and X causes the material existence and accidents of Y, then it is reasonable to think that X must do the same thing to itself and therefore it must be itself material. Therefore, there seems to be no reasonable way through which we can justify the existence of two kinds of mathematical entities of the same nature such that one of them needs and is caused by the other. But if the separate mathematical objects are not the principles of their sensible counterparts, why do we need them at all? It seems that in the sensible objects we have everything which can be found in the separate objects, in addition to some other things (e.g., power) which the separate objects lack. Thus, compared to the sensible mathematical objects, the separate mathematical objects are deficient. If so, why should we assume that such useless objects exist? We do not have any convincing argument for the existence of separate mathematical objects. Nor do they play

[1] Avicenna (2005, Chapter VII.3, sec. 5). I have substantively revised the translation of the last sentence of the passage.

any irreplaceable explanatory role in our picture of the ontology of mathematics. So we can safely conclude that they simply do not exist.[1]

1.7. Conclusion

I have argued that Avicenna's description of Plato's philosophy of mathematics differs from both Plato's original view and the view that Aristotle attributes to him. Both the original Plato and Aristotle's Plato believe in (SF), (SM), and (PM), but Avicenna's Plato believes only in (SF) and does not endorse (SM) and (PM). Like Aristotle, Avicenna believes that mathematical objects for Plato are intermediates between immaterial Platonic forms and physical objects. However, contrary to Aristotle, Avicenna understands the intermediacy of mathematical objects in a sense which does not imply their immateriality in the extramental world. Indeed, his rendition of intermediacy entails a naïve version of his own view about the ontological status of mathematical objects as presented in his discussions of the division of the sciences. This shows that Avicenna himself does not consider his criticism of (SM) and (PM) as attacks on Plato.

As we saw, Avicenna criticizes some arguments he attributes to the defenders of (SM) and (PM). Moreover he puts forward two subtle and complex arguments against these two theses. I close this chapter by highlighting three brilliant ideas which he proposes in these arguments. First, he establishes the existence of sensible mathematical objects, without endorsing the Platonist idea that sensible objects cannot be as perfect as the objects of mathematics. Second, he disputes the existence of separate mathematical entities whose nature is totally different from what we can see in the sensible world. By endorsing a version of concept empiricism in the context of mathematics, he argues that if separate mathematical objects are so special that no sensible thing has the same nature as them, then we have no

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[1] The arguments offered in the last two passage can easily be reconstructed as ones against the principleness and priority of Platonic Forms over their sensible instances. So they can be understood as having a more general aim.

epistemic access to them and no knowledge of either their properties or even their existence. Third, he shows that we do not really need separate mathematical objects, because there is nothing in them that we cannot find in their sensible counterparts. Belief in separate mathematical objects is simply a redundant assumption. What more do we need to be convinced that we should give up mathematical Platonism? Now it is time to dicuss the positive aspect of Avicenna's ontology of mathematics which clarifies what mathematical objects *are*.

2. On the Nature of Mathematical Objects

Some scholars have proposed that Avicenna considers mathematical objects, i.e., geometric shapes and numbers, to be mental existents completely separated from matter. In this chapter, I will show that this description, though not completely wrong, is misleading. Avicenna endorses, I will argue, some sort of literalism, potentialism, *and* finitism*.*

2.1. Introduction

In this chapter, I will confine myself to the positive aspect of Avicenna's ontology of mathematics, and will try to shed new light on Avicenna's views concerning the nature of mathematical objects (e.g., numbers and geometric shapes).[1]

What are mathematical objects? Do they exist at all? If yes, where? What is their nature? These are some of the questions to which I will try to identify Avicenna's answer. Some authors, without addressing the details of his views on the philosophy of mathematics, have hastily concluded that Avicenna's position regarding the nature of mathematical objects is simply Aristotelian.[2] These authors have overlooked two facts: on the one hand, in the absence of a careful inspection of Avicenna's writings on the ontology of mathematics, it is perilous to ascribe a full-blown Aristotelian position to him. There are many topics on which Avicenna's views differed, either in part or in full, from those of Aristotle. Therefore, only a detailed textual analysis can reveal whether or not the ontology of mathematics is one of those topics. On the other hand, there is a wide range of different, even mutually inconsistent, positions ascribed to Aristotle concerning the ontology of mathematics.[3] These positions vary from a *fictionalist* one (according to which mathematical objects do not exist in any sense) to a *literalist* position (according to which such objects do literally exist in the material world).[4] Merely stating that Avicenna is Aristotelian does not help us to situate Avicenna's philosophy

[1] For the sake of simplicity, I do not tackle his views concerning the arithmetic of other rational or irrational numbers. These problems are briefly discussed by Rashed (1984, 2008, sec. 2).

[2] See Al-Daffa and Stroyls (1984, p. 90) and McGinnis (2006, p. 68).

[3] For a classic work on Aristotle's philosophy of mathematics, see Apostle (1952). For a recent work on this topic, see Bostock (2012). Franklin (2014) defends a modern reconstruction of an Aristotelian philosophy of mathematics.

[4] Lear (1982) and Hussey (1991) attribute variations of fictionalism to Aristotle. Mueller (1970, 1990) defends a literalist interpretation of Aristotle. The strengths and weaknesses of these interpretations have been discussed by Corkum (2012). White (1993) discusses a spectrum of miscellaneous interpretations of the nature and location of mathematical objects in the framework of Aristotle's philosophy.

of mathematics in relation to this diverse set of views on the ontology of mathematical objects. More substantial clarification is in order.

There is a growing tendency in the scholarship on Avicenna to defend an interpretation according to which he believes that mathematical objects are mental existents. Jon McGinnis, Mohammad Ardeshir, Allan Bäck, and Hassan Tahiri uphold this interpretation.[1] They believe that "Avicenna's ontology implies that mathematical objects are mental objects"[2] and that he sees these "objects as mental constructs abstracted from concrete physical objects."[3] Given this understanding of Avicenna, mathematical objects are mental entities purely abstracted and separated from matter. Although they are not abstract Platonist entities with extramental independent (or autonomous) existence, they are mental constructions and intentional objects entirely separated from matter.[4] Some of the proponents of this position

[1] See, respectively, McGinnis (2006), Ardeshir (2008), Bäck (2013), and Tahiri (2016). While McGinnis believes that Avicenna is fully Aristotelian concerning the nature of mathematical objects, Bäck and Tahiri distinguish Avicenna's view from Aristotle's. It seems that Bäck, like Hussey (1991), considers interpreting Aristotle in a fictionalist framework to be tendentious (Bäck, 2013, p. 100), but Tahiri attributes a *potentialist* position to Aristotle (Tahiri, 2016, sec. 3.3) according to which mathematical objects (at least numbers) only potentially exist. Fictionalism and potentiallism are two distinct, though not necessarily incompatible, positions.

[2] Ardeshir (2008, p. 43).

[3] McGinnis (2006, p. 68).

[4] I borrow the phrase 'intentional objects' from Tahiri's (2016) preferred terminology. According to his understanding of Avicenna, mathematical objects, and particularly numbers, "are intentional objects, the product of a specific intentional act that makes it possible to generate objects beyond the sensible experience such as infinite numbers" (2016, p. 41). Tahiri believes that *intentionality* is the most substantial notion in Avicenna's metaphysics: "If there is one word that can sum up Ibn Sīnā's al-Ilāhiyāt, it is without doubt *intentionality*" (2016, p. 69). Tahiri's understanding of intentionality seems very similar to Crane's (2001, 2013) view, according to which all mental phenomena are intentional. I seriously doubt the reliability of such an interpretation of Avicenna. In particular, I think that Tahiri overestimates the significance of the notion of intentionality (in the sense mentioned) in interpreting Avicenna. However, I avoid further discussion on this issue here. See Banchetti-Robino's (2004) and Black (2010) for Avicenna's treatment of intention and intentionality.

have no hesitation in interpreting Avicenna's philosophy of mathematics as a *constructivist* or *intuitionist* philosophy, in the modern senses of these notions.[1]

There is an underrepresented view, on the other hand, which says that mathematical objects "are always [i.e., even in our minds] mixed with matter, but not, however, with a specific kind of matter [...]. As objects of mathematical knowledge, they undergo a degree of abstraction whereby the mathematician will consider their properties dissociated from any specific kind of material, but not, however, from any matter whatsoever."[2] Both of these two kinds of interpretations are *to some extent* true. But I will show that, as interpretations of Avicenna, they are imprecise.

In the following section, I will draw a general sketch of Avicenna's views on the nature of mathematical objects. I will show that in his philosophical system geometric shapes and numbers are accidents of material substances existing in the physical world. They are associated with specific kinds of matter in the extramental world but, in our minds, they can be separated from matter to different degrees. In sections 2.3 and 2.4, I will clarify that geometric shapes and numbers differ with respect to the mode and degree of their separability from matter.[3] Although both are separable from specific kinds of matter in our minds, geometric shapes, contrary to numbers, are inseparable from materiality itself. Geometric shapes have some sort of *ontological admixture* with material forms that is

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[1] See McGinnis (2006, p. 64) and Tahiri (2016, sec. 5.2.1). While they emphasize the affinities between Avicenna's *ontology* of mathematical objects and the modern constructivist/intuitionist ontology of mathematics, Ardeshir (2008, pp. 57–58) highlights similarities between Avicenna's *epistemology* of mathematics and the modern intuitionist epistemology of mathematics.

[2] See Marmura's introduction to his translation of *The Metaphysics of the Healing* (2005, p. xix). Marmura (1980) defends the same position.

[3] None of the aforementioned studies on Avicenna's philosophy of mathematics has investigated distinctions between the nature of geometric objects (i.e., geometric shapes) and the nature of arithmetical objects (i.e., numbers). Although Ardeshir (2008) discusses some general points about the subject matter of geometry (sec. 2.1), his main discussion on the ontology of mathematical objects is focused on the nature of numbers (sec. 2.2). Tahiri (2016) confines himself even more to the nature of numbers. Nonetheless, in a few footnotes, he briefly discusses the views of Al-Fārābī (p. 20, n. 20), Avicenna (p. 33, n. 17), and Averroes (p. 54, n. 6) concerning the distinctions between numbers and geometric shapes. I will return to his note on Avicenna later in this chapter.

retained, even in the mind. Numbers, on the other hand, can be separated from materiality and all material forms in the mind. But, inasmuch as they are the subject of mathematical studies, they should still be *considered* as receptive of the accidents they (i.e., numbers) may have only when they are in numbered material things. Numbers, therefore, have some sort of *epistemological admixture* with materiality. In section 2.5, I will show that Avicenna endorses the existence of *perfect* mathematical objects in the external world. I will argue that there is no serious obstacle preventing us from attributing a full-blown literalism to him. Independently of the accuracy of such an attribution, the number of mathematical objects that do *actually* exist, in either the extramental or the mental realm, is finite, or so I will argue. There are an infinite number of mathematical objects that only *potentially* exist. So, the attribution of some sort of *finitism* and *potentialism* to Avicenna is unavoidable. In the last section, I conclude by discussing the main points on which I diverge from the mainstream understandings of Avicenna's philosophy of mathematics.

2.2. Mathematical Objects: A General Picture

Do mathematical objects exist? One may consider this question to be a paraphrased form of a more specific question: *Are mathematical objects mind-independent substances?* Nonetheless, from the perspective of Avicenna's philosophy, we should distinguish these two questions. His answer to the former question, but not the latter, is *trivially* positive. In Avicenna's philosophy, *existence* (*wujūd*) and *thingness*/*objecthood* (*shay'īya*) are distinct but coextensive concepts.[1] This justifies the interchangeable use of 'existent' and 'object' in the context of Avicenna's philosophy.[2] It also entails that not only mathematics but all

[1] See, for example, Avicenna (2005, sec. I.5). For a comprehensive study on Avicenna's treatment of the notion of *shay'iyya*, see Wisnovsky (2000).

[2] However, this view raises some controversial problems. For example, quiddity (*māhīya*) as quiddity is something, so it should have some sort of existence. But this result seems in tension with one of Avicenna's famous doctrines, according to which quiddity is neutral relative to existence. The solution lies in the fact that existence can be qualified in different modes. See Marmura (1979, 1992), Black (1999), and Bertolacci (2012) for more discussions on this issue.

sciences are pre-scientifically committed to the existence of their subject matters, i.e., their objects. Every science, inasmuch as it is a science, studies some things or objects. Moreover, thingness/objecthood and existence are coextensive, such that science studies some existents and, consequently, carries ontological commitments to its subject matter. Subject matters of all sciences exist, but that does not entail that they exist in the same way. Existence can be qualified in many different ways, and everything exists in a certain way. Avicenna believes that mathematical objects do exist, but not as mind-independent substances. Hence, his answer to the second question is negative.

Mathematical objects or subject matters of mathematical studies are *quantities* (*kammīyāt*). They are either (a) *continuous* (*muttaṣil*) quantities or magnitudes (*maqādīr*), which are geometric objects (or shapes), or (b) *discrete* (*munfaṣil*) quantities or numbers (*aʿdād*), which are arithmetical objects. Both of these two groups of mathematical objects are *accidents* of material substances, which have *mind-independent* existence, but as accidents dependent on material substances, rather than as autonomous substances.[1] Therefore, mathematical objects are not primarily mental constructions. However, we can separate them, in our minds, from the particular material substances to which they are attached in the extramental realm. Nonetheless, even in our minds, they have some sort of dependency on matter and materiality. A careful analysis of Avicenna's writings on the classification of the sciences reveals that only subjects of metaphysical studies can be completely released from all sort of dependencies on matter and materiality.[2]

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[1] In chapter III.3 of *The Metaphysics of the Healing*, Avicenna argues that numbers are accidents. In the next chapter of the same book, he argues that magnitudes are accidents too.

[2] Avicenna discusses this issue in several places. See, among others, chapter I.2 of *Isagoge* (1952c), chapters I.1-3 of *The Metaphysics of the Healing* (2005), and chapters 1-2 of *The Metaphysics of ʿAlāʾī Encyclopedia* (1952d). The idea of classification of the sciences according the ontological status of the objects that they study goes back to Aristotle (*Metaphysics* VI.1, 1026a13–19) and has been discussed by Al-Fārābī in his *The Aims of Aristotle's Metaphysics*, which Avicenna explicitly mentions, in his autobiography (2014, pp. 17–18), having read. The original Arabic text of *The Aims of Aristotle's Metaphysics* can be found in Al-Fārābī (1890, pp. 34–38). Gutas (2014, pp. 272–275) and McGinnis and Reisman (2007, pp. 78–81) provide partial English translations of this work. Its complete English translation can be found in Bertolacci (2006, pp. 66–72). Cleary (1994) has

According to Avicenna's categorization of the sciences, two sciences are distinct either because they study objects with different natures or because they study objects with the same nature but from different aspects (*ḥaythīyāt*).[1] He believes that theoretical sciences are divided into natural sciences, mathematics, and metaphysics. Every object of a natural science is mixed with a specific kind of matter in both the external world and the mind. It *may* be possible to abstract this object, in the mind, from the specific kind of matter with which it is mixed. However, if we do so, the abstracted object cannot be the subject of theoretical studies in a natural science, but should instead be studied by mathematics or metaphysics.[2] Every object of a natural science, inasmuch as it is the subject of theoretical studies in a natural science, is associated with a specific kind of matter. The objects of mathematics are similarly mixed with specific kinds of matter in the external world. Nevertheless, we can separate these objects, in our minds, from all particular kinds of matter. Nonetheless, this does not mean that mathematical objects are completely separated from materiality itself and that they have no dependency on matter. An object free from any kind of dependency on materiality cannot be the object of mathematical study; it should be studied in metaphysics. Given this classification, mathematical objects are separable from any specific kind of matter, but they still have some sort of dependency on materiality itself. The following passage supports this understanding:

> **TEXT # 2.1:** The various kinds of the sciences therefore either [(a)] treat the consideration of existents inasmuch as they are in motion, both in cognitive

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discussed Aristotle's classification of the sciences. See also Marmura (1980) and Gutas (2003) for two modern commentaries on Avicenna's classification of the sciences.

[1] Marmura (1980, p. 240) believes that Avicenna appeals to an *ontological* basis for his categorization of the sciences. Admittedly, there are some phrases in Avicenna's writings that seemingly support this claim. But a detailed investigation of his writings shows that his classification is grounded on an intertwined group of *ontological* and *epistemological* criteria. Sometimes he distinguishes two sciences because of the different objects they study; this is an ontological ground. But he also, as we will see, accepts that two distinct sciences may study the same object from different aspects; this can be considered to be an epistemological ground.

[2] For a recent work on Aristotle's treatment of the notion of abstraction, see Bäck (2014); for studies on different aspects of Avicenna's theory of abstraction, see Hasse (2001) and McGinnis (2007c).

> apprehension (*taṣawwuran*) and in subsistence, and are related to materials of particular species; [(b)] treat the consideration of existents inasmuch as they separate from materials of a particular species in cognitive apprehension, but not in subsistence; or [(c)] treat the consideration of existents inasmuch as they are separated from motion and matter in subsistence and cognitive apprehension.
>
> The first part of the sciences is natural science. The second is pure mathematical science, to which belongs the well-known science of number, although knowing the nature of number inasmuch as it is number does not belong to this science. The third part is divine science [i.e., metaphysics]. Since the existents are naturally divided into these three divisions, the theoretical philosophical sciences are these.[1]

According to this passage, the objects of natural science are mixed with specific kinds of matter in both the extramental world and the mind. Mathematical objects are similarly associated with specific kinds of matter in extramental reality, but they can be separated from all specific kinds of matter in the mind. This passage does not explicitly say whether mathematical objects still have some sort of materiality or dependency on materiality in the mind. However, there is a hint that this is the case. It seems that if we purify number of all characteristics of materiality, then the result is number inasmuch as it is number which, as Avicenna says in the above text, is the subject of metaphysical, not mathematical, studies. Admittedly, we need more persuasive evidence to support the dependency of mathematical objects on materiality in the mind. The nature of this dependency (if there is such) is itself unclear. So, it is better to address the subtleties of Avicenna's view about geometric shapes and numbers. In the next section, I will discuss his views on geometric shapes.

[1] Avicenna (1952c, chap. I.2, p. 14, ll. 3-10). English translations of all passages from chapter I.2 of *Isagoge*, I.2 are Marmura's in his (1980) paper, unless otherwise specified.

2.3. Geometric Objects

Avicenna believes that geometric shapes, even in our minds, have some sort of *necessary* association with materiality. They are separable from all specific materials in our minds, but not from materiality itself. I will try to establish and expand this rendition of Avicenna by gleaning textual evidence for it from his various works. I start by analyzing a passage from the *Isagoge*:

> **TEXT # 2.2:** The things existing in external reality whose existence is not by our choice and action are first divided into two divisions: [(I)] one consists of things that are mixed with motion; [(II)] the second of things that do not mix with motion, for example, mind and God. The things that mix with motion are of two modes. They are either [(Ia)] such that they have no existence unless they undergo admixture with motion, as for example, humanness, squareness and the like; or [(Ib)] they have existence without this condition. The existents that have no existence unless undergoing admixture with motion are of two divisions. They are either [(Ia-1)] such that, neither in subsistence nor in the estimation (*al-wahm*) would it be true for them to be separated (*tujarrada*) from some specific matter (*māddatan muʿayyanah*) as for example, the form of humanness and horseness; or else, [(Ia-2)] this would be true for them in the estimation but not in subsistence, as for example, squareness. For, in the case of the latter, its acquisition as a form (*taṣawwuruhu*) does not require that it should be given a specific kind of matter (*nawʿ mādda*) or that one should pay attention to some state of motion.[1]

If we consider what Avicenna says in this text about the quiddity (*māhīya*) of squareness as his general view about quiddities of geometric shapes, then we should conclude that for him these quiddities have no existence unless undergoing admixture with motion and matter.[2]

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[1] Avicenna (1952c, chap. I.2, from p. 12, l. 11 to p. 13. l. 4).

[2] I have supposed that the admixture with *motion* is equivalent to the admixture with *matter*. Many authors have endorsed this equivalency in Avicenna's writings. For example, Hasse (2013, p. 115, n. 28) writes: "In the *Introduction* to *al-Shifā*, Avicenna differentiates beings mixed with motion (matter) from those unmixed, for

In other words, it is impossible for them to be fully detached from materiality. Geometric shapes, *inasmuch as they are geometric shapes*, are necessarily mixed with materiality (because they lie under one of the subdivisions of the group (Ia) mentioned in the text). Nonetheless, according to this text, our *estimation* (*wahm*) has the ability to separate geometric shapes from all specific kinds of matter with which they may be mixed in the external world (because they lie under the group (Ia-2) of objects mentioned in the text). Geometric shapes, inasmuch as they are geometric shapes, are not necessarily associated with a specific kind of matter, though they are mixed with materiality.[1] A square, inasmuch as it is square, is not necessarily mixed with gold, wood, or any other specific kind of matter, but it is necessarily associated with materiality. Therefore, contrary to the concepts of *non-wooden triangle* or *non-golden triangle*, which are easily intelligible, the concept of *immaterial triangle* is a self-contradictory and unintelligible concept, just as impossible as *round square*.[2] Materiality is integrated with the quiddities of geometric shapes. Avicenna says that the core of the truth about geometric shapes, which Platonists and Pythagoreans have not ascertained, is that:

> **TEXT # 2.3**: [T]he *definitions* of geometric [shapes] among mathematical [objects] do not *utterly* dispense with matter, even though they can do without any given species of matter.[3]

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which he gives 'the intellect and God' as examples." McGinnis (2010a, p. 37) offers the same treatment of these two notions. TEXT # 2.5 confirms that Avicenna uses these two notions equivalently. But, according to some commentaries, movability is not equivalent to materiality for Aristotle. See Porro (2011, pp. 278–279).

[1] In contrast with geometric shapes, the quiddity of humanness, inasmuch as it is quiddity of humanness, is mixed with a specific kind of matter; i.e., flesh and blood. So, it cannot be abstracted from either materiality or even this specific kind of matter.

[2] Humanness is inseparable from not only materiality, but also the particular kind of matter from which human beings are constituted; i.e., flesh and blood. Therefore, the concept of *immaterial humanness* and the concept of *humanness separated from flesh and blood* are both self-contradictory.

[3] Avicenna (2005, chap. VII.2, p. 249, ll. 2-4). I have modified Marmura's translation by putting 'shapes' instead of 'figures,' 'do not utterly' instead of 'absolutely do not,' and 'species' instead of 'kind.' The italics are mine.

Here, Avicenna explicitly embraces the notion that association with materiality is a characteristic of not only the extramental existence of geometric shapes, but also of their definitions. This text explicitly shows that Ardeshir's reading of Avicenna, according to which mathematical objects are "not combined with matter in definition but with matter in existence," is misleading if not wrong.[1] Geometric shapes, even in our minds, are connected to matter. However, it remains obscure how it is possible for a mental existent to be mixed with matter but not with a specific kind of matter. Obviously, when we consider a geometric shape in our minds as an object of our cognition, it is fully separated from the materiality that exists in the physical world. So, the materiality from which we cannot separate geometric shapes in our estimation is not the former kind of materiality existing in the extramental world.[2] Geometric shapes are associated with some sort of *estimative* or, in Aristotelian terms, *intelligible matter* which may be considered as the cognitive counterpart to the perceptible materiality in the physical world.[3] We can say, at least metaphorically, that geometric shapes are mixed with some sort of estimative or intelligible matter, which is neither any specific kind of matter we have in the external world, nor separable from geometric shapes. A significant consequence of this inseparability from intelligible matter is

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[1] See Ardeshir (2008, p. 45).

[2] Having a mental concept of something in the mind does not necessarily guarantee that that thing is separable from matter. Consider Eiffel Tower and its mental counterpart, i.e., the concept EIFFEL TOWER. These two things, according to Avicenna, have the same quiddity; the quiddity of Eiffel Tower, which can accept two distinct modes of extramental and mental existence. The concept of EIFFEL TOWER, inasmuch as it is a concept, is mental and therefore, in a trivial sense, separated from matter. In this sense, *anything* of which we have a concept is trivially separated from matter in the mind. However, this is definitely not what Avicenna means by separability from matter in the mind. It seems, rather, that, according to Avicenna, X is separable from Y in the mind if and only if it is possible to conceive X without Y. We can conceive squareness without woodenness. For we can conceive a non-wooden, say golden, square. Therefore, squareness is separable from woodenness. However, according to Avicenna, we cannot conceive squareness without materiality (as we will see, it means: without the intelligible matter or material form). Consequently, squareness is inseparable from materiality in the mind. For a recent study on Avicenna's understanding of the notions of immateriality and separability, see Porro (2011).

[3] Porro (2011, p. 294) upholds this interpretation.

that geometric shapes are necessarily associated with material forms (*ṣuwar māddīya*). Avicenna says:

> **TEXT # 2.4:** [The subject matter of geometry, i.e., magnitude (*miqdār*)] does not separate from matter except in the act of estimation and *does not separate [even in the estimation] from the form that belongs to matter*.[1]

Geometric objects, inasmuch as they are what they are, are necessarily attached to intelligible matters. They are always in the forms of material objects.[2] So, they have some sort of *ontological admixture* and *association* with (or dependency on) materiality, or, more precisely, on material forms.[3] In our estimation, we can separate them from all the particular matters mixed with which they may exist in the physical world; nevertheless, they remain attached to their material forms.[4] It is impossible to conceive of geometric objects as being separated from their material forms. I shall now turn to an investigation of the nature of numbers.

2.4. Numbers

Avicenna's position on the status of numbers differs slightly from his views on the nature of geometric objects. Numbers have no necessary association with material forms, but, inasmuch as they are the subject of arithmetical studies, they still have some sort of dependency on materiality. This long passage sets out the main characteristics of numbers:

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[1] Avicenna (2005, chap. III.4, p. 84, ll. 31-32). I have slightly modified Marmura's translation. Particularly, I prefer to translate '*miqdār*' into 'magnitude,' not 'measure.' The italics are mine.

[2] By 'the *form* (*ṣūra*) that belongs to matter,' Avicenna means nothing more than the *shape* of material objects, or so it seems. Geometric objects are inseparable from intelligible matter. Therefore, they cannot be conceived without material shape. It is worth remembering that Avicenna had no understanding of geometry in dimensions higher than three. See Avicenna (2005, chap. III.4, from p. 89, l. 25 to p. 90, l. 7). If he had, his view on the necessary association of geometric objects with material form might have changed.

[3] See also Avicenna (2005, chap. III.4, p. 85, ll. 14-16 and from p. 86, l. 34 to p. 87 l. 2).

[4] See also Avicenna (2009a, chap. I.8, sec. 6, p. 59).

TEXT # 2.5: Regarding those things that can mix with motion, but have an existence other than this, these [include] such things as individual identity (*al-huwīya*), unity, plurality and causality [...]. These are either: [(a)] regarded inasmuch as they are [the things] they are (*min ḥaythu hiya hiya*), in which case viewing them in this way does not differ from looking at them inasmuch as they are abstracted—for they would then be among [the things examined through] the kind of examination that pertains to things not inasmuch as they are in matter, since these, inasmuch as they are themselves (*min ḥaythu hiya hiya*) are not in matter; or, [(b)] regarded inasmuch as an accidental thing that has no existence except in matter has occurred to them. This latter is of two divisions, It is either the case [(b1)] that that accident cannot be apprehended by the estimative faculty as existing except in conjunction with being related to specific matter and motion—for example considering one inasmuch as it is fire or air, plurality inasmuch as it is the [four] elements, causality inasmuch as it is warmth or coldness, and intellectual substance inasmuch as it is soul, that is, a principle of motion even though it in itself (*bi-dhātihi*) is separable—or [(b2)] that that accident, even though it cannot occur except in relation to matter and mixed with motion, is such that its state can be apprehended by the estimation and discerned without looking at the specific matter and motion in the aforementioned way of looking. The example of this would be addition and subtraction, multiplication and division, determining the square root and cubing, and the rest of the things that append (*talḥaqu*) to number. For all this attaches to number either in men's faculty of estimation, or in the existents that are subject of motion, division, subtraction and addition. Apprehending this as a form (*taṣawwuru dhālik*), however, involves a degree of abstraction that does not require the specifying of matters of certain species.[1]

[1] Avicenna (1952c, chap. I.2, from p. 13, l. 4 to p. 14. l. 2). I have slightly modified Marmura's translation of the Arabic phrase '*mawjūdāt mutaḥarrika munqasima mutafarriqa wa mujtami'a*.' His translation is 'existents that move, divide, separate and combine,' while my translation is 'existents that are subject of motion, division,

This text reveals that numbers, inasmuch as they are numbers, are mixed with neither any specific kind of matter, nor even materiality itself. Therefore, they can be found and considered in three different forms, considered inasmuch as: (1) they are what they are, fully separated from materiality (i.e., category (a) of the text), (2) they are accidents of material things, associated with a specific kind of matter (i.e., category (b1) of the text), (3) they are accidents of material things, but dissociated from any specific kind of matter (i.e., category (b2) of the text). Subject matters of arithmetic are numbers only when they are considered in the latter form.

Numbers mixed with some specific kinds of matter should be studied in natural science. For example, the number four, inasmuch as it is accidentally true of the four elements, should be studied in natural science. On the other hand, numbers, inasmuch as they are what they are, fully separated from materiality, should be studied in metaphysics. To be the subject of arithmetical studies, numbers should be considered as accidents of material things. They are associated with materiality, but not necessarily with a specific kind of matter. The concept of *immaterial number*, contrary to the concept of *immaterial triangle*, is plausibly intelligible.[1] intelligible.[1] When Avicenna discusses numbers in his metaphysics, he does indeed have a fully immaterial conception of numbers in mind.[2]

This discussion shows that there is a dissimilarity between geometric shapes and numbers. Geometric shapes, inasmuch as they are what they are, are necessarily associated with materiality or material forms. As we saw in TEXT # 2.1, TEXT # 2.2, and TEXT # 2.3, Avicenna believes that inseparability from materiality is included in the definition of geometric shapes. Therefore, the dependency of geometric objects on materiality is an *ontological* dependency,

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subtraction and addition.' I think that my translation is more faithful the context of this passage, which is about mathematics and mathematical operations.

[1] The concept of immaterial triangle is self-contradictory, at least if by 'materiality' we mean 'association of material form.'

[2] Therefore, numbers are similar neither to certain objects of metaphysics, e.g., God and mind, which are necessarily immaterial, nor to certain objects of natural science or mathematics, e.g., humanness and squareness, which are necessarily material. Numbers are *contingently* mixed with matters.

in contrast to the dependency of numbers on materiality, which is only *epistemological*. Numbers, inasmuch as they are what they are, have no necessary accompaniment with materiality. But, inasmuch as they are the subject of arithmetical studies, we should *consider* or *regard* them as things dependent on materiality (i.e., accidents of material things). Therefore, numbers are not *ontologically* intertwined with materiality. It is only the consideration (*naẓar*) of the arithmetician that preserves numbers in association with materiality (not the ontological status of numbers inasmuch as they are numbers). Hence, numbers, inasmuch as they are the subject of arithmetical studies, have some sort of *epistemological* dependency on materiality. I will now show why we need to consider this dependency for numbers, and why numbers, inasmuch as they are numbers, cannot be the subject of mathematical studies:

> **TEXT # 2.6:** [N]umber can be found in separable things and in natural things [...]. Number whose existence is in things separate [from matter] cannot become subject to any relation of increase or decrease that may occur but will only remain as it is. Rather, it is only necessary to posit it in such a way that it becomes receptive to any increase that happens to be, and to any relation that happens to be when it exists in the matter of bodies (which is, potentially, all modes of numbered things) or when [number] is in the estimative faculty. In both these states, it is not separable from nature.
>
> Hence the science of arithmetic, inasmuch as it considers number (*yanẓuru fi al-ʿadad*), considers it only after [number] has acquired that aspect possessed by it when it exists in nature. And it seems that the first consideration [or theoretical study] of [number that the science of arithmetic undertakes] is when it is in the estimative faculty having the description [mentioned] above; for this is an estimation [of number] taken from natural states subject to addition and subtraction and unification and division.
>
> Arithmetic is thus neither a consideration [or study] of the essence of number nor a consideration [or study] of the accidents of number inasmuch as it is absolute number, but [it is a study] of its accidental occurrences with respect to its attaining a

state receptive of what has been indicated [above]. It is either material, then, or [it] pertains to human estimation dependent *(yastanidu)* on matter.[1]

This text contains at least three important points: First, it corroborates the view that number can be found in both inseparable and separable things.[2] This means that number, inasmuch as it is number, is neutral with respect to materiality. As we saw, geometric shapes are not neutral in this respect. This alone, however, is not enough for us to conclude that numbers, inasmuch as they are the subject of arithmetical studies, are neutral with respect to materiality. Here we find the second point I want to make. Number, inasmuch as it is the subject matter of arithmetic, should be capable of participating in relations of decrease, increase, addition, subtraction, etc. And it is so only when it exists as accidents of material things.[3] In our estimative faculty, we can separate numbers from all particular materials while preserving those aspects that they have acquired only after admixture with materiality. We separate numbers from materiality in our estimation, but *consider* them *as if* they had some material aspects and capabilities.

In another phrase, very similar to the view Avicenna expresses in the first sentence of TEXT # 2.6, he says that number "would apply to [both] sensible and non-sensible things. Thus, inasmuch as it is number, it is not attached to sensible things."[4] Ardeshir concludes from this that "discussion about number and its relations should be understood as abstracted from sensible objects, not when it may belong to sensible objects. So, discussion about numbers is not about sensible objects."[5] But this interpretation is misleading. As we saw, discussion of numbers, *inasmuch as they are numbers*, should be understood as abstracted from sensible

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[1] Avicenna (2005 chap. I.3, from p. 18, l. 18 to p. 19, l. 8). I have slightly modified Marmura's translation. More precisely, I prefer to translate '*naẓar'* and '*yanẓuru'* respectively to 'consideration' and 'considers,' rather than 'theoretical studies' and 'studies.' I have also translated '*an tajtamiʿ va taftariq'* into 'subject to addition and subtraction,' rather than 'subject to combination and separation.'

[2] See also Avicenna (2009a, chap I.8, p. 57), where he says that the identity (*huwīya*) of number "does not require any dependence relation upon either natural or non-natural things."

[3] See also Avicenna (2009a, chap I.8, p. 57).

[4] Avicenna (2005, chap. I.2, p. 8, ll. 15-17).

[5] Ardeshir (2008, p. 46).

objects, but discussion of numbers and their *mathematical* relations, *inasmuch as they are the subject of arithmetical studies*, are not completely independent of sensible or, more precisely, material objects.

The third interesting point in this text is that Avicenna does not say that numbers in our estimation are attached to (or associated with) materiality or nature; he says only that they have some sort of dependency (*istinād*) on nature, which seems weaker than an ontological association with materiality. That is what I call 'epistemological' dependency on matter.

What Avicenna says in TEXT # 2.6 is intended to *refute* the claim of a person who might say: "The purely mathematical things examined in arithmetic and geometry are also 'prior to nature'—particularly number, for there is no dependency at all for its existence on nature because it cannot be found in nature."[1] Tahiri confusingly considers this passage to be something that Avicenna *believed*. He writes:

> This specificity of arithmetic is stressed by many 19th century mathematicians like Gauss who strikingly expressed a similar view in his letter to Olbers (1817) following the discovery of non-Euclidean geometries: "geometry must not stand with arithmetic which is purely a priori" (Gauss 1900, vol. VIII, p. 177). Ibn Sīnā would wholly agree with Gauss, since for him the concept of number is so pure that even time is not essential to its construction.[2]

My analysis of Avicenna's view regarding the natur of numbers indicates that the above interpretation is clearly false. According to Avicenna, arithmetic has an epistemological dependency upon nature. Numbers are not as pure as Tahiri suggests. They have some sort of admixture with materiality. To summarize:

(I) Geometric shapes, inasmuch as they are geometric shapes, are mixed with estimative matter (or, consequently, with material forms). They have an ontological dependency on materiality.

[1] Avicenna (2005, chap. I.3, p. 17, ll. 10-13).

[2] Tahiri (2016, p. 33, n. 17).

(II) Numbers, inasmuch as they are numbers, are not mixed with materiality; but, inasmuch as they are the subject of arithmetical studies, they should be considered as mixed with materiality. They have an epistemological dependency on materiality.

Numbers, are, in a sense, more abstractable or more separable than geometric shapes. The subject matter of arithmetic is somehow closer to the subject matter of metaphysics. Hence, it is plausible to expect that the difference in the ontological status of numbers and geometric shapes would lead to further epistemological differences between geometry and arithmetic. However, I will refrain from engagement in this debate, which merits independent study.

In his discussion of the division of sciences, McGinnis says that, for Avicenna, "[t]hose existents that can be conceptualized without matter, even though they are necessarily mixed with motion and never subsist without matter, are the subject of mathematical sciences."[1] My analysis shows that this picture of the nature of mathematical objects suffers from imprecision. If by 'matter' McGinnis means particular kinds of matter existing in the physical world, then he is right. Mathematical objects can be conceptualized without any particular matter. However, if he intends to convey materiality *qua* materiality by 'matter,' then his interpretation is false. Mathematical objects, inasmuch as they are the subject of mathematical studies, cannot be abstracted from the materiality itself.

Unfortunately, we cannot arrive at a reliable understanding of Avicenna's view on the subject matter of the theoretical sciences by consistently fixing one of these two meanings of 'matter' in McGinnis's book. He writes:

> [T]here are three major branches of theoretical sciences: the natural sciences, the mathematical sciences, and the science of metaphysics. These divisions correspond respectively with whether the objects investigated by the science must necessarily subsist as well as be conceptualized together with motion and matter; necessarily

[1] McGinnis (2010a, p. 36).

> subsist together with matter and motion but need not to be conceptualized as such; or need neither subsist nor be conceptualized together with matter and motion.[1]

As we saw, if by ‘matter’ he means materiality itself, then mathematical objects cannot be conceptualized without matter. On the other hand, if by ‘matter’ he means particular matters existing in the physical world, then subsisting and being conceptualized without matter (in this new sense) is not sufficient for being the subject of metaphysical investigation. The sufficient condition for being an object of metaphysics is being separated from all material forms and from materiality itself, not merely from special kinds of matter.[2]

2.5. Actual and Potential Perfect Objects

Mathematical objects, as we saw, are primarily accidents of material things in the external world. Mathematical enquiry is, therefore, primarily about accidents of material objects, not about independent immaterial entities. We can obtain a universalized conception of mathematical objects by abstracting them, via our faculty of estimation, from all particular materials with which they may be mixed in the extramental world. This purification procedure can, in principle, end in the production of some intelligible forms of *exact* and *perfect* mathematical objects that are not easily perceptible in tangible objects.[3] But does this necessarily mean that there are no *perfect* mathematical objects in the physical world? Does

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[1] McGinnis (2010a, p. 37).

[2] Admittedly, the source of this ambiguity is Avicenna’s own writings, where he uses the term ‘matter’ equivocally. See, for example, Avicenna (1952d, pp. 4–5).

[3] What we perceive in our ordinary perceptual experiences only approximates the ideal shape of celebrity geometric objects, e.g., perfectly straight lines, circles, parabolas, etc. So, at first glance, it might seem that there should exist nothing in nature with exactly these shapes. If so, this fact provides a strong motivation for the view that perfect mathematical objects (i.e., objects which exactly satisfy the mathematical definition of those ideal shapes) are merely mental constructions that do not really exist in the extramental world. Nonetheless, Avicenna believes that these perfect objects can—and some of them really do—exist in the extramental world, or so I will argue.

it necessarily mean that perfect mathematical objects are merely mental constructions that have no counterpart in the external world? I will argue, in this section, that Avicenna endorses the existence of at least some perfect mathematical objects in the extramental world.

The most important evidence for Avicenna's agreement with some sort of literalism is that, wherever he discusses the nature of mathematical objects in his writings, he affirms that mathematical objects exist in the external world in association with determinate kinds of matter (as accidents of specific material particulars). However, he simultaneously insists that mathematical forms are not sensible natural forms.[1] Some might object that, although this provides strong evidence for the existence of mathematical objects in the extramental world (albeit not as independent substances), this cannot count as evidence that those objects are perfect. I think that this objection is untenable. If it stood, then Avicenna would need to distinguish between two sorts of perfect and imperfect mathematical objects, such that the latter could exist in the external world mixed with matter, but the former could exist only in the mind. He would need to say that, for example, *quasi* circular objects (which approximate the ideal shape of a circle but do not really satisfy the mathematical definition of a circle) could exist in the extramental world but *quiet* circular objects (which perfectly satisfy the mathematical definition of a circle) could exist only in the mind. This he does not do.[2] As we will see, he denies that what we *see* in the external world are perfect mathematical objects, but this is so just because mathematical forms are not sensible (visible) forms, not because we see a mathematical form that is imperfect. Paying attention to the epistemological formalities that Avicenna proposes for grasping mathematical forms and producing mathematical concepts will show how mathematical literalism can, by and large, be compatible with Avicenna's philosophy.

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[1] See, for example, Avicenna (2005, chap. III.4, p. 85, ll. 10-16) and (2009a, chap I.8, pp. 57-58). Mathematical forms can exist in sensible things but they are not themselves sensible forms.

[2] Recall that in his attack on Platonism, he defends the view that the geometric shapes that exist in the external world and the intelligible geometric forms that we have in our minds have the same quiddities. See Avicenna (2005, chap. VII.3, especially pp. 249-250).

Interestingly, the mental faculty involved in apprehending mathematical concepts is *estimation*. Discussing the details of the role estimation plays in forming mathematical concepts and attaining mathematical knowledge is outside the scope of this chapter.[1] But consideration of some other objects of the estimative faculty may help us to reach a better understanding of the existential mode of mathematical objects in the extramental world. According to Avicenna's epistemology, estimation is a bodily faculty with a distinct and unique cognitive power that lies between imagination and intellect in the hierarchy of cognitive faculties. Some of its activities are common to both humans and animals, while others are exclusively human. Looking at one of its activities will give us a better understanding of the role of estimation in perceiving mathematical objects in the external world; this activity is likened to *incidental perception.*[2] When somebody perceives the sweetness of a yellow cake just by seeing it, she has an incidental perception. She has perceived a sensible form without employing the right perceptive faculty that we usually use to apprehend similar sensible forms. She has perceived the sweetness of the cake without tasting it. Such an apprehension, according to Avicenna, is feasible only because of our estimative faculty. It enables us to 'see' the sweetness of a yellow cake. Supposing that her apprehension is reliable and that the cake really is sweet, she has apprehended, by the aid of her estimation, something which *really exists* in the external world but is imperceptible by the sense she uses.[3] It can be argued, analogously, that when we see a triangular wooden shape or a group of three balls we apprehend perfect mathematical objects (a perfect triangle or the number three[4]) that really exist in the extramental world as accidents of material

[1] A detailed account of the role of estimation in the epistemology of mathematics will be presented in in the last chapter.

[2] For more on Avicenna's treatment of estimation and the other roles that he attributes to this faculty, see Black's (1993) seminal paper and my discussion in the last chapter.

[3] For more on the details of the mechanism of incidental perception, see Black (1993, pp. 25–27).

[4] Of course, it is more precise to say that what exist in the physical world are *instantiations* of triangularity and threeness.

objects; but they are not visible, or available, to our sensory faculties.[1] Estimation, among its other roles, enables us to apprehend those things that actually exist in the extramental world but are invisible.[2]

The existence of numbers as perfect mathematical objects in the external world seems more defensible than the existence of perfect geometric objects. The twoness of two tomatoes is as perfect as the twoness of two books or the twoness of one imaginary Santa Claus and one (hopefully real!) Christmas gift. They are different instantiations of the same universal concept, i.e., twoness. Twoness is not directly sensible, like whiteness or warmth, but we apprehend it thanks to the estimative faculty, and it is as real as the existence of any ordinary accidents that material objects may have. Hence, there is no serious hindrance to the attribution of arithmetic literalism to Avicenna.

The existence of perfect geometric objects, on the other hand, might seem more improbable. At first glance, it seems obvious that there is no perfect geometric object in the physical world. Since everything there has width, length, and depth, there is no perfect line with no width; hence, there is no perfect triangle. In fact, there are some passages from which

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[1] Admittedly, there are some dissimilarities between the roles estimative faculty plays in incidental perception and apprehension of mathematical objects. In incidental perception, our estimation enables us to perceive something that is the proper object of the sense-perceptual faculty X using data we receive instead from the faculty Y. Therefore, estimation performs what the faculty X can normally perform. In apprehension of mathematical objects, however, estimation performs what no sense-perceptual faculty can perform, because mathematical objects are not sensible at all. So, it might seem better to analogize mathematical perception to the apprehension of some non-sensible intentions, such as pleasantness, goodness, friendship, and hostility. Avicenna believes that, although these intentions are not themselves sensible, they can be perceived through the perception of some sensible forms, albeit by the aid of estimation. For example, we can apprehend the goodness of a friend through what we perceive by our senses from her. Nonetheless, some dissimilarities rise again. Some of these intentions, contrary to mathematical objects, are not necessarily associated with materiality. They can be properties of some immaterial objects (e.g., God is good). The moral is that each analogy has limitations.

[2] Tahiri has, strangely, overlooked the significant role Avicenna accords to the estimative faculty in attaining knowledge of mathematical objects. His translation of 'uhām al-nās' to 'people's beliefs' (2016, p. 31, n. 13) is just one of the signs of his negligence.

someone may conclude that Avicenna endorses this view, i.e., anti-literalism. For example, he accepts that, when we want to prove a geometric theorem based on a composition of geometric shapes that we have drawn on a piece of paper, what we have drawn are *not* perfect geometric objects, and what we are trying to prove is *not* about those *visible* figures. He endorses a view that he attributes to Aristotle:

> **TEXT # 2.7:** The drawn line and the drawn triangle are not drawn because the demonstration needs them. The demonstration [of a geometric theorem] is [demonstrated] on a line which is really [i.e., perfectly] straight and width-less; and [it is demonstrated] on a triangle which has really [i.e., perfectly] straight sides with the same length. This triangle and that line [drawn on the paper] are rather for preparation of the mind to imagine. Demonstration is [demonstrated] on the intelligible, not sensible or imaginable (*mutakhayyal*) [forms].[1]

There is no doubt that perfect geometric objects cannot be drawn. Moreover, they are completely invisible. But this does not necessarily entail that there is no perfect geometric object in the external world.[2] There is some evidence that supports the claim that Avicenna accepts the existence of perfect geometric objects in the external world, though not as sensible things. It can be argued, compatibly with Avicenna's philosophy, that the role of estimation is not merely to construct perfect mental objects that have no counterpart in the extramental world. Estimation, at least in some cases, helps us to apprehend some sort of *non-sensible perfection* that really exists in the extramental world. There are some passages that support this construal. For example, Avicenna says that the ascertained doctrine is that:

[1] Avicena (1956, chap. II.10, p. 186). The translation is mine.

[2] Marmura, in a note on his translation of *The Metaphysics of The Healing*, writes: "Geometer's circle is a partial abstraction by the estimative faculty of circles that exist in sensible matter. This need not exclude the existence of 'perfect' circles in material things, a notion rejected by atomists" (Avicenna, 2005, p. 397, n. 5 of chap. III9). It seems that in (at least some of) his arguments against atomism, Avicenna presupposes the possibility of the existence of perfect geometric objects in the external world. I will return to this issue in the last chapter.

TEXT # 2.8: Point exists only in line, which is in surface, which is in body, which is in matter.[1]

From one perspective, this text is a criticism of Platonism. Avicenna believes that perfect geometric objects have no immaterial independent existence. From another perspective, it is a confirmation of the actual existence of perfect mathematical objects (e.g., point and line) in the extramental world. It is worth noting that the actual existence of point, line, and surface in the external world do not entail their separability from each other outside the mind. Avicenna emphasizes that we can separate point from line, line from surface, and surface from body (*jism*) *only* in our estimation.[2] They exist in the external world, but they are not distinctly perceptible and cannot be separately predicated upon material particulars. Analogously, we can say that perfect triangles exist in the external world (e.g., in triangular bodies), but they are not distinctly perceptible and cannot be separately predicated upon material particulars. Given these considerations, the actual existence of *at least some* perfect geometric objects in the physical world seems, by and large, compatible with the tenor of Avicenna's writings on the nature of mathematical objects. So, it is not incautious to say that he endorses some sort of geometric literalism.

Undoubtedly, a great deal of work is needed to establish literalism as a plausible view about the nature of mathematical and especially geometric objects. But my concern here is the compatibility and consonance of literalism (in the sense described) with Avicenna's philosophy, rather than the plausibility of the view itself. My arguments show that he, by and large, endorses some sort of literalism.[3]

Even if we accept the actual existence of some perfect mathematical objects in the physical world, it is undeniable that most mathematical objects do not exist in the physical world. If the number of material objects is finite, then there are some large numbers (larger than the

[1] Avicenna (2005, chap. VII.3, p. 254, ll. 25-27). I have slightly modified Marmura's translation.

[2] See Avicenna (2005, III.4, 86-87).

[3] Nonetheless, if literalism is the actual existence of mathematical objects as physical *substances*, then Avicenna expressly rejects the doctrine. For Avicenna, mathematical objects, inasmuch as they are *accidents* of material things, exist in the extramental realm.

number of all objects that we have in the extramental world) that are not accidents of any group of numbered material objects; so, those numbers do not literally exist. On the other hand, as Avicenna admits, there are many geometric objects that do not exist in the physical world.[1] These objects, by the aid of the imaginative and estimative faculty, can, in principle, be constructed in the mind.[2] But, obviously, there are infinitely many of these objects that have never been constructed. Consider a very large number (larger than the number of physical objects and larger than the largest number we have ever thought of), or a very strange geometric shape that nobody has never thought about. These objects do not exist, either as accidents of material objects in the physical world or as objects constructed by imagination and estimation in a human mind. Of course, if someone decides to construct such an object in her mind, she *may* succeed. But before that, these objects do not actually exist. Nevertheless, we can still attribute some sort of *potential* existence to these objects. We have the *potentiality* to create them in our minds, so they *potentially* exist. Therefore, Avicenna is somehow a *potentialist*: at least some mathematical objects only potentially exist. More precisely, some mathematical species exist only in a potential sense of existence.[3]

[1] See Avicenna (1957 chap. III.7, pp. 336-337). Interestingly, he does not say that no geometric shape exists in the physical world; he says that many geometric shapes do not exist in the physical world. This means that he accepts the actual existence of at least some of geometric objects in the external world. However, it does not automatically entail that those objects (which exist in the exteramental real) are necessarily perfect.

[2] In the last chapter, I will go through the details of the mechanisms Avicenna proposes for constructing such objects.

[3] I do not claim that Avicenna commits to a vast ontology of non-existent objects. The quantifier 'there are' in the sentence that 'there are some mathematical objects that only potentially exist' should not be read as having ontological claim. My aim in this section is simply to emphasize that, although Avicenna believes in the existence of mathematical objects (i.e., numbers and geometric shapes) in the external world, he does not believe that all numbers and all geometric shapes one can, in principle, think about do really exist in the external world. Contrary to a mathematical Platonist who endorses the actual existence of an infinite number of mathematical objects (though not as concrete objects), Avicenna's philosophy allows only a finite number of mathematical species, either in the external world or even in the mind, to exist. He nonetheless accepts the possibility of creating *new* mathematical objects. This is what I mean by attributing potentialism to Avicenna.

2.6. Conclusion

According to Avicenna, mathematical objects are, in the first instance, accidents of material objects, so they exist in the extramental realm mixed with particular materials. Nonetheless, they are not themselves material or natural forms. In our minds, we can separate them, by our estimation, from all the determinate kinds of matter with which they may be associated. However, the degrees of separability of geometric objects differ from that of numbers. Geometric objects, inasmuch as they are what they are, are inseparable from materiality *qua* materiality. We can separate them from all specific kinds of matter, but not from materiality itself; such objects are necessarily attached to material forms, even in our estimation. So, geometric objects have some sort of *ontological dependency* on materiality. Numbers, on the other hand, are completely separable from matter. Inasmuch as they are what they are, they have no dependency on materiality. They can be found in association with, or separate from, materiality. But numbers as the subject of arithmetical studies should be receptive to decrease and increase, and should have the capability of being subject to addition, subtraction, multiplication, and addition. Numbers are receptive to such accidents only when they are applied to material things. So, if we want to have a conception of number, inasmuch as it is the subject of arithmetical studies, we should *consider* it as something associated with matter. Number in its nature has no ontological dependency on materiality but, as a subject of mathematical studies, should be considered as mixed with matter. So, numbers, inasmuch as they are the subject matter of arithmetic (but not inasmuch as they are numbers) should be considered in accompaniment with materiality. Therefore, they have some sort of *epistemological dependency* on materiality.

Avicenna endorses the existence of perfect mathematical objects in the external world. He believes that mathematical objects can literally exist in the extramental world as accidents of material things, though not as independent substances. They are not sensible forms but they can be perceived by the aid of the estimative faculty.

In any case, the number of mathematical species that actually exist in either the extramental or mental world is finite. Most mathematical objects only potentially exist. They have no actual existence, whether extramental or mental. They can, in principle, be constructed either in the extramental world by creating new objects and increasing the number of material objects in the world, or in the mind with the aid of imagination and estimation. So, Avicenna is somehow a *literalist*, a *finitist* and a *potentialist*. He does not think that mathematical objects can be released from all the ontological or epistemological dependencies they may have on materiality; this is what distinguishes my view from that of McGinnis, Ardeshir and Tahiri. However, geometric objects and numbers have different degrees or different kinds of dependencies on materiality, and this is what distinguishes my view from that of Michael Marmura.

3. On Mathematical Infinity

Avicenna believed in mathematical finitism. He argued that magnitudes and sets of ordered numbers and numbered things cannot be actually infinite. In this chapter, I discuss his arguments against the actuality of mathematical infinity. A careful analysis of the subtleties of his main argument, i.e., The Mapping Argument, *shows that, by employing the notion of* correspondence *as a tool for comparing the sizes of mathematical infinities, he arrived at a very deep and insightful understanding of the notion of mathematical infinity, one that is much more modern than we might expect. I argue moreover that Avicenna's mathematical finitism is interwoven with his literalist ontology of mathematics, according to which mathematical objects are properties of existing physical objects.*

3.1. Introduction

The problem of infinity in the Aristotelian tradition has two distinct aspects. Its negative aspect includes various arguments for the impossibility of the *actual* existence of infinity. Its positive aspect, on the other hand, justifies the merely *potential* existence of infinity and explains how something can have the potentiality of being infinite (*apeiron*), although this potentiality can never be actualized. Avicenna had some innovative ideas with respect to both of these aspects. Compared to most other Aristotelian philosophers, he had a more flexible approach to the impossibility of actual (*bi-l-fiʿl*) infinity (*lā nihāya*).[1] Specifically, he preserves the possibility of the actual existence of a very specific type of infinity. He believes that an infinite *non-ordered* (*ghayr murattab*) set of immaterial objects, e.g., angels or souls, can (and, indeed, does) actually exist.[2] Nonetheless, this view does not take him very far from Aristotle's own position on the ordinary types of physical or mathematical infinity we usually think about in scientific inquiry. Avicenna, like Aristotle, believes that they exist *only* in a potential (*bi-l-quwwa*) sense. The significance of Avicenna's views on physical or mathematical infinity lies, therefore, in the subtle and insightful ideas he adds to both the negative and the positive aspects of the problem to support the core idea of Aristotelian infinity, rather than in a rejection of Aristotle's view.

This chapter aims to explain Avicenna's views on the negative aspect of the problem of mathematical infinity and to clarify their significance and novelty from the perspective of the history and philosophy of mathematics. I should, therefore, first specify what exactly I mean by 'mathematical infinity'. In the next section, I discuss this issue and elaborate the relation between mathematical and physical infinity in Avicenna's philosophy. Knowing about this relation helps us to a better realization of how Avicenna's arguments for the impossibility of the actual existence of mathematical infinity are interwoven with his arguments against the actuality of the physical infinite. Moreover, it sheds a new light on why Avicenna discusses mathematical infinity in the *Physics* parts of his works. In section 3.3, I briefly review two of Avicenna's arguments against the actuality of infinity. The first, *The Collimation Argument*

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[1] Nawar (2015, p. 2355, n. 15).

[2] See Marmura (1960), Rashed (2005, pp. 298–299), and McGinnis (2010b).

(*burhān al-musāmita*), appeals to the notion of *motion*, while the other, *The Ladder Argument* (*burhān al-sullam*), does not engage such physical notions. Our study of Avicenna's views on the negative aspect of the problem of mathematical infinity is completed in section 3.4, by investigating the details of his main argument against the actuality of mathematical infinity; this is *The Mapping Argument* (*burhān al-taṭābuq* or *al-taṭbīq*). I will show that only this argument can be applied to the case of numerical (discrete) infinity. Elucidation of the philosophical and mathematical presuppositions of this argument reveals that the affinity between Avicenna's understanding of the notion of infinity and our modern understanding of this notion is stronger than we might have expected, or so I will argue there. I close, in section 3.5, with some concluding reflections.

3.2. The Notion of Mathematical Infinity

There are two important points to make concerning the notion of mathematical infinity, before going through the details of Avicenna's views about this notion. First, I explain exactly what I mean by 'mathematical infinity'. Second, I discuss the connection of this notion with the notion of physical infinity from the perspective of Avicenna's philosophy.

Aristotle defines infinity as something that 'if, taking it quantity by quantity, we can always take something outside.'[1] Avicenna accepts this definition and mentions it in many different places of his oeuvre. For example, in *The Physics of the Healing* he says that infinity is 'that which whatever you take from it—and any of the things equal to that thing you took from it—you [always] find something outside of it.'[2] Nonetheless, Avicenna's treatment of the more specific notion of mathematical infinity is not in complete accordance with that of

[1] *Physics* II.6, 207a7-8.

[2] Avicenna (2009b, chap. III.7, sec. 2). See also Avicenna (2009b, chap. III.7, sec. 3, and chap. III.9, sec. 1). As another example, in the letter to the vizier Abū Saʿd, Avicenna defines infinity as 'a quantity or something possessing a quantity that if you take something from it, you still find something other than what you took and you never reach something beyond which there is nothing of it [i.e., of that infinity]' (2000, p. 28).

Aristotle. In the Aristotelian tradition, the problem of mathematical infinity has been studied by analyzing three different yet interrelated phenomena: (a) the infinity of numbers[1] (i.e., the subject matters of arithmetic), (b) the infinity of magnitudes (i.e., the subject matters of geometry), and (c) the infinite divisibility of magnitudes.[2] What I mean by 'mathematical infinity' is restricted to the two former types of infinity. I do not go, therefore, into the details of Avicenna's views on the latter type of mathematical infinity. This departs somewhat from the general approach of contemporary Aristotle scholars, most of whom have paid more attention to the infinite divisibility of magnitudes.[3] Specifically, they have tried to clarify Aristotle's view about (a) and (b) by scrutinizing his views about (c). I think, nonetheless, that following the opposite strategy is more plausible, at least in the context of Avicenna's philosophy. Let me justify why.

Aristotle believes that a thing—whatever it is—may be 'infinite either by addition or by division.'[4] As a result, the first two kinds of mathematical infinity (i.e., the infinity of numbers and magnitudes) are cases of being *infinite by addition* while the latter (i.e., the infinite divisibility of magnitudes) is a case of being *infinite by division*. There are two things which motivate Aristotle scholars to base their discussions of mathematical infinity on the infinite by division rather than the infinite by addition. First, some of Aristotle's texts can, in principle, be interpreted in a way that represents him as believing that the problem of the

[1] Recall that throughout this dissertation, by 'numbers' I just mean ordinary natural numbers.

[2] This classification is inspired by Aristotle's *Physics* III, 206a9-12. However, in that passage he speaks of the infinity of *time* rather than *magnitude*. He confirms there that the infinite divisibility of magnitudes and the infinity of numbers and time are, *in a sense*, undeniable. Given the so-called 'isomorphism thesis' according to which one-dimensional magnitudes, one-dimensional motions, and time have the same mathematical structure, (b) and (c) have some strong connections, respectively, to the infinity of time and the infinite divisibility of temporal intervals. Fred Miller (1982, sec. 5) has argued that Aristotle endorses this thesis. See Newstead (2001) for a discussion of this thesis from the perspective of modern mathematical theories of continuum. McGinnis (1999) shows that Avicenna endorses this thesis and his temporal theory rests on it.

[3] See, among others, Hintikka (1966), Bostock (1973, 2012), Lear (1980), Bowin (2007), and Coope (2012).

[4] *Physics* III, 206a14-8.

infinite by addition is reducible to the problem of the infinite by division.[1] Consequently, the latter notion has a priority over the former in discussion. Second, Aristotle's discussions of the infinity of numbers and magnitudes lack any *mathematically* remarkable feature and cannot, therefore, attract the attention of modern historians and philosophers of mathematics. By contrast, none of these points is true of Avicenna's account of mathematical infinity. On the one hand, the reducibility of the notion of infinity by addition to that of infinity by division does not play any influential role in Avicenna's discussion of the infinity of numbers and magnitudes. To be precise, he *is* sympathetic to the idea of discussing infinite divisibility prior to the infinite largeness of numbers and magnitudes,[2] *but* the plausibility of his discussions of the latter does not really depend on that of the former. There is no obvious inferential connection between them. On the other hand, and perhaps more importantly, his discussions on the infinity of numbers and magnitudes include the introduction and application of some mathematical notions and the presentation of some arguments which are worthy of investigation by historians and philosophers of mathematics, or so I believe. Therefore what motivates Aristotle scholars is weakened in the context of Avicenna's philosophy of mathematics. Additionally, it seems that, at the end of the day, without a correct conception of the infinity of numbers it is impossible to illustrate the infinite divisibility of magnitudes. The infinite divisibility of a line is nothing other than that the number of divisions we can make in that line is infinite; ergo, the infinity of numbers should be discussed prior to the infinite divisibility of magnitude. These considerations are enough to show that we can base our study of Avicenna's views about mathematical infinity on (a) and (b), rather than (c). I think that Avicenna's views about the infinite divisibility of

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[1] See, for example, Lear (1980, p. 195) who interprets *Physics* III, 206b3-4 in this way. Bowin (2007, sec. III) not only confirms this approach, but also construes *Physics* III, 207b10-13 as claiming the strong epistemological thesis 'that our ability to think of ever larger natural numbers also depends upon the infinite divisibility of magnitudes' (2007, p. 244).

[2] In *The Physics of the Healing*, Avicenna says that 'before we speak about finite bodies and their states with respect to largeness, we should speak about the finite and infinite with respect to smallness and divisibility' (2009b, chap. III.2, sec. 1). But, as we will see, his discussion of the former has no argumentative connection to that of the latter.

magnitudes should be studied in connection with the notion of the *mathematical continuum*, and I postpone this to an independent further work.

Another point worth mentioning about the notion of mathematical infinity is its relation to physical infinity. Aristotelian philosophers in general believe that the mathematical realm is connected to the physical realm, although the nature of this connection can be (and indeed is) construed in many different ways. Avicenna's views on the ontology of mathematics and the nature of mathematical objects show how strong this connection is for him. Familiarity with these views helps us to attain a more comprehensive understanding of Avicenna's position on mathematical infinity. Therefore, I briefly sketch his theory on the ontological status of mathematical objects. I discussed the details and subtleties of this theory in the previous chapter.

According to Avicenna, mathematical objects, i.e. numbers (*aʿdād*) and geometrical shapes (or magnitudes (*maqādīr*) in general), are neither Platonic forms, nor independent material objects, nor even purely mental existents completely separated from matter. They are, in the first instance, accidents of actually existing material objects.[1] They are, therefore, mixed with particular materials (or, in other words, with particular kinds of matter) in the extramental realm. They are predicated upon the physical objects. By the aid of our estimation (*wahm*), we can separate mathematical objects, in our minds, from all those determinate kinds of matter to which they are attached outside the mind.[2] Nonetheless, mathematical objects cannot be separated from materiality itself. Even in the mind they are mixed with materiality.

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[1] See Avicenna (2005, chaps. III.3-4).

[2] According to Avicenna's theory of knowledge, estimation is a bodily cognitive faculty which plays a protagonist role in the epistemology of mathematics. For a magnificent discussion on the other human or animal functions of this faculty, see Black (1993). See also Hall (2006) for a more recent study on the role of the estimative faculty in Avicenna's psychology. I will discuss the role of the faculty of estimation in the process of forming mathematical concepts in the last chaper.

They ‘absolutely do not dispense with matter, even though they can do with some kind of matter.’[1]

The ontological status of mathematical objects, as the objects studied by mathematics, may become clearer in contrast with the ontological status of the objects studied and investigated by natural sciences and metaphysics. Whether in the external world or in the mind, the objects studied by natural sciences are inseparable from not only materiality itself but also from the particular matters with which they are mixed. We cannot separate humanness, for example, from either materiality in general or even the particular matter it is mixed with, i.e., flesh and blood. Therefore, even in our estimation, we cannot detach humanness from flesh and blood. On the other hand, the objects studied by metaphysics, though they *may* be mixed with some particular matters in the external world, are separable from not only those particular matters but also from materiality itself in the mind.[2] With respect to separability from matter, mathematical objects lie between these two groups of objects. More precisely, with respect to separability from determinate kinds of matter, mathematical objects are similar to the objects studied by metaphysics and dissimilar to the objects studied by natural sciences. In our minds, we can separate mathematical objects from those particular matters with which they are mixed in the extramental world. But with respect to separability from materiality itself, mathematical objects are similar to the objects studied by natural sciences and dissimilar to the objects studied by metaphysics, because even in our mind we cannot separate them from materiality itself. Mathematical objects, inasmuch as they are the subject matters of mathematical studies, are inseparable from the material form (*al-ṣūra al-*

[1] Avicenna (2005, chap. VII.2, sec. 21). I have corrected an oversight in Marmura’s translation by putting the second ‘with’ in the above quote to replace his ‘without’. To be precise, Avicenna says: ‘*al-hindisīyāt min al-ta‘līmīyāt lā tastaghnī ḥudūdahā ‘an al-mawādd muṭlaqan, wa ’in istaghanat ‘an naw‘ mā min al-mawād*.’

[2] Some objects studied by metaphysics, e.g., God and mind, are necessarily separated from matter. Therefore they cannot be mixed with matter. Others, by contrast, e.g. numbers, can in principle be mixed with matter. However, if we consider them as mixed with matter then our study is no longer metaphysical. See Marmura’s diagram of the classification of the objects studied by the different sciences at the end of his (1980) paper.

māddīya).[1] Even in our estimation, we should consider them as accidents of material objects and, therefore, attached to matter, albeit not to a specific kind of matter.[2]

[1] See Avicenna (2005, chap. III.4, sec. 2). By 'material forms', or more precisely 'forms that belong to matter', Avicenna seems to mean the form of corporeality which is common to all corporeal things. Mathematical objects cannot be conceived separated from the corporeal form. Shihadeh (2014) discusses Avicenna's views on the corporeal form and its reception in the twelfth century. See especially p. 367 for an explanation of the inseparability of mathematical objects from the corporeal form. The corporeal form can be conceptually separated from prime matter. Therefore, one might suggest that mathematical objects, though inseparable from the corporeal form, can be completely separated from matter in the estimation. If so, when Avicenna says that mathematical objects 'do not dispense with matter, even though they can do without some kind of matter', by *dispensability with matter* (*istighnā 'an mādda*) he does mean nothing more than *separability from the corporeal form*. In other words, mathematical objects in the estimative faculty are detached from matter, though still attached to the corporeal form. However, it seems that mathematical objects should have a stronger connection to materiality. It is possible (and mathematicians often need to) consider different but qualitatively indistinguishable instances of each kind of geometrical shapes, e.g., two distinct circles of the same radius or two distinct squares of the same size. But the corporeal form cannot be the distinguishing feature of these distinct instances. This is because, as Shihadeh (2014, p. 385) clarifies, 'Avicenna does not speak of multiple 'corporeities'.' All corporeal things, Avicenna believes, share the same corporeal form; corporeality itself is not quantifiable. Therefore, indispensability with matter seem to be something more than mere attachment to the corporeal form. It is ratter attachment to an unqualitative indeterminate kind of matter which we can call it, in Aristotelian term, intelligible matter (or estimative matter, to have a more Avicennian flavor). This intelligible matter (to which mathematical object are attached in the estimative faculty) plays the role of the distinguishing element of distinct instances of the same kind of geometrical shapes. Admittedly, this account should be discussed and developed in more detail. Nonetheless, doing so is beyond the scope of this chapter.

[2] As I showed in detail in the previous chapter, these views concerning the ontological status of the objects studied and investigated by the different sciences are deduced from Avicenna's discussions on the classification of the sciences. To be precise, for Avicenna the theoretical sciences are divided primarily according to whether or not the objects they study are related to *motion* (i.e., whether or not they are *movable*). But an object is movable if and only if it is associated with matter in the external world. That is how *movability/immovability* can be replaced with *inseparability/separability from matter* as a criterion for classifying the objects of the sciences. There is another thing which encourages me to be focused mainly on the latter distinction in my explanation of Avicenna's position about the nature of the objects studied by the different sciences. (In)separability from matter can be divided into two finer-grained kinds of (in)separability—i.e., (in)separability from specific kinds of matter and (in)separability from materiality itself—which play

According to the above picture, Avicenna should be described as a literalist with respect to the ontology of mathematics. He believes that mathematical objects are accidents and properties of physical objects that literally exist in the external world. The existence of such properties does not depend upon the specific kinds of matter of the objects of which they are properties. To study these properties, we can therefore ignore those specific kinds of matter. Mathematical objects are then abstracted by the estimative faculty from all specific kinds of matter. Nonetheless, it does not mean that they are studied *as if* they are not material properties. Mathematics for Avicenna is a specific way of studying a very specific group of physical properties. One might oppose this rendition of Avicenna's view by putting forward that mathematical objects are mental objects that are constructed by the abstraction mechanism. The quantity investigated by mathematics, the objector might discuss, is not the extramental quantity.[1] However, I do not find this position convincing. Mathematical abstraction for Avicenna is not a machinery for creating the objects that otherwise do not exist. It is a cognitive mechanism which provides us with a suitable conceptual framework for thinking about some specific physical properties in a specific way. But the objects of mathematics are those physical properties themselves, rather than their conceptual/mental counterparts constructed by the abstraction mechanism. The existence of mathematical objects does not depends on the human mind.[2]

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important roles in Avicenna's discussion of classification of the sciences but there is no parallel division with respect to (im)movability.

[1] There is a parallel debate on Aristotle's ontology of mathematics. Mueller (1970, 1990) provides a literalist interpretation of Aristotle's view about mathematical objects. Lear (1982), on the other hand, emphasizes on the role of mathematical abstraction in constructing mathematical objects.

[2] Admittedly, there are still some issues to be clarified. For instance, as I mentioned in the previous chapter, there are *peculiar* or *perfect/exact* geometrical shapes that can be (and are indeed) studied by mathematics but do not exist in the extramental world; i.e., there are no physical objects of those shapes. One might say, for example, that there is no perfect circle or no closed-shape-with-5326-straight-sides in the physical world, but these objects are (or, at least, can be) studied by mathematics. Therefore, the objector might conclude, that literalism is false. I think, however, that there are plausible answers for these objections in Avicenna's philosophy of mathematics. I dealt with some of these objections in the previous chapter. Some others will be discussed in the next chapter.

Avicenna thus believes that mathematical objects have some sort of dependency on or inseparability from materiality. However, for the case of geometrical objects (and magnitudes in general) this dependency is in some sense stronger than for numbers. Numbers, unlike magnitudes, are in principle separable from materiality itself. But numbers separated from materiality cannot be subject to increase and decrease. Consequently, they cannot be the subject of mathematical studies, and should therefore be studied in metaphysics. In other words, numbers, inasmuch as they are numbers, are not inseparable from materiality; but inasmuch as they are the subject of mathematical studies (i.e., inasmuch as they are the subject matters of mathematics) they should be considered as mixed with materiality. Numbers emancipated from any dependency on materiality are, therefore, the subject of metaphysical studies rather than mathematical studies. Otherwise put, Avicenna accepts that *numerosity* can in principle find a way into the domain of immaterial objects which are subjects of metaphysical studies; but he denies that this sort of numerosity can be the subject of mathematical studies.[1]

Contrary to numbers, however, magnitudes and geometrical shapes, inasmuch as they are themselves (and, *a fortiori*, inasmuch as they are the subject of mathematical studies), are inseparable from materiality. They cannot be detached from the material form. In other words, although there can be *numerous* immaterial objects, there cannot be any immaterial magnitude or, for example, immaterial triangle. This is the case, at least, if we interpret immateriality as meaning being separated from materiality itself, even from estimative matter, as well as from the material form; not, therefore, simply as being separated from the particular matters upon which they are predicated in the extramental realm. In brief, there is no place for magnitudes in the realm of immaterial objects. This contrast between the ontological status of numbers and magnitudes has some interesting consequences for Avicenna's views about the problem of infinity to which I return at the end of this chapter.

[1] Numbers separated from matter are not, according to Avicenna, receptive to decrease and increase. They are not capable of being subject to addition, subtraction, and other mathematical operations and, consequently, they cannot be the subject of mathematical studies. See Avicenna (2005, chap. I.3, secs. 17-18).

But insofar as our concern is merely with the relationship between mathematical and physical infinity, the aforementioned contrast between numbers and magnitudes has no importance. The only important thing is that mathematical objects, either numbers or magnitudes, inasmuch as they are the subject matters of mathematics, are (or, at least, should be considered as being) inseparable from materiality.[1] Therefore, it is necessarily true that there actually exists a mathematically infinite magnitude if and only if there exists a physically infinite object upon which that magnitude is predicated. Similarly, it is necessarily the case that there actually exists an infinite set of numbers if and only if there exists an infinite set of numbered physical objects upon which those numbers are predicated. In this sense, the problem of mathematical infinity for Avicenna is a special case of the problem of physical infinity.

3.3. Two Arguments against the Actuality of Infinity

Avicenna investigated the problem of infinity extensively, in all of his main encyclopedic works as well as in several other places.[2] He proposed several arguments, some of which are more faithful to the structure of Aristotelian arguments against an actual infinity. Almost all of Aristotle's arguments are based on (a) the application of physical notions such as *motion* and *traversability*, and (b) the presupposition of certain Aristotelian doctrines in physics and cosmology. As a result, they hold little interest for someone who is looking at the problem of

[1] For two alternative interpretations of Avicenna's views about the nature of mathematical objects see Ardeshir (2008) and Tahiri (2016), who believe that mathematical objects are 'mental' (Ardeshir's term) or 'intentional' (Tahiri's term) objects completely separated from materiality. The arguments of the previous and present chapters show that these interpretations are not sufficiently accurate.

[2] These are some of the places in which Avicenna *directly* discusses the problem of infinity: (1) *The Physics of the Healing* (2009b, chaps. III.7-11), (2) *The Physics of the Salvation* (1985, chap. IV.2, pp. 244-252), (3) *The Physics of Pointers and Reminders* (1957, chap. I.11, 160-167), (4) *The Metaphysics of ʿAlāʾī Encyclopedia* (1952d, chap. 16, pp. 58-61), (5) *The Physics of Fountains of Wisdom* (1980, chap. 3, pp. 19-20), and (6) *The Letter to the Vizier Abū Saʿd* (2000, pp. 27–36). As we will mention shortly, there are other places in which Avicenna *indirectly* considers this problem.

infinity from a purely mathematical perspective.[1] Moreover, most of those Aristotelian physical or cosmological presuppositions have lost plausibility for modern readers. This is why contemporary philosophers of mathematics usually avoid discussing the details of Aristotle's argument against mathematical infinity.[2] By contrast, in addition to some (by and large Aristotelian) physical arguments, Avicenna proposed some mathematical arguments in which he does not appeal to physical notions and Aristotelian presuppositions. In this section I briefly review a physical argument in which Avicenna appeals to the notion of *circular motion*, and a mathematical argument. In the fourth section I focus on another mathematical argument, this being Avicenna's main argument against the actuality of mathematical infinity, which includes the introduction and application of some mathematically significant notions and methods.

3.3.1. The Collimation Argument

There are some places in which Avicenna provides an indirect discussion of the problem of infinity. For example, in his discussions of the void in *The Physics of the Healing*[3] and *The Physics of the Salvation*,[4] he appeals to the impossibility of circular motion in an *infinite* void as one of his premises in arguing for the impossibility of the void. To justify this premise, he proposes an auxiliary argument that is known as *The Collimation Argument* (*burhān al-*

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[1] It is true that, according to Avicenna, ontology of mathematics somehow depends on the ontology of physics, but this does not entail either that methodology of mathematical studies is the same as that of physical studies or that every physical notion has something to do with mathematics. We can look at mathematical properties of physical objects by employing a methodology which does not appeal to some physical notions such as mass, weight, and motion. In this sense, physical arguments can be separated from mathematical arguments, even if mathematical ontology cannot be entirely detached from physical ontology.

[2] For example, David Bostock's reluctance to discuss these arguments was expressed in this way: 'I shall not rehearse his [i.e., Aristotle's] arguments [for the claim that there is a definite limit even to the *possible* sizes of things], which—unsurprisingly—carry no conviction for one who has been brought up to believe in the Newtonian infinity of space. I merely note that this is his view' (2012, pp. 479–480).

[3] Avicenna (2009a, chap. II.8, sec. 8).

[4] Avicenna (1985, chap. IV.2, pp. 233-244).

musāmita) or *The Parallelism Argument* (*burhān al-muwāzāt*). This argument likely originates in Aristotle's *De Caelo*.[1] Aristotle's original argument was proposed to show that the infinite 'cannot revolve in a circle; nor could the world, if it were infinite.' Avicenna extensively revised this argument to show, primarily, that circular motion in an infinite void is impossible. Coupling this result with the claim that the void, if it exists, cannot be finite, Avicenna concludes that the void does not exist.[2] But in other places such as *The Physics of Fountains of Wisdom*,[3] he also proposed this argument as an independent argument against the actual infinitude of intervals (*ab'ād*). The argument goes as follows:

Consider the line **L** which is infinite in one direction; it starts from the center **O** of a finite circle **C**, intersects the circumference of the circle, and extends infinitely. Consider, moreover, another line **L′** which is parallel to but distinct from **L**, and extends infinitely in both directions. Now, suppose that the circle **C** together with **L** start to rotate around **O**, while **L′** remains motionless and fixed. As a result of this circular motion, these two lines intersect. Therefore, there is a moment of time in which these lines are parallel and there is a moment of time in which they intersect with each other. From this fact, Avicenna concludes that there should be a moment of time **T** and, accordingly, a point **P** on **L′** in which these lines intersect each other for the first time (after the beginning of the circular motion). But there is obviously no such point. For every point **P** which we consider as the first intersection point of these lines, there are infinitely many points on **L′** prior to **P** which would have been passed and intersected by **L** (Fig. 1). Since Avicenna believes that circular motion undeniably can happen, he concludes that what should be rejected is the existence of infinite lines and intervals.[4]

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[1] *De Caelo* I.5, 272a8-20.

[2] For an explanation of this argument, as it appeared in *The Physics of the Healing*, see McGinnis (2007a).

[3] Avicenna (1980, chap. 3, p. 20).

[4] A variation of this argument was employed by Abū Sahl Al-Quhī (940-1000) to show that a principal characteristic of Aristotelian infinity, i.e., the claim that 'the infinite magnitude will not be traversed in a finite time' (*Physics* VI.7, 238a20-31), is wrong. I will shortly sketch how this argument can show that the infinite is traversable in a finite time. Rashed (1999) has discussed the details of Al-Quhī's argument.

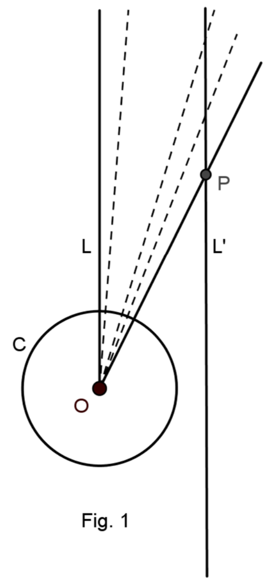

Fig. 1

Although this argument is intuitively powerful, it is not convincing. The argument is not only invalid from the point of view of modern mathematics, but is also incompatible with some of Avicenna's own views. From the fact that there is a moment of time in which **L** and **L′** are parallel and there is another moment in which they intersect, we cannot conclude that there is a moment in which these lines intersect with each other for the first time, nor that there is a point on **L** in which these lines intersect for the first time. Here Avicenna seems to suppose that the set of all temporal moments in which **L** and **L′** are intersected has a least element (with respect to the natural order and succession we consider for temporal moments). Correspondingly, there is a first point of intersection on these lines.[1] However, this supposition is false. It is not the case that every subset of temporal moments or every subset of the set of points on a straight line has a least or first element.[2]

[1] By 'the first point of intersection' I mean the point on which those lines intersected each other for the first time.

[2] It is accepted in modern mathematics that the set of points on a straight line that is infinite in both directions and the set of temporal moments with their own natural orders and successions of their elements are isomorphic to the set of real numbers with its natural order. The above supposition can therefore be

Interestingly, while Avicenna endorses the above supposition in the context of *The Collimation Argument*, he seems to reject it in his discussion of time and linear motions. If the above supposition were true, we could argue similarly against the possibility of even *linear* motions on *finite* spatial intervals. Consider a finite interval **AB**. Suppose that an object **X** is first motionless on **A**, then it moves and goes without stopping from **A** to **B**, and finally stops on **B**. Therefore, there is a time in which **X** is at rest and there is a time in which **X** is in motion. Now, if every subset of the points on a line (a spatial or temporal interval) had a least member, then there would be a first intermediary point of **AB** on which **X** does not stop (i.e., it merely passes that point). Correspondingly, there would be a moment in which **X** is on an intermediary point for the first time. But, given the continuity of time, space and motion—which Avicenna holds—there is neither such point nor such moment.[1] Therefore, if the supposition under discussion were true, we should conclude, in a way parallel to what we had in *The Collimation Argument*, that it is impossible for **X** to move from **A** to **B**. It means that not only is circular motion in an infinite space impossible, but linear motion on a finite magnitude is impossible as well. However, Avicenna accepts, not surprisingly of course, that continuous linear motion is possible. It indicates that he should reject the supposition that every subset of temporal moments or every subset of points on a line has a least or first element. He does so, indeed. McGinnis has elegantly shown that, according to Avicenna's theory of time, 'if one takes some instant ***t*** as a limit, then for any other instant ***t´***, no matter how close one wants to take ***t´*** to ***t***, then there is another instant ***t´´*** that is not identical with ***t***, but is closer to ***t*** than ***t´***. Since this same analysis will be true of ***t´´***, ***t´´´*** and so on, one can get indefinitely close to ***t*** without actually being at ***t***.'[2] It means that the set of all temporal moments between ***t*** and ***t´*** has no least number. So the aforementioned supposition is

paraphrased in the language of modern mathematics as the claim that every subset of real numbers has a least element with respect to their own natural order. Equivalently, the natural order on real numbers makes this set *well-ordered* in its technical sense. However, this claim can be mathematically proven to be false.

[1] See Avicenna's discussions of the continuity of magnitudes and motions in, for example, *The Physics of the Healing* (2009b, chaps. III.2 and IV.8).

[2] McGinnis (2004, p. 60).

rejected.[1] But rejecting this supposition renders *The Collimation Argument* invalid. It seems therefore that this argument is controversial even with respect to Avicenna's own philosophical framework. At least, Avicenna owes us an explanation of why this argument is based on a supposition that he rejects in another context.

Avicenna could however slightly revise this argument such that it validly entails the impossibility of the actual infinity, at least in an Aristotelian framework which would be acceptable for him. Avicenna accepts, following Aristotle, that the infinite cannot be traversed in a finite time.[2] Therefore, he could argue that if the circle **C** is finite, then it takes a finite time for it to rotate once around **O**. Accordingly, it takes a finite time for **L** to rotate once around **O**. But in each round of rotating around **O**, **L** traverses the whole line **L′**. This means that the infinite length of **L′** can be traversed in a finite time (equal to half the time of **L**'s rotating around **O**). This argument shows that the conjunction of (a) the principle that the infinite cannot be traversed in a finite time, (b) the possibility of having infinite intervals or lines, and (c) the possibility of circular motion, entails a contradiction. Avicenna could, therefore, argue that since (a) and (c) are obviously true (according to him), (b) must be rejected; there is no actually infinite interval. This argument seems perfectly sound, at least in an Aristotelian framework. However, Avicenna did not propose such a revised version of *The Collimation Argument*, and, as I mentioned, his own original version is problematic.

A more careful inspection of the original version of *The Collimation Argument* (especially an inspection of how this argument is related to the argument I proposed against the impossibility of linear motion) reveals some interesting aspects of Avicenna's understanding

[1] Considering the isomorphism between the structures of time, space and motion, the rejection of the aforementioned supposition with respect to one of them entails its rejection with respect to the others. For discussions of Avicenna's theories of time and motion, see, respectively, McGinnis (1999) and Ahmed (2016). Ahmed has explicitly pointed out that Avicenna rejects that supposition with respect to motion—i.e., he holds that there is no first part of motion (2016, p. 236, n. 50).

[2] Avicenna (2009b, chap. III.4, sec. 1, and chap. III.8, secs. 5-6).

of the notion of continuity; but discussing this issue would take us too far from our main concerns.[1] I turn, therefore, to another argument against the actuality of infinite magnitudes.

3.3.2. The Ladder Argument

The idea of *motion* plays an important role in *The Collimation Argument*; that argument should therefore be categorized as a physical argument in the aforementioned sense. Now we briefly review a mathematical argument against the actuality of infinite intervals that does not appeal to such physical notions. This argument, known as *The Ladder Argument* (*burhān al-sullam*), is first proposed in *The Physics of the Healing*[2] as a potential rehabilitation of one of Aristotle's physical arguments in *De Caelo*.[3] *The Ladder Argument* is also Avicenna's only argument against the actuality of infinity in *Pointers and Reminders*.[4] The argument, as appeared in *The Healing*, goes as follows:

> **TEXT # 3.1.** Let us posit a certain interval between two opposite points on two lines extending infinitely. Now, let us connect the [points] by a line that is a chord of the intersecting angle. So, because the extension of the two lines, which is infinite, is proportional to the increase of the interval [that is, the length of the chord], the increases to that interval are infinite. [Those increases] can also exist together equally, because the increases that are below will actually be joined to those that are above. For instance, the [amount that] the second increases the first will belong to the third, together with any other increase. So the infinite increases must actually exist in one of the intervals, and that is because the increases actually exist, and every actual

[1] After Avicenna, many influential figures in Islamic philosophy discussed this argument, and from many different perspectives. For example, Abū 'l-Barakāt Al-Baghdādī (1080-1165), Naṣīr Al-Din Al-Ṭūsī (1201-1274), and Al-Ḥillī (1250-1325) criticized the argument. On the other hand, Fakhr Al-Din Al-Rāzī (1149-1209) and Mulla Ṣadrā (1572-1640) defended the argument.

[2] Avicenna (2009b, chap. III.8, secs. 5-7).

[3] *De Caelo* (I.5, 271b26-272a7).

[4] Avicenna (1957, chap. I.11, pp. 160-167).

increase will exist and so will belong to a certain one [of the intervals]. In that case, it necessarily follows that some interval will exist in which there is an actual infinity of equal increases. So that interval would increase the first finite [interval] by an infinite [amount], in which case there would be an infinite interval [...]. This infinite can exist only between two lines, in which case it is finite and infinite, which is absurd.[1]

To have a more diagrammatic understanding of this argument, suppose that **L** and **L′** are two distinct lines that start at the same point **A** and extend infinitely to make an acute angle with infinite sides. Now, consider two arbitrary points $\mathbf{D_1}$ and $\mathbf{E_1}$ on **L** and **L′** respectively. As a result, $\mathbf{D_1E_1}$ is an interval which lies between **L** and **L′**. Furthermore, consider all intervals $\mathbf{D_2E_2}$, $\mathbf{D_3E_3}$, $\mathbf{D_4E_4}$, etc., parallel to $\mathbf{D_1E_1}$, such that $\mathbf{D_i}$ and $\mathbf{E_i}$ (for every natural number $1\leq\mathbf{i}$) lie respectively on **L** and **L′** and the difference between the lengths of every two consecutive intervals is constant. If we suppose that this difference is **d**, then for every natural number $1\leq\mathbf{i}$, $\mathbf{D_{i+1}E_{i+1}} - \mathbf{D_iE_i} = \mathbf{d}$. So we have a hierarchy of intervals in which every interval is formed by adding an interval of the length **d** to the previous interval.[2] For short, every interval is formed by an *increase* to the previous interval. Therefore, every interval is formed by a number of increases to $\mathbf{D_1E_1}$. Moreover, if an increase belongs to an interval, then all the previous increases belong to the same interval too. As a result, every interval somehow includes all the previous intervals. For instance both the first and the second increases belong to $\mathbf{D_3E_3}$ and, in a sense, it includes both $\mathbf{D_1E_1}$ and $\mathbf{D_2E_2}$, i.e.,

$$\mathbf{D_3E_3} = \mathbf{D_2E_2} + \mathbf{d} = \mathbf{D_1E_1} + \mathbf{d} + \mathbf{d}.$$

The number of the increases is infinite, and all of them actually exist, since otherwise there is an upper bound for the lengths of $\mathbf{D_iE_i}$ and, as a result, **L** and **L′** would be finite. From these premises, Avicenna concludes that there should be an interval **BC** (in such a way that **B** and **C** lie respectively on **L** and **L′**) which includes all of the infinitely many increases. **BC** is,

[1] Avicenna (2009b, chap. III.8, sec. 7).

[2] To be more precise, each interval is formed by adding an interval of the length **d** to *a copy of* the previous interval. However, Avicenna does not consider any difference between an interval and its copies. For the sake of simplicity, I follow the same practice in my discussion on *The Ladder Argument*. So instead of saying that, for example, the interval $\mathbf{I_2}$ includes *a copy of* the interval $\mathbf{I_1}$, I simply say that $\mathbf{I_2}$ includes $\mathbf{I_1}$.

therefore, larger than any $\mathbf{D_iE_i}$ we consider. It should be itself infinite. However, **BC** is restricted to **L** and **L′**, and terminates at them, which indicates that it is finite (Fig. 2). Consequently, it is both finite and infinite. Contradiction. Ergo, there are no infinite lines such as **L** and **L′**.

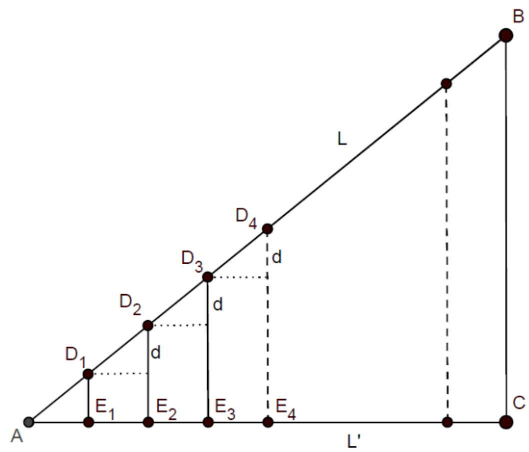

Fig. 2

This argument again seems to be controversial. It is based on three premises:

(1) Every increase belongs to an interval. In other words, for every increase there is a $\mathbf{D_iE_i}$ to which the increase belongs.

(2) If an increase belongs to an interval, then all of the previous increases also belong to the same interval. In other words, if the $\mathbf{n^{th}}$ increase belong to $\mathbf{D_iE_i}$, then all the first, the second, ..., and the $\mathbf{(n\text{-}1)^{th}}$ increases also belong to the same interval.

(3) All of the infinitely many increases actually exist.

From these premises, which seem to be uncontroversial, Avicenna then concludes that:

(4) The infinite increases all together must actually exist in one of the intervals. In other words, some interval will exist in which there is an actual infinity of equal increases.

It is, however, far from clear how (4) can be validly entailed from premises (1)-(3). These premises indicate that there exists an actually infinite hierarchy of finite intervals with increasing lengths. Each interval is longer than the previous by the amount **d**, and each interval is formed by a finite number of increases to **D_1E_1**. Nonetheless, these facts do not seem to imply that there is an actually infinite interval which includes all the increases.

As I mentioned earlier, *The Ladder Argument* is discussed in *The Physics of the Healing* as a rehabilitation of one of Aristotle's physical arguments which Avicenna finds weak. However, it seems that *The Ladder Argument* is still vulnerable to one of the very objections that he himself puts forward against Aristotle's argument. Roughly speaking, Aristotle's argument goes as follows:[1] Consider a finite circle in an infinite space and suppose that it takes a finite time for it to rotate once around its center. Suppose, moreover, that two distinct radii of this circle are extended infinitely. Now, Aristotle argues that there should be an infinite interval—e.g., an infinite arc parallel to the circumference of the circle—between these two radii. But if so, this infinite interval would be traversed in a finite time by one of the two infinitely extended radii during the rotation of the circle. This however leads to a contradiction, since the infinite is not traversable in a finite time. Given the assumption that circular motion is possible, Aristotle concludes that space cannot be actually infinite. Consequently, there is no magnitude. In criticizing Aristotle's argument, Avicenna says:

> **TEXT # 3.2.** It is not the case that an infinite interval [[e.g., an infinite arc]] must occur [[between two infinitely extended radii of a circle]]; rather, the increase [[of the lengths of the intervals between those radii]] will proceed infinitely, where every increase will involve one finite [amount] being added to another, in which case the interval will be finite. This is just like what you learned concerning number—namely, that [number] is susceptible to infinite addition, and yet, any number that occurs is finite without some number actually being infinite, since any given number in an infinite sequence exceeds some earlier number [in that sequence] only by some finite [number].[2]

I think, however, that the same point can be made concerning *The Ladder Argument*. By borrowing Avicenna's own wording, we can say: The increase of the length of the intervals

[1] See Aristotle's *De Caelo* (I.5, 271b26-272a7) and Avicenna (2009b, chap. III.8, sec. 5).

[2] Avicenna (2009b, chap. III.8, sec. 6). Words inside the single and double-square-brackets are added by McGinnis and me respectively.

D_iE_i will proceed infinitely, where every increase will involve one finite interval of the length **d** being added to another finite interval, in which case the outcome will be finite. Therefore, although we have actually infinite hierarchy of finite intervals with increasing lengths, it is not the case that there is an actually infinite interval terminated at **L** and **L′** which includes all of the intervals **D_iE_i**. This is exactly what happens in the case of numbers. The supposition that all numbers actually exist does not imply that there actually exist a number which is infinite. Even if the chain of consecutive numbers extends infinitely and all numbers actually exist, this does not entail that infinity itself is a number and occupies a place in the chain of numbers. Similarly, the actual existence of infinitely many intervals terminated at **L** and **L′** does not necessarily entail the actual existence of an infinite interval terminated at these two lines.

It is not clear how Avicenna intended to save *The Ladder Argument* from the above objection. This is why the soundness of this argument was the subject of a longstanding discussion in the post-Avicennan Islamic philosophy.[1] But regardless of this historical debate, the argument does not seem to be convincing for modern readers. From the perspective of contemporary mathematics, the above objection is quite compelling and reveals a fatal flaw in *The Ladder Argument*. Avicenna seem to believe that the existence of an infinite plane entails the existence of an infinite triangle **ABC** whose sides are infinite. He argues, then, that this is impossible. On the one hand, all three sides of this triangle should be infinite; so, **BC** is infinite. On the other hand, **BC** is restricted to **AB** and **AC**, since it terminates at **B** and **C**; so, **BC** is finite. Consequently, **BC** is both finite and infinite. Contradiction. Ergo, there is no infinite plane and no infinite interval at all. However, this is unsound. The existence of an infinite plane does not imply the existence of an infinite triangle, or any other infinite shape with a closed boundary on a plane. Therefore, even this mathematical argument—which, unlike *The Collimation Argument*, does not appeal to physical notions such as motion—does

[1] For example, Abu'l-Barakāt Al-Baghdādī criticized the argument, and Naṣīr Al-Dīn Al-Ṭūsī and Mullā Ṣadrā defended it. See McGinnis 2018 for a detailed discussion of the different aspects of the historical debate concerning *The Ladder Argument*.

not work, at least from our modern point of view. I now turn to Avicenna's main argument for the non-actuality of mathematical infinity.

3.4. Avicenna's Main Argument: The Mapping Argument

Concerns may be raised about the aforementioned arguments. One of them has to do with the genuine relation of these arguments to the problem of mathematical infinity. Considering the *contexts* of these arguments (which appear mostly in the *Physics* parts of Avicenna's works), it might seem that the target of these discussions is *exclusively* a rejection of the actual existence of infinitely large bodies (or a rejection of the infinity of the world). Therefore, one might conclude that these discussions have no decisive outcome for the problem of *mathematical* infinity. One might claim, therefore, that although Avicenna rejects the actuality of physical infinity, we have no evidence to suppose that he does not accept the actual mathematical infinite. We cannot say based merely on these arguments that Avicenna rejects the actual existence of an infinite set of numbers or an infinitely long line (as an object of geometry).

Admittedly, there are some phrases that *might* motivate an interpretation of Avicenna as believing that mathematical infinity should not be discussed in *Physics*. For example, he says: '*fa'inna al-naẓar fī al-'umūr ghayr al-ṭabīʿīya, wa annahā hal takūn ghayr mutanāhiya fī al-ʿadad aw fī al-quwwa, aw ghayr dhālik, falays al-kalām fīhā lāʾiqa bihādhā al-mawḍiʿ*.'[1] Jon McGinnis has translated this phrase as, 'For now, this [i.e., *The Physics of the Healing*] is not the place to investigate things outside of natural philosophy—that is, to discuss whether there is an infinite with respect to number, power, or the like.' McGinnis adds, in a footnote, that 'the proper place for such a discussion would seem to be the science of metaphysics, and while Avicenna has no appreciable discussion of the infinite in number in book 3 of his *Ilāhīyāt*, which is his most extended account of the philosophy of mathematics, he does have scattered, extended discussions of the infinite in book 6 (particularly chapters 2 & 4) where

[1] Avicenna (2009b, chap. III.7, sec. 1).

he discusses causes.'[1] Therefore, it seems that McGinnis interprets Avicenna as believing that *Physics* is not the proper place for discussing mathematical infinity.

Despite my undeniable debt to McGinnis's works on Avicenna, my interpretation differs from his. My discussion in the second section shows that mathematical infinity is not something entirely distinct from physical infinity. According to Avicenna, mathematical objects cannot exist independently from physical objects. Therefore, if the numbers and magnitudes of physical objects cannot be actually infinite (non-actuality of physical infinity), then numbers and magnitude, inasmuch as they are mathematical objects, cannot be actually infinite either (non-actuality of mathematical infinity). This is because, according to Avicenna, there is no number or magnitude fully separated from physical objects which can still be considered as a mathematical object (i.e., a subject of mathematical study). Therefore, the claim that the above arguments cannot be employed to attack the actuality of mathematical infinity seems to be untenable.

My disagreement with McGinnis's view arises, among other things,[2] from the different ways we translate the Arabic phrase cited. According to his translation, Avicenna believes that the infinity of numbers, which are subject matters of mathematics and therefore stand outside natural philosophy, should be discussed somewhere other than *Physics*. I think, however, that the phrase should be translated as something like this: 'This [i.e., *The Physics of the Healing*] is not the place to investigate non-natural (*ghayr al-ṭabīʿīya*) things, and to discuss whether they are infinite with respect to number, power, or the like.' Therefore, according to my translation, Avicenna simply claims that the infinity of non-natural things (which I understand to mean *things completely separated from matter*, not *things outside of natural*

[1] Avicenna (2009b, chap. III.7, p. 320, n. 1).

[2] One of these things may be his view about the nature of mathematical objects. McGinnis believes that Avicenna sees 'mathematical objects as mental constructs abstracted from concrete physical objects' (2006, p. 68), and that Avicenna 'invokes an account of conceptual analysis and mathematical objects that has certain affinities with the thoughts of some contemporary modal metaphysicians and mathematical constructivists or intuitionists (or perhaps better, "anti-Platonist mathematicians")' (2006, p. 64). I agree that Avicenna is anti-Platonist in his ontology, but he does not believe that mathematical objects are mental objects/constructs, or so it seems to me. Therefore, his ontology of mathematics cannot be interpreted as a constructivist ontology.

philosophy[1]) with respect to numbers (not *the infinity of numbers inasmuch as they are the subject matters of mathematics*) should be discussed elsewhere. As we will see, Avicenna believes that there are some infinite sets of fully immaterial objects. What he wants to clarify here, therefore, is simply that his argument against the actuality of infinity does not apply to the objects that are completely separated from matter and, in this sense, non-natural (*ghayr al-ṭabīʿīya*). It seems to me, therefore, that 'non-natural' does not refer to numbers and magnitudes which are the subject matters of mathematics and, therefore, attached to matter. As a result, the phrase quoted is not evidence against the idea that the afore-discussed arguments can be applied against the actuality of mathematical infinity.

There is yet another concern about *The Collimation Argument* and *The Ladder Argument*. It can be convincingly argued that the actual existence of an infinite number of physical objects entails the actual existence of an infinite interval; therefore, one of the indirect conclusions of these two arguments (albeit, if they were sound) could be to reject the actuality of numerical infinity. But it should be admitted that these arguments are *not* intended to be *directly* applied to the case of numerical (i.e., discrete) infinity. These arguments are primarily against the actuality of infinite magnitudes and intervals, rather than the infinity of numbers or the infinity of a set of numbered things. Now, one might wonder if Avicenna has proposed any argument which can be *directly* applied to the case of numerical infinity. Fortunately, the answer is positive.

Avicenna's main argument against actual infinity is *The Mapping Argument* or *The Correspondence Argument*[2] (*burhān al-taṭābuq* or *al-taṭbīq*), which is proposed and

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[1] Mathematical objects are, by definition, the subject matter of mathematics. Therefore, in this sense, they lie outside of natural philosophy. However, they are not completely separable from materiality. As a result, they are not entirely non-natural. Thus the claim that the (in)finitude of non-natural things should be discussed somewhere other than *Physics* has no immediate consequence for the problem of infinity and whether it can be discussed in *Physics*.

[2] Here I use 'mapping' and 'correspondence' synonymously.

discussed in many different places in his oeuvre.[1] This argument is a substantially revised version of an argument originally proposed by Al-Kindī (c. 801-873).[2] A significant advantage of *The Mapping Argument* over the two aforementioned arguments, and over Al-Kindī's original argument, is that *The Mapping Argument* can be applied simultaneously to both numbers and magnitudes. Almost whenever Avicenna mentions this argument, he explicitly states that he intends to show (by this argument) that both *numbers* and *magnitudes* (in addition to some other things) cannot be infinite. For example, in his discussion of this argument in *The Physics of the Healing*, Avicenna says:

> **TEXT # 3.3.** The first thing we say is that it is impossible that there exist as wholly actualized some unlimited (*ghayra dhī nihāya*) magnitude, number, or [set of] numbered things having an order in either nature or position (*waḍʿ*).[3]

Similarly, in the beginning of his discussion of *The Mapping Argument* in *The Physics of the Salvation*, Avicenna says:

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[1] See, for example, (1) *The Physics of the Healing* (2009b, chap. III.8, sec. 1), (2) *The Physics of the Salvation* (1985, chap. IV.2, pp. 244-245), (3) *The Metaphysics of ʿAlāʾī Encyclopedia* (1952d, chap. 16, pp. 58-60), and (4) *The Physics of Fountains of Wisdom* (1980, chap. 3, pp. 19-20).

[2] Al-Kindī's argument has appeared, with different wordings, in his *On First Philosophy* and three other essays which are exclusively dedicated to the discussion of infinity: (1) *On the Quiddity of What Cannot Be Infinite, and What is Said to Be Infinite*, (2) *On the Oneness of God and the Finiteness of the Body of the World*, and (3) *Al-Kinī's Epistle to Aḥmad ibn Muḥammad Al-Khurāsānī, Explaining the Finiteness of the Body of the World*. For an English translation of these works, see Al-Kindī (2012). Rescher and Khatchadourian (1965) have discussed Al-Kindī's views about mathematical infinity by translating and analysing the third essay. Shamsi (1975) provides a translation of the first essay and discusses Al-Kindī's views on the finitude of the world and the time. For a more detailed discussion of the various aspects of Al-Kindī's position on infinity, see Adamson (2007, chap. 4).

[3] Avicenna (2009b, chap. III.8, sec. 1). I have slightly revised McGinnis's translation. He has translated '*tartīb*' into 'ordered position', but I prefer to translate it simply as 'order'. The translation should not incautiously induce that there is a necessary connection between *being ordered* and *having a position* (*waḍʿ*); especially if, as McGinnis does (2010b, p. 217), one interprets *position* as one of concomitants that follows only upon matter.

TEXT # 3.4. I say that there does not arise an infinite continuous quantity (*kam muttaṣil* [= *miqdār* = magnitude]) that exists essentially possessing a position (*waḍʿ*); there is also no ordered infinite number that exists all together.[1]

It is particularly noteworthy that, in the first text, he distinguishes number (*ʿadad*) from numbered things (*maʿdūdāt*) and claims that number and magnitude (i.e., the subject matters of mathematics) cannot be infinite.[2] He does not, therefore, restrict himself just to those physical objects upon which numbers and magnitude are predicated. This shows that, in his discussions of *The Mapping Argument*, Avicenna 'deliberately' considers not only physical but also mathematical infinity—even if we suppose that the two aforementioned arguments are exclusively targeted at physical infinity. We should not be misled, therefore, by the fact that this argument appears mostly in the *Physics* parts of his works.[3] Nonetheless, there might still be some remaining concerns. In his discussions of TEXT # 3.4 McGinnis argues that:

> Avicenna consistently uses 'position' (*waḍʿ*) as one of the concomitants that follows upon matter, and in fact in the version of the proof as it appears in *al-Ishārāt wa al-Tanbīhāt*, Avicenna make[s] clear that the argument merely proves that "corporeal extension (*al-imtidād al-jismānī*) must be finite." In it[s] simplest terms the mapping argument, Avicenna seems to think, merely shows that there can be no material instantiation of an actual infinite.[4]

By appealing to this line of reasoning, one might claim that *The Mapping Argument* has nothing to do with numerical infinity. I disagree however. Avicenna's argument in *al-Ishārāt*

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[1] Avicenna (1985, chap. IV.2, p. 244); my translation. He repeats the same claim at the beginning of his discussions on this argument in *The Metaphysics of ʿAlāʾī Encyclopedia* (1952d, chap. 16, pp. 58-59) and *The Physics of Fountains of Wisdom* (1980, chap. 3, p. 19).

[2] See also Avicenna's *The Notes* (1973, 38) for the claim that numbers are not actually infinite, though they are potentially infinite.

[3] Its appearance in *The Metaphysics of ʿAlāʾī Encyclopedia* is an exception.

[4] McGinnis (2010b, p. 217). For the sake of consistency with my other transliterations in this dissertation, I have revised McGinnis's transliterations.

wa al-Tanbīhāt is definitely not a version of *The Mapping Argument*. As I mentioned above, it is a version of *The Ladder Argument*. Therefore, even if we accept that Avicenna's argument in that book can be applied merely to corporeal extension, this does not entail that the target of *The Mapping Argument* is restricted to the same thing, and thus it is not a direct argument against the actual infinity of numbers. TEXT # 3.3 and TEXT # 3.4, which appear in the introductions to Avicenna's discussions of *The Mapping Argument*, explicitly show that he believes that this argument can be applied not only to magnitudes but also to numbers.[1] Therefore, one of the main functions of *The Mapping Argument* is to reject the actuality of mathematical infinity—in its general sense—in a direct way.

The two aforementioned texts reveal some other important points about the function of *The Mapping Argument* to which we shall shortly return. Before touching on these points we should first clarify the structure of the argument.

3.4.1. The Structure of the Mapping Argument

Consider the straight line **AB** which starts from the point **A** and extends infinitely in the direction of **B**. **AB** represents a one-dimensional magnitude infinitely extended in one direction, or an infinite set of numbered objects possessing an order, the first element of which is placed on **A** while the other elements are successively lined up on some discrete

[1] As Euclid has shown in books 7-9 of his *Elements*, numbers can, in a sense, be constructed from magnitudes. They can be treated as the sets of units of magnitudes. Therefore, not only *The Mapping Argument*, but every argument against the actual infinity of magnitudes can, in principle, be considered as an *indirect* argument against the actual infinity of numbers. However, Avicenna seems to believe that, contrary to the other arguments he discusses, *The Mapping Argument* is applicable to the case of numbers in a more *direct* manner. This is why although he is silent about the applicability of the other arguments to the case of numerical infinities, he explicitly mentions that *The Mapping Argument* is applicable to the case of numbers and numbered things. I am thankful to an anonymous reviewer of *Archiv für Geschichte der Philosophie* for encouraging me to clarify this point.

points on the rest of **AB**.[1] Suppose that we take the finite part **AC** from **AB**. Avicenna argues that:

> **TEXT # 3.5.** [I]f some amount equal to **CB** were mapped on or parallel to **AB** (or you were to consider some other analogous relation between them), then either [**CB**] will proceed infinitely in the way **AB** does, or it will fall short of **AB** by an amount equal to **AC**. If, on the one hand, **AB** corresponds with **CB** [in proceeding] infinitely, and **CB** is a part or portion of **AB**, then the part and the whole correspond [with one another], which is a contradiction. If, on the other hand, **CB** falls short of **AB** in the direction of **B** and is less than it, then **CB** is finite and **AB** exceeds it by the finite [amount] **AC**, in which case **AB** is finite; but it was infinite. So it becomes evidently clear from this that the existence of an actual infinite in magnitudes and ordered numbers is impossible.[2]

Here, Avicenna argues that after taking the finite part **AC** from **AB**, we can compare the sizes of **CB** and **AB** by mapping (*aṭbaqa*) something equal (*musāw*) to the former on the latter, i.e., by mapping a copy of **CB** on **AB** in a way that the first point of the copy of **CB** corresponds with **A**. Avicenna uses the term '**CB**' (*jīm bā'*) equivocally in referring to both **CB** and its copy (i.e., the thing equal to **CB**). For clarity, we use the term **C*B*** to refer to the copy of **CB**. Therefore, Avicenna believes that we can compare the sizes of **CB** and **AB**, by mapping **C*B*** onto **AB** in a way that **C*** corresponds with **A**. After this mapping, either **C*B*** extends infinitely in the direction **AB** does (Fig. 3a) or **C*B*** falls short of **AB** (Fig. 3b). In the former case, **C*B*** corresponds with **AB**. But **C*B*** is equal to **CB**. Therefore, **CB** corresponds with **AB**. This entails that a whole (i.e., **AB**) totally corresponds with its part (i.e., **CB**). Avicenna

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[1] Before proposing the details of *The Mapping Argument* in the *Physics* part of *The Healing* (2009b, chap. III.8, sec. 1) and *Fountains of Wisdom* (1980, chap. 3, p. 19), Avicenna briefly argues that if something is infinite in more than one dimension or direction, then we can restrict ourselves to the first dimension and consider a single direction in which that thing is infinite. It seems that he wants to justify why *The Mapping Argument* is applied only to either one-dimensional magnitudes or numbers (or numbered things) ordered on a line and infinitely extended in one direction. If we show that nothing can be actually infinite in the chosen direction, then by generalization of this result we can claim that there is no direction in which something can be actually infinite. Unless otherwise mentioned, by 'magnitude', I mean only *one-dimensional* magnitude.

[2] Avicenna (2009b, chap. III.8, sec. 1). I have slightly revised McGinnis's translation.

believes that this is a contradiction. Now we should check the other horn. Suppose that **C*B*** falls short of **AB**. This means that **C*B*** corresponds with a part of **AB** which starts at **A** and terminates at a determinate point on **AB**. Call this latter point **D** (Fig. 3b). **AD** is an interval which terminates at two determinate points **A** and **D**; therefore, it has determinate limits and is finite. Now, since **B*C*** corresponds with **AD**, **B*C*** is finite too. On the other hand, the amount by which **C*B*** falls short of **AB** is equal to **AC,** for the fact that **C*B***=**CB** implies that **AB**-**C*B*** =**AB**-**CB**=**AC**. This means that the sum of **C*B*** and **AC** is equal to **AB**. But **C*B*** and **AC** are both finite. Therefore, their sum, which is equal to **AB**, is finite. As a result, we should accept that **AB** is finite. This contradicts our first supposition. Avicenna concludes that an infinity like **AB** cannot actually exist.

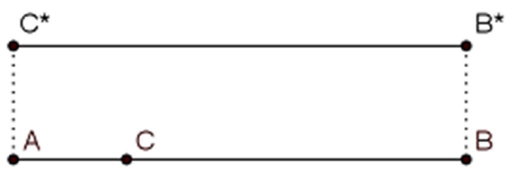

Fig. 3a

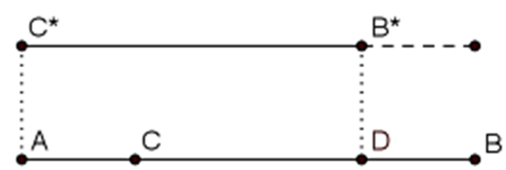

Fig. 3b

This clarification of *The Mapping Argument*, which brings some geometrical diagrams (such as Fig. 3) to mind, shows how this argument is intended to be applied to magnitudes. But the last sentence of TEXT # 3.5, in addition to the aforementioned evidence, confirms that Avicenna believes that this argument can also work perfectly against numerical infinity and show 'that the existence of an actual infinite in [...] ordered numbers is impossible.' Therefore, one might ask how this argument works in the case of numbers or numbered things. To the best of my knowledge, Avicenna has not explicitly replied to this question, at least not in his major works. He has astutely realized that the mapping technique can be employed against the actual infinity of numbers and numbered things, but it seems that his own *explanations* of the application of this technique are more perfectly matched to the case of magnitudes, rather than that of numbers. Fortunately, it is not very difficult to guess what he had in mind for the case of numbers.

If we consider **AB** as an infinite set of numbers or numbered objects possessing an order the first element of which is placed on **A** and the other elements are successively lined up on some discrete points of the rest of **AB**, then **AC** can be considered as the finite subset of the initial elements of **AB** placed from **A** to **C**. Accordingly, mapping **C*B*** (i.e., a copy of **CB**) onto **AB** pairs the first element of **C*B*** with the first element of **AB**, the second element of **C*B*** with the second element of **AB**, and so on (i.e., for every natural number n, pairing the n^{th} element of **C*B*** with the n^{th} element of **AB**). If this pairing procedure ends at some finite stage by pairing the last element of **C*B*** with an element of **AB**, this means that **C*B*** and consequently **CB** are finite. Therefore **AB**, which is the union of **AC** and **CB**, would be finite too (Fig. 4b). On the other hand, if the elements of **C*B*** extend infinitely, then it is possible to set a one-to-one correspondence between **C*B*** and **AB** by pairing every n^{th} element of the former with the n^{th} element of the latter. Therefore, **AB** corresponds with **B*C*** and, consequently, with **BC** (Fig. 4a). This means that a whole (i.e., **AB**) corresponds with one of its proper parts (i.e., **BC**). Since Avicenna sees the correspondence between a whole and its proper part as a contradiction, this line of argument can establish for him that there cannot be any actually infinite set of numbers or numbered things possessing an order.

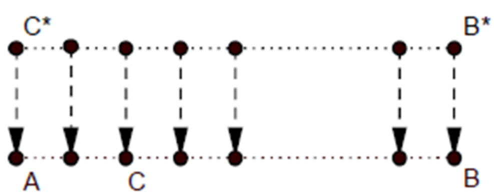

Fig. 4a

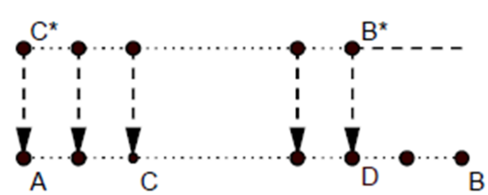

Fig. 4b

However, as far as I know, Avicenna himself has nowhere clarified the details of the application of *The Mapping Argument* to the case of numerical infinity. His explanations, as I discussed, highlight only the application of this argument to the case of geometrical infinity. He clearly claims that this argument rejects the actuality of numerical infinity, but he does not say how it really works in that case. Nonetheless, it does not seem implausible to accept that Avicenna had in mind something very similar to what is set out in the above paragraph. At least four different considerations support this claim. The first consideration is that the only natural way to develop the notion of mapping from the context of continuous geometrical magnitudes to the context of sets containing discrete elements seems to be interpreting mapping between these sets as one-to-one correspondence between their elements. Since Avicenna believes that the mapping technique can successfully work in the case of numbers and numbered things, it is highly probable that he has such a natural understanding of the notion of mapping in the context of numbers. The second consideration comes from an abstruse passage in *The Discussions* presenting a brief version of *The Mapping Argument* in response to a question probably raised by Ibn Zayla (c. 983-1048). It is argued there that it is possible to have two potential infinities of different sizes (i.e., one of them is bigger than the other). But it is impossible to have such infinities in actuality, because:

> **TEXT # 3.6.** When it [i.e., an infinity] became concurrent with and parallel to it [i.e., another infinity] in terms of connectedness or in terms of orderedness, or when it [i.e., one of those infinities] became a part of the other, then one of them would come to an end in one side and a remnant of one of them would remain at the other side. Therefore, the finitude of [… the shorter infinity] is necessary [and this is absurd].[1]

The passage seems to be saying that after mapping one of those infinities on the other, they become parallel to each other in terms of connectedness *or* in terms of orderedness (*muḥādhāt fī ittiṣāl aw fī tartīb*). This disjunctive phrase suggests that being parallel in terms of connectedness/continuity differs from being parallel in terms of order/orderedness. If the mapping technique was applicable only to continuous magnitudes, then we would have only one kind of parallelism. All continuous magnitudes are isomorphic to each other. Consequently, their being parallel to each other cannot be of more than one kind. Even if Avicenna thought that all parallelisms of magnitudes are of both mentioned kinds, he would have to use a conjunctive phrase rather than a disjunctive one. This he does not do. It indicates that the mapping technique is applicable to not only continuous magnitudes but also numerical infinities. The distinction between these two kinds of parallelism brings to mind the two different ways of the application of the mapping technique I explained above. By applying this technique to infinite magnitudes they become parallel to each other in terms of connectedness and continuity. By contrast, the application of this technique to the case of numerical infinities makes them parallel to each other in terms of order and orderedness.

[1] Avicenna (1992, VI, § 588). Avicenna accepts that potential infinities can be of different sizes. For example, time is potentially infinite and the time passed until last year is a potential infinity smaller than the time passed until this year. See section 4.3 for why *The Mapping Argument* is not applicable to time.

Therefore, 'parallelism in terms of orderedness' could be interpreted as Avicenna's term for the notion of *one-to-one correspondence*.

The third consideration supporting that Avicenna was aware of the application of the mapping technique to the case of numerical infinities is this: even early commentators on Avicenna (who are obviously closer than we are to the historical context of Avicenna's discussions) have had the same interpretation of the intended function of Avicenna's mapping argument in the case of numbers. For example, Fakhr Al-Din Al-Rāzī, in his commentary on Avicenna's *Fountains of Wisdom*, explains this function of *The Mapping Argument* along exactly the same lines as the interpretation mentioned above. Consider two sets of numbers: (1) A proper subset of natural numbers, for example, the set of natural numbers bigger than 10, and (2) the set of all natural numbers from one to infinity. Al-Rāzī writes that one can argue, based on the mapping technique, that:

> **TEXT # 3.7.** We put the first position of this [latter] set in front of the first position of that [former] set, and the second of this in front of the second of that, and so on successively. So, if the remainder does not appear [i.e., if by following this procedure nothing of the latter set remains unpaired], then the more is identical to the less. And, if the remainder appears at the end of the positions [of the former set, and some positions of the latter set remains unpaired], then it entails the finitude of number in the direction of its increase; and it is self-evident for the intellect that this is impossible.[1]

The text evidently shows that Al-Rāzī understands Avicenna's mapping notion to be one-to-one correspondence in the case of numbers and numbered things.[2] The plausibility of the

[1] Al-Rāzī (1994, p. 53).

[2] It should be mentioned, however, that Al-Rāzī does not believe that *The Mapping Argument* can successfully prove that the infinite set of natural numbers cannot actually exist. He thinks that although this argument works against the actual infinity of numbered things, it does not succeed in showing the non-actuality of the infinite set of natural numbers. He does so because he believes that (1) numbers are merely mental and completely separated from matter, and (2) *The Mapping Argument* has nothing to do with immaterial entities. It seems that Al-Rāzī attributes (wrongly, as I believe) this position to Avicenna too. Therefore, he does not interpret

attribution of this position to Avicenna is reinforced by a fourth consideration: the explanatory power of this interpretation as to why Avicenna emphasizes that possessing an order, in either nature or position, is a necessary condition for the applicability of *The Mapping Argument* in the case of numbers and numbered things. If we understand Avicenna as believing that correspondence between sets of discrete elements is one-to-one correspondence between their elements, then we have a very convincing explanation of why non-ordered sets of discrete objects cannot be the subject of *The Mapping Argument*. I come back to this issue in section 4.3 below.[1] Now it is time to discuss the significance of the insights behind the surface structure of *The Mapping Argument*.

3.4.2. Insightful Ideas behind the Surface Structure

An exhaustive analysis of Avicenna's explanation of *The Mapping Argument* reveals some remarkably interesting aspects of his treatment of the concept of infinity. The general structure of this argument is as follows: Avicenna argues that the actual existence of an infinity entails the equality of a whole to some of its parts. He denies the equality of whole and part, at least for physical objects and mathematical objects which are, as he believes, properties of physical objects. He concludes therefore that the infinite cannot exist in actuality. Before Avicenna, some other philosophers had correctly noticed that the existence

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Avicenna as believing that *The Mapping Argument* can successfully reject the non-actuality of the infinity of numbers. In fact, Al-Rāzī considers what is expressed by TEXT # 3.7 as an objection to the soundness of *The Mapping Argument*. According to this objection, if the form of *The Mapping Argument* is valid then one might reject the actual infinity of numbers by appealing to the lines of argument expressed by TEXT # 3.7. Al-Rāzī discusses this objection as the 11th question he proposes about *The Mapping Argument*. In his reply to this question, he says that this argument, though successful against numbered material things such as causes, does not work against the numbers themselves which are, according to him, completely mental and immaterial (Al-Rāzī, 1994, p. 57). My concern, however, is not Al-Rāzī's view about the nature of mathematical objects and the soundness of *The Mapping Argument* in the case of numbers. I merely want to highlight his understanding of how Avicenna's notion of *correspondence* should be interpreted in the case of numbers or numbered things.

[1] See note 1 on page 123.

of an actual infinity entails the equality of a whole to some of its parts;[1] and indeed, some of them had appealed to this fact to reject the actuality of infinity. For example, Al-Kindī followed such a line of reasoning in parts of his previously mentioned argument against infinity, from which *The Mapping Argument* was inspired.[2] Both Al-Kindī and Avicenna employed the mapping technique to compare the sizes of infinities. They however had two different understandings of the notion of *equality* (*tasāwī*). According to Al-Kindī, things with an equal distance between their limits (*ḥudūd*), or things whose dimensions between their limits are the same, are equal. This understanding is however controversial for three

[1] John E. Murdoch (1982, p. 569, n. 13) refers to some ancient sources (by philosophers such as Plutarch, Philoponus, Alexander of Aphrodisias, Proclus and Lucretius) in which this issue is discussed. However, I have suspicions that at least some of the cited arguments—for example, Philoponus's argument in his *De aeternitate mundi contra proclum* (2004, chap. I, sec. 3, pp. 24-25)—are not explicitly grounded on the absurdity of the whole-part equality. Philoponus believed that the Aristotelian doctrine that the universe has no beginning entails that the size of some infinities can (and indeed do) increase. The number of years until this year is an infinity smaller than the infinite number of years until next year. As a result, there is an infinity such that there exists something larger than it. Since Philoponus believed that the existence of something larger than infinity is absurd, he concludes that Aristotle is wrong about the infinity of the past. It may seem, at first glimpse, that there are some commonalities between the general structure of Philoponus's argument and that of *The Mapping Argument*. If the claim that there is nothing larger than an infinity can be construed as the claim that all infinities are of the same size, then we can understand Philoponus as arguing as follows: (1) if the past is infinite, then the set of the years until this present year and the set of the years until the next year are both infinite. (2) Since all infinities are of the same size, both of these sets must be of the same size. Nonetheless, (3) the former set is a proper subset of the latter; i.e., the latter set is a whole and the former is one of its parts. Therefore, (4) infinity of the past entails the equality of a whole to some of its parts. Finally, (5) since the equality of a whole to one of its parts is absurd, the infinity of the past is unjustified. This construal might lead one to think that the general structure of Philoponus's argument is, by and large, similar to those of Al-Kindī and Avicenna. But this view is not, in my idea, compelling. There is nothing in Philoponus's argument to show that he is attacking the equality of a whole and its part. He just wants to rule out the possibility of infinities of different lengths or of something outside an infinity. The sophisticated connection between this goal and the rejection of the equality of whole and part is not something made by Philoponus himself. He does not see the issue under this latter guise. For a detailed discussion of Philoponus's views against the Aristotelian doctrine of infinity and eternity, see Sorabji (2010). See particularly pp. 213-4 for the argument I have discussed here.

[2] See Al-Kindī (2012, pp. 20–21).

reasons. First, Al-Kindī defines equality in terms of itself. It seems therefore that the definition is question-begging and somehow uninformative. One can take the question of equality one step further and legitimately ask how the equality of the dimensions between the limits of things can be investigated. Al-Kindī seems not to have a non-circular convincing answer for this question. Second, it is not clear how this notion of equality can be applied to infinite magnitudes. If a magnitude is infinite, then it has no limit, at least in one direction. Therefore, the sizes of infinite magnitudes cannot be compared based on the equality of the dimensions between their limits. Third, even if we accept that the sizes of infinite magnitudes can be compared based on this understanding of the notion of equality, it is far from clear how this understanding can be applied to numerical infinities. Talking about the dimensions between the limits of infinite sets of numbers or discrete objects does not seem to be meaningful.[1] The latter point is one of the reasons why Al-Kindī's argument, despite having the same general structure as that of Avicenna, should not be interpreted as directly applicable to the case of numerical infinities.

Avicenna, on the other hand, employs the notion of *mapping* or *correspondence* (*taṭābuq*) to compare the sizes of infinities. The idea of understanding the notion of equality in terms of a *mapping* or *correspondence* relation goes back to Euclid. According to his fourth common notion in the first book of *The Elements*, 'things which fit onto/coincide/correspond with (*ta epharmozonta*) one another are equal to one another.'[2] This common notion is translated into Arabic in the so-called Isḥaq–Thābit version as '*allatī lā yufaḍḍal ʾaḥaduha ʿalā al-ākhar idha inṭabaq baʿḍuhā ʿalā baʿḍ fa-hiya mutasāwiya*.'[3] 'Things which none of them exceeds another, when some of them are mapped on some others, are equal.' Avicenna appeals to this simple principle to compare the sizes of different infinite magnitudes or different infinite

[1] See McGinnis (2010b) for a discussion on the shortcomings of Al-Kindī's definition of equality.

[2] In his translation of *The Elements*, Heath prefers to use the notion of *coincidence*. See Euclid (1908, vol. 1, p. 155).

[3] See a copy of the Isḥaq–Thābit version in the codex now numbered 6581/1 of the Majlis Shawrā Library in Tehran. The PDF version of this manuscript can be found here (accessed 19 October 2017): http://dlib.ical.ir/faces/search/bibliographic/biblioFullView.jspx?_afPfm=uzzsmt9nt
The essential features of this manuscript has been described by De Young 2015.

sets of numbers or numbered things.[1] He elegantly develops the application of this principle to the context of infinities. The significance of this strategy becomes more evident when we consider that most of the tools we usually use to compare the sizes of finite things are ineffective in the case of infinities.[2]

The equality of the sizes of two different sets of objects is usually understood as the equality of the numbers of their elements. Since these sets are finite, we can count and determine the number of their elements. If these numbers, which are obtained by two distinct counting procedures, are equal to each other, then we can say that those sets are of the same size; otherwise, they are of different sizes and the larger set is the one whose number of elements is larger. Similarly, we can measure the lengths of two different finite magnitudes separately and one by one, then we can compare the numbers obtained to decide whether they are equal or not. But it is obviously impossible to follow this approach in the case of infinities. Enumerating the elements of an infinite set or measuring the length of an infinite magnitude is impossible.[3] By definition, the sizes of infinities cannot be described by a finite number. Therefore, comparison between the sizes of different infinities must be grounded on something else: Avicenna's creative suggestion is returning to Euclid and borrowing his notion of *correspondence.* Avicenna shows that appropriate interpretations of this notion can be successfully applied to the case of infinities. I will return to this issue in section 4.3.

The Mapping Argument indicates, therefore, that the existence of an actual infinity entails the correspondence between that infinity and some of its infinite parts—which is, according to

[1] It does not mean, however, that Al-Kindī's definition of equality does not originate from the same principle. It could be an imprecise construal of this principle, perhaps based on an inaccurate Arabic translation of the Greek.

[2] I am extremely thankful to an anonymous reviewer of *Archiv für Geschichte der Philosophie* for drawing my attention to the possible link between Euclid's *ta epharmozonta* and Avicenna's *taṭābuq*.

[3] There are some studies—originating from Piaget (1952)—showing that the ability to put different sets of objects in one-to-one correspondence with each other is more fundamental than the ability to count. This means that we cannot count the number of the elements of even a finite set of objects without understanding the notion of one-to-one correspondence and without having the ability to put the elements of that set into a one-to-one correspondence with numerals.

Avicenna, simply contradictory.[1] A careful investigation of the notion of *correspondence*, especially when it applies to the infinite sets of numbers or numbered things, makes it clear that Avicenna's comprehension of the notion of infinity is much more modern than we might expect. In the previous section I argued that what Avicenna means by correspondence between two sets of numbers or numbered things is one-to-one correspondence between their elements, in such a way that every element of each of those sets corresponds with one and only one element of the other set. So, if I am right, Avicenna believes that on the one hand, (1) correspondence between sets is correspondence between their elements, and on the other hand, (2) the existence of an infinite set entails its correspondence with some of its proper subsets. The moral is that Avicenna sees infinite sets of numbers or numbered things as sets which are in one-to-one correspondence with some of their proper subsets. But this is exactly what Dedekind proposed in 1888 as a definition for infinite sets. Erich Reck's reading of Dedekind's definition is as follows:

> A set of objects is infinite—"Dedekind-infinite", as we now say—if it can be mapped one-to-one onto a proper subset of itself. (A set can then be defined to be finite if it is not infinite in this sense.)[2]

It is definitely surprising that more than eight centuries earlier, Avicenna had had a similar conception of the notion of infinity. This does not mean, however, that the above definition is Avicenna's own *definition* of infinity. He was aware, if I am right, that every infinite set of numbers or numbered things has the property of being in one-to-one correspondence with some of its proper subsets (borrowing the first letters of the names of Avicenna and Dedekind, I would like to call this property 'AD-property'), but he never explicitly proposed

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[1] See TEXT # 3.5 above. See also *The Physics of the Salvation* (1985, chap. IV.2, p. 245) where Avicenna, discussing *The Mapping Argument*, says that if the whole and the part 'correspond [with each other] in extension, then the greater and the lesser are equal; while it is absurd.'

[2] Reck (2016, sec. 2.2.). Dedekind's original definition is this (1963, p. 63): 'A system ***S*** is said to be *infinite* when it is similar to a proper part of itself [...]; in the contrary case ***S*** is said to be a *finite* system' (emphasis in the original). Before proposing this definition he clarifies, by some consecutive definitions and theorems, that what he means by 'similarity' between systems is exactly the existence of a one-to-one mapping between the elements of those two systems.

having this property as a definition for being infinite.[1] Another important dissimilarity between Avicenna and Dedekind's views concerning infinity is that Dedekind, contrary to Avicenna, does accept the actual existence of some infinite sets.[2] In other words, for Avicenna no actual thing can *instantiate* or *exemplify* the AD-property.[3] So, I do not want to exaggerate the commonalities between Avicenna's view and our post-Dedekindian, post-Cantorian understanding of infinity. My claim is merely that Avicenna had correctly noticed that every infinite set of objects has the AD-property, and this is enough to show that there is a strong connection between his views about infinity and ours. I do not know of any philosopher before Avicenna who was aware of this property of infinite sets.[4]

Jon McGinnis believes that, before Avicenna, the Ṣābian mathematician Thābit Ibn Qurra al-Ḥarrānī (d. 901) recognized the 'modern definition of infinity as a set capable of being put into one-to-one correspondence with a proper subset of itself.'[5] I disagree. To justify his position, McGinnis refers to two sections of a series of questions addressed by Abū Mūsā ʿĪsā Ibn Usayyid to Thābit Ibn Qurra in which Thābit argues that there is an infinite number of different sizes that an infinite set may have.[6] I believe, however, that a careful analysis of the

[1] As I mentioned at the beginning of the second section, Avicenna's definition of infinity is an Aristotelian definition according to which infinity is something that whatever we take from it, we always find something outside of it.

[2] Dedekind claims and tries to prove that his 'own realm of thoughts, i.e., the totality ***S*** of all things, which can be objects of [his] thought, is infinite' (1963, p. 64).

[3] One might therefore suggest that AD-property is not a *real* property for Avicenna; it is like the property *being a partner of God* that can never actually be exemplified.

[4] It has been argued by Netz *et al* (2001) that Archimedes was aware of the one-to-one mapping technique and implicitly employed it, in his *Method*, to show that two infinite sets of objects are *equal in multitude*. I have some doubts about the plausibility of this claim which cannot be discussed here. But even if this claim is true, it does not entail that Archimedes was aware—even implicitly—that infinite sets can be put in one-to-one correspondence with some of their subsets. Moreover, no evidence has been presented to show that his method is grounded on an explicitly intentional application of the notion of correspondence, as proposed by Euclid and consciously and deliberately employed by Avicenna.

[5] McGinnis (2010b, p. 211, n. 32).

[6] McGinnis refers to sections 13 and 14 from Sabra's translation of those questions (1997, pp. 24–25). For an earlier discussion of Thābit's views on numbers and mathematical infinity, see Pines (1968).

relevant passages to which McGinnis has referred reveals nothing confirming that Thābit recognized the idea of one-to-one correspondence and the AD-property of infinite sets. This quotation expresses the core of Thābit's claim and clarifies the structure of his argument:

> **TEXT # 3.8.** We [i.e., Ibn Usayyid and his friends] questioned him [i.e., Thābit] regarding a proposition put into service by many revered commentators, namely that an infinite cannot be greater than an infinite. – He pointed out to us the falsity of this (proposition) also by reference to numbers. For (the totality of) numbers itself is infinite, and the even numbers alone are infinite, and so are the odd numbers, and these two classes are equal, and each is half the totality of numbers. That they are equal is manifest from the fact that in every two consecutive numbers one will be even and the other odd; that the (totality of) numbers is twice each of the two [other classes] is due to their equality and the fact that they (together) exhaust (that totality), leaving out no other division in it, and therefore each of them is half (the totality) of numbers. – It is also clear that an infinite is one third, or a quarter, or a fifth, or any assumed part of one and the same (totality of) numbers. For the numbers divisible by three are infinite, and they are one third of the totality of numbers; [...] and so on for other parts of (the totality of numbers). For we find in every three consecutive numbers one that is divisible by three, [...] and in every multitude of consecutive numbers, whatever the multitude's number, one number that has a part named after this multitude's number.[1]

Here, Thābit is providing an elegant argument that there are infinities of different sizes. He argues that from every two consecutive numbers one is even and the other odd. Therefore, the totality of even numbers is equal to the totality of odd numbers; since the union of these totalities is equal to the totality of numbers, each of the totalities of even or odd numbers is equal to a half of the totality of numbers. By similar lines of argument, Thābit shows that the totality of numbers divisible by three is equal to one third the totality of numbers, and the totality of numbers divisible by four is equal to a quarter the totality of numbers, and so on. But he does not say anything confirming that any of those proper subsets of numbers are

[1] Sabra (1997, pp. 24–25).

equal to the totality of numbers; nor does he mention the idea of one-to-one correspondence. Even his argument for the equality of the set of even numbers to the set of odd numbers is not grounded on the one-to-one correspondence of the elements of these two sets. Paolo Mancosu is right to say:

> When ibn Qurra states that odd numbers and even numbers have the same size one should be careful not to immediately read his argument as being the standard one based on one-to-one correspondence, for the motivation adduced does not generalize to other arbitrary infinite sets. Rather, it would seem that some informal notion of frequency (how often do even numbers (respectively odd numbers) show up?) is in the background of ibn Qurra's conception of infinite sizes ("we find in every three consecutive numbers one that is divisible by 3").
>
> What would he have replied to the possible objection that there are as many even numbers as natural numbers based on a one-to-one correspondence between the two collections? *The text is silent on this issue*.[1]

Without doubt, Thābit's discussion has genuinely innovative aspects.[2] But he aims only at proving the existence of different sizes of infinite sets, not at showing the equality of infinite sets to some of their proper subsets by appealing to the notion of one-to-one correspondence. We do not have enough evidence, therefore, to claim that Thābit had recognized that infinite sets of numbers have the AD-property.[3] At least, the passages cited by McGinnis do not provide such evidence.

It is worth mentioning that neither Thābit nor Avicenna nor any other scholar before Georg Cantor recognized the (either actual or potential) existence of infinite sets of different

[1] Mancosu (2009, p. 615). My emphasis.

[2] Mancosu says that '[t]he first occurrence I know of a defense of the existence of different sizes of infinity given in terms of collections of natural numbers comes from the Islamic philosopher and mathematician Thābit ibn Qurra', and then he quotes TEXT # 3.8 (2009, p. 614).

[3] Despite this deficiency, Thābit's account of infinity has an important advantage over Avicenna's. By contrast to Avicenna who does not provide any explicit numerical example, Thābit grounds his discussions on some concrete examples of numerical sets.

cardinalities. From the standpoint of modern mathematics, two sets are of the same cardinality if there is a one-to-one correspondence between their elements. Therefore, the existence of infinite sets of different cardinalities means the existence of infinite sets which cannot be put in one-to-one correspondence with each other. Avicenna, as I argued, was aware that infinite sets of numbers can be put in one-to-one correspondence with some of their proper subsets, but he did not know that there are some infinite sets that cannot be put in one-to-one correspondence with the set of natural numbers.[1] So he did not know about infinite sets of different cardinalities. *A fortiori,* Thābit (who, as I showed, was not familiar with the notion of one-to-one correspondence) did not know about the different cardinalities of infinite sets. This means that Thābit's claim that the totality of even numbers is equal to the half of the totality of all numbers should not be understood as the claim that the cardinality of the former set is less than the latter's.[2]

In sum, Avicenna employs the notion of *mapping* or *correspondence* as a tool for the comparison of the different sizes of infinite magnitudes or infinite sets of numbers or numbered things possessing an order. By *The Mapping Argument* he shows that an infinite magnitude is necessarily in correspondence with some of its proper parts. He believes and indeed explicitly states that the argument is appropriate to the case of numerical infinities, but he himself does not discuss the details of this application. By (1) appealing to the most natural development of the notion of correspondence to the case of numbers and numbered objects, and (2) relying on some of Avicenna's own texts and some early commentaries on

[1] The fact that some infinite sets cannot be put in one-to-one correspondence with the set of natural numbers was proved by Georg Cantor in 1891. His proof is based on the so-called *diagonalisation* technique. The complexity of this technique was far beyond the boundaries of mathematical knowledge in the time of Avicenna.

[2] Jon McGinnis (2010b, p. 221), in explaining Philoponus's criticism of the Aristotelian doctrine of the eternity of the world, says: 'Such a thesis, argued Philoponus, violated a number of Aristotelian dicta concerning infinity, as, for example, the impossibility of an actual infinity being realized, an infinite's being traversed, the ability to increase an infinite as well resulting in infinities of varying *cardinality* [...]' (My emphasis. See also p. 205 of the same paper). I think, however, that this interpretation suffers from imprecision. As I mentioned in note 1 on page 110, Philoponus did argue about the different sizes of infinity, but what he means by 'different sizes of infinities' is by no means the same as 'infinities with different cardinalities' in its modern sense.

his works which confirm that Avicenna had such a development in mind, we can say that, according to Avicenna, correspondence between two sets of numbers or numbered things means nothing other than one-to-one correspondence between the elements of these sets. Coupling this fact with Avicenna's insistence on the applicability of *The Mapping Argument* to the case of numbers and ordered objects, we can conclude that he was aware, like Dedekind, that every infinite set of objects possessing an order has the AD-property. However, by contrast to Dedekind, he did not propose the having of this property as a definition for being infinite. Moreover, he did not know anything about infinities with different cardinalities, in the modern sense of the term 'cardinality'. Finally, contrary to Dedekind and most other modern set-theorists, Avicenna rejects the actual existence of infinite sets of numbers or numbered objects. He believes that, on the one hand, (I) every mathematical infinity corresponds with some of its proper subsets, and on the other hand, (II) it is impossible for a set of *actually existent* mathematical objects to correspond with some its proper subsets. Therefore, he concludes that it is impossible for a magnitude or a set of discrete mathematical objects to be infinite. His justification for (I) comes from *The Mapping Argument*, but he himself does not provide any justification for (II); he simply accepts it. I will argue, in the next two sections, that Avicenna's position regarding the ontology of mathematics justifies (II).

There is still another problem that we have not yet touched on. According to TEXT # 3.3 and TEXT # 3.4, *The Mapping Argument* works against the actual existence of infinite magnitudes and infinite sets of numbers or numbered things *having an order*. But we have not yet clarified why having an order is a necessary condition for the elements of an infinite set of objects to be the subject of *The Mapping Argument*. In the following section, I discuss this issue and show how it is related to the possibility of having an actual infinity of immaterial objects.

3.4.3. Non-Ordered Can Be Actually Infinite

In the following passage, from *The Physics of the Salvation*, Avicenna explicitly says that *The Mapping Argument* does not work against infinities such that either their parts do not exist totally together at the same time (e.g., the infinite time line and infinite motions) or they cannot be ordered (e.g., immaterial objects such as angels and devils):

> **TEXT # 3.9.** [There are two situations in which we may have infinites:] either when [(1) the totality of] the parts are infinite and do not exist all together—therefore, it is not impossible for them to exist one before or after another, but not all together—or when [(2)] the number [i.e., the set of numbered things] itself is not ordered in either nature or position—therefore, there is nothing preventing it [i.e., the non-ordered set of numbered things] from existing [with] all [its members] together. There is no demonstration for its impossibility, rather there is a demonstration for its [actual] existence. As for the first kind [of these infinites], time and motion are proven to be such. As for the second kind, [the existence of] a multiplicity of angels and devils—that are infinite with respect to number—is proven to us, as will become clear to you [too]. All of this [i.e., the set of angels and devils] is susceptible to increase, but this susceptibility does not make the [the application of] the mapping [technique] permissible; for what has no order in either nature or position is not susceptible to [the use] of the mapping [technique]; and [the use of this technique] in the case of what does not exist all together is even more impermissible.[1]

In this passage Avicenna discusses two conditions for the applicability of *The Mapping Argument*. Following McGinnis, I call these conditions respectively (1) the 'wholeness condition' and (2) the 'ordering condition'. According to the wholeness condition, *The Mapping Argument* is applicable to a set or totality only if its elements exist all together at the same time. According to the ordering condition, the argument is applicable to a set or totality only if its elements have an order in either nature or position. Avicenna seems to believe that both magnitudes and sets of numbers (or numbered things) must satisfy the

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[1] Avicenna (1985, chap. IV.2, 245-246). My translation. At least at first glance, there seems to be a tension between Avicenna's claim in this text that '[a]ll of this [i.e., the set of angels and devils] is susceptible to increase' and his view in *The Metaphysics of the Healing* that '[n]umber whose existence is in things separate [from matter] cannot become subject to any relation of increase or decrease that may occur but will only remain as it is' (2005, chap. I.3, sec. 17). However, discussing this issue would take us too far afield from our main concerns. See also note 2 on page 123 for a related issue.

wholeness condition to be the subject of *The Mapping Argument*. For him it is nonsense to speak about the (non-)equality of things—whether magnitudes or numbers or numbered things—that do not actually exist. This is why time cannot be the subject of the mapping argument. Time does not satisfy the wholeness condition because temporal moments do not exist simultaneously and all together. In each moment there actually exists only one point of the temporal line. Therefore, the infinity of the temporal line and the eternity of the world do not entail the *actual* existence of an infinity. Since temporal moments do not exist all together at the same time, it is nonsense for Avicenna to say that the whole of the temporal line corresponds with—and is therefore equal to—some of its parts. As a result, *The Mapping Argument* is not applicable to time and cannot reject its potential infinity.[1]

The case of the ordering condition is more complicated. Is having an order is a necessary condition for both numbers and magnitudes to be the subject of mapping argument? In his discussions of *The Mapping Argument* Avicenna repeatedly says that this argument rejects the actual infinity of magnitudes and numbers or numbered things having an *order* in either nature or position (*tartīb fi al-ṭabʿ aw al-waḍʿ*). Accordingly, in the above passage, he says that no argument, including *The Mapping Argument*, can reject the actual existence of an infinite set of non-ordered numbered things. It is therefore undisputable that, for Avicenna, the ordering condition is a necessary condition for numbers and numbered things to be the subject of mapping argument. Is it also a necessary condition for the case of magnitudes? McGinnis argues for a positive answer to this question. He believes that (1) there is a sense in which magnitudes are (or at least can be) ordered and (2) having an order is a necessary condition for magnitudes to be the subject of *The Mapping Argument*.[2] The former claim seems to be defensible. However, I think that there are reasons to be suspicious of the latter. To justify these claims, we should first have a clear understanding of what exactly Avicenna means by having an order. In the introduction to his discussion of *The Mapping Argument* in the *Metaphysics* part of *ʿAlāʾī Encyclopedia* Avicenna writes:

[1] Since Al-Kindī does not consider the wholeness condition as a necessary condition for the applicability of the mapping technique, he rejects the eternity of the world based on the very argument from which Avicenna's mapping argument is inspired.

[2] McGinnis (2010b, p. 218).

TEXT # 3.10. Beforeness and afterness (*pīshī wa sepasī*) is either by nature—as in number (*shumār*)—or by supposition (*farḍ*)—as in measures (*andāza-hā*) [i.e., one-dimensional magnitudes]—that you can start from any direction you may want. And, everything that either there is beforeness and afterness in it by nature or it is a magnitude (*miqdār*) which has parts that exist all together is finite.[1]

A comparison of this passage with TEXT # 3.3 and TEXT # 3.4 shows that Avicenna uses the Arabic phrase '*tartīb*' as an equivalent for the Persian phrase '*pīshī wa sepasī*'. It indicates that, for Avicenna, a set or totality of things have an order only if for every two members of this totality one of them is, in a sense, before (or after) the other. As Avicenna explicitly confirms in the above passage, numbers have a natural and essential order. This is because for every two numbers one of them is before or less than the other. One-dimensional magnitudes (i.e., lines), on the other hand, do not have a natural order. This is because they lack essential directionality. They can be traversed in two opposite directions. But as soon as we fix one of these directions, we have a beforeness/afterness (or priority/posteriority) relation between the parts of this one-dimensional magnitude. Therefore, every one-dimensional magnitude can in principle have two different suppositional orderings, depending on the direction we consider for it. This demonstrate (1); at least in the case of one-dimensional magnitudes. Nonetheless, it does not show that satisfaction of this condition is necessary for magnitudes to be the subject of *The Mapping Argument*. Indeed, although they are ordered, it is not in virtue of this orderedness that *The Mapping Argument* is applicable to them. Let me explain why.

According to Avicenna, correspondence between two geometrical magnitudes **L** and **L′** is nothing more than projection and coverage: **L** corresponds with **L′** if and only if it is possible to map **L** onto **L′** in such a way that no part of either **L** or **L′** remains uncovered. It seems, therefore, that the notion of correspondence could be understood by appealing merely to our geometrical intuitions without the aid of the notion of ordering (or, equivalently, beforeness/afterness). It is in virtue of the continuity and connectedness of geometrical magnitudes—as suggested by TEXT # 3.6—that the mapping technique and the notion of

[1] Avicenna (1952d, chap. 16, 58-59). My Translation.

correspondence in the sense of coverage and projection are applicable to them. Considering the case of two-dimensional magnitudes shows us that the application of the mapping technique to the case of magnitudes is not in virtue of their orderedness. The sizes of two-dimensional figures can be compared by defining equality in terms of coverage. Consider two triangles drawn on a paper. By mapping one of them on the other and checking whether or not they completely cover each other, we can compare their areas. But there is no obvious/natural order among the points of two-dimensional magnitudes/shapes. More precisely, the natural positions of the points on a two-dimensional magnitude do not impose a beforeness/afterness relation on them. Those points are not ordered by their natural positions. It shows that the applicability of the mapping technique in the sense of coverage and projection is something completely independent from orderedness in the sense of having beforeness/afterness. That is why Avicenna mentions the ordering condition only for the case of numbers. Orderedness does not play any crucial role in the applicability of *The Mapping Argument* to the case of magnitudes.

Quite differently, correspondence between two sets of discrete objects **A** and **B** is one-to-one correspondence between their elements.[1] But the notion of one-to-one correspondence cannot be reduced to geometrical projection and coverage; for the elements of **A** and **B** may be of different sizes, different shapes and have different places (if they have these properties at all). If **A** and **B** are ordered then we can compare their sizes by pairing the first element of **A** with the first element **B**, the second element of **A** with the second element of **B**, and so on (i.e., for every natural number n, pairing the n^{th} element of **A** with the n^{th} element of **B**). In fact, Avicenna seems to believe that the possibility of being ordered is a necessary condition for the possibility of being put in a one-to-one correspondence. He thinks that if a set of objects cannot be ordered, it cannot be put in one-to-one correspondence to other sets.[2] In other words, the possibility of picking elements of a set one-by-one to put them in a one-to-one correspondence with the elements of another set is equivalent to the possibility of

[1] As TEXT # 3.6 suggests, correspondence in the case of numerical infinities is being parallel in terms of orderedness.

[2] By generalizing this idea we may arrive at the modern idea of the priority of the notion of *ordinality* to the notion of *cardinality*.

putting an order on the elements of that set. Therefore, if it is impossible for a set to be ordered, then it is equivalently impossible for it to be put in a one-to-one correspondence. Accordingly, the size of such a set—that cannot be ordered—cannot be compared to the sizes of other sets, including its own proper subsets. As a result, it cannot be shown that that set corresponds with some of its subsets. So, no contradiction arises and *The Mapping Argument* does not work in such a case.[1]

Apparently, it is by following this line of argument that Avicenna claims that *The Mapping Argument* is not applicable to the case of immaterial objects such as angels and devils; for he believes that the set of those objects cannot be ordered. More precisely, they do not have a natural and essential order, nor it is possible to posit a conventional order for them (at least it seems inconceivable how they might have such an order). The actual infinity of such sets cannot be rejected by appealing to *The Mapping Argument*. Avicenna seems to believe that (1) other arguments against the actual infinity are of no use in the case of sets of discrete elements, and (2) there are some arguments for the actual infinity of the set of angels and demons. Consequently, he believes that such infinite sets are uncontroversially actual.[2]

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[1] On the one hand, the above explanation can precisely clarify why non-ordered sets cannot be the subject of *The Mapping Argument*. On the other hand, the plausibility of this explanation is itself dependent on the interpretation of the correspondence between sets of discrete elements as one-to-one correspondence between their elements. Since there seems to be no defensible rival for the above explanation, the explanatory power of this scenario about the role of ordering in *The Mapping Arguments* can itself be considered as evidence for the claim that, according to Avicenna's philosophy, the correspondence between two sets of discrete elements should be understood as a one-to-one correspondence between their elements.

[2] It should be noted that here we can find a serious problem for Avicenna's philosophy. The infinity of a set of immaterial objects depends on the numerical individuality of its elements. If its elements are not individuated, then it seems nonsense to believe in their numerosity and, *a fortiori*, their infinity. Therefore, if Avicenna accepts the actual existence of an infinity of immaterial individuals of the same kind, then he has no way out but to reject the Aristotelian principle that objects of the same kind are numerically individuated by their matters. This is one of Averroes's (1126-1198) criticisms of Avicenna. See Marmura (1960, sec. II, pp. 173-174).

But what will happen if—by contrast to the case of angels and devils—it is possible to put a conventional order on a set of immaterial objects? Can we still argue, based on the mapping technique, that such a set cannot be actually infinite?

For example, Al-Ghāzalī (1058-1111) argued that the eternity of the world implies the eternity of the species human being. This means that every moment of time is preceded by an infinite number of people who have died. Therefore every moment of time is preceded by an infinity of human souls who have been separated from matter but still actually exist. Moreover, Al-Ghāzalī believes that we can put a conventional order on (at least a subset of) the set of the souls who have been separated from matter until now.[1] Suppose that the first element of the set **A** is the last soul who was separated from her body before the present time, the second element of **A** is the last soul who was separated from her body before the end of yesterday, the third one is the soul who was separated from its body before the end of the day before yesterday, and so on.[2] The eternity of the world and human beings implies the actual existence of and the infinity of the set **A**. But, since **A** is ordered, its actual infinity can be rejected by *The Mapping Argument*. This is simply a contradiction. Based on this line of argument, Al-Ghāzalī concludes that Avicenna's doctrine of the eternity of the world and human being contradicts his rejection of the actual infinity.[3]

I think that we have a strong strategy to rebut this objection on behalf of Avicenna. Having an order guarantees that we can employ the mapping technique to argue that an infinite set of discrete objects corresponds with some of its proper subsets. But it cannot itself guarantee that such a correspondence is a contradictory conclusion. Avicenna can say that although the correspondence of a whole to its proper parts is a contradiction in the case of material

[1] The core of Al-Ghāzalī's idea is to produce an order based on temporal successions and regressions. Nonetheless, the order I will propose, though faithful to Al-Ghāzalī's general approach, is not identical to his proposal.

[2] For the sake of simplicity, I have assumed that (1) for every day there has been at least one soul who has been separated from her body, and (2) in every temporal moment at most only one soul has been separated from her body. Even if we reject these assumptions, we can propose some more sophisticated conventional orders for some infinite subsets of the set of immaterial souls who have been separated from their matter until now.

[3] For more discussion on the problem of the infinity of souls, see Marmura (1960).

objects and their properties (e.g., mathematical objects), such a correspondence is not controversial in the case of immaterial objects. In other words, the equality of the whole and its part is unacceptable for physical and mathematical objects, but it is acceptable for immaterial objects, because the mereology of the realm of materiality is not similar to that of the realm of immateriality. The principle of the impossibility of the equality of the whole and its proper part is not necessarily valid in the latter realm. Therefore, it is in principle possible for some ordered infinite set of objects to exist actually; but they cannot be material or dependent on materiality.[1]

According to this solution, there are two crucial steps in the application of *The Mapping Argument*: (1) employing the mapping technique to show that a whole corresponds with some of its proper subsets, and (2) extracting a contradiction from such a correspondence. Having an order, either natural or conventional, guarantees that the first goal can be accomplished. The second one depends on the ontology of the objects under discussion. If they are material, such a correspondence is absolutely unacceptable. But if they are immaterial, it *may* be justified. It is worth emphasizing, however, that I do not claim that there is no such immaterial whole-part correspondence that is contradictory.

In sum, Avicenna believes that *The Mapping Argument* (his only argument against the actuality of discrete infinity) does not work in the case of those sets of immaterial objects which cannot be ordered. It is therefore possible, in principle, that some such sets are actually infinite. However, such an infinite immaterial numerosity cannot be the subject of mathematical studies, because Avicenna believes that numerosity completely separated from matter should be studied by metaphysics, not mathematics. Moreover, it is not conceivable, at least from Avicenna's standpoint, that one could undertake mathematical studies on a set that is not ordered and cannot be put in a simple one-to-one correspondence. It is also worth mentioning that there is no immaterial infinite *continuity*; there is no immaterial infinite magnitude. This is because magnitudes, according to Avicenna's ontology of mathematics, by contrast with numbers, cannot be separated from matter. They have an

[1] To show that *The Mapping Argument* cannot reject the actual infinity of numbers, Al-Rāzī follows this approach. See note 2 on page 108.

ontological dependency on matter. Therefore, there is neither finite nor infinite immaterial continuity.[1]

3.5. Conclusion

Avicenna endorses some sort of mathematical finitism by rejecting the actual existence of infinite magnitudes and infinite sets of ordered numbers and numbered things. His main argument against the actuality of mathematical infinity, *The Mapping Argument*, is grounded on the whole-part inequality axiom. He shows that the existence of an actual infinity implies the equality of that infinity with some of its proper parts. Since Avicenna believes that such an equality is absurd, at least in the case of mathematical objects, he concludes that no mathematical infinity can actually exist. Therefore, his main argument for mathematical finitism has two principal elements: (1) the whole-part equality in the case of mathematical infinities, and (2) the absurdity of the whole-part equality in the case of mathematical objects.

To argue for (1) Avicenna employs the notion of correspondence as a tool for comparing the size of different mathematical infinities and determining whether they are equal. The concept of correspondence, in turn, is reduced to the concept of geometrical projection and coverage in the case of magnitudes and to the concept of one-to-one correspondence in the case of numbers and numbered things. The sizes of all one-dimensional magnitudes can be compared to each other using the notions of projection and coverage. It is in virtue of their continuity and connectedness that they can be compared in this way. But sets of discrete objects, e.g., numbers and numbered things, can be put in one-to-one correspondences only if they can be ordered.

I have argued that (2) can be justified by Avicenna's preferred ontology for mathematical objects. He believes that mathematical objects are properties of physical objects and,

[1] Remember that number can be separated from matter; but if it is, it cannot be the subject of mathematical studies, but rather of metaphysical studies. By contrast, magnitudes cannot be detached from materiality.

consequently, have some sort of dependency on materiality. It seems that this dependency on materiality—which ties the mereology of mathematical objects to that of material objects—is what renders the whole-part equality absurd in the case of mathematical infinities. This shows that Avicenna's mathematical finitism is heavily founded on his views about the nature of mathematical objects. The map shown in Fig. 5 is the Avicennan strategy which leads to mathematical finitism.

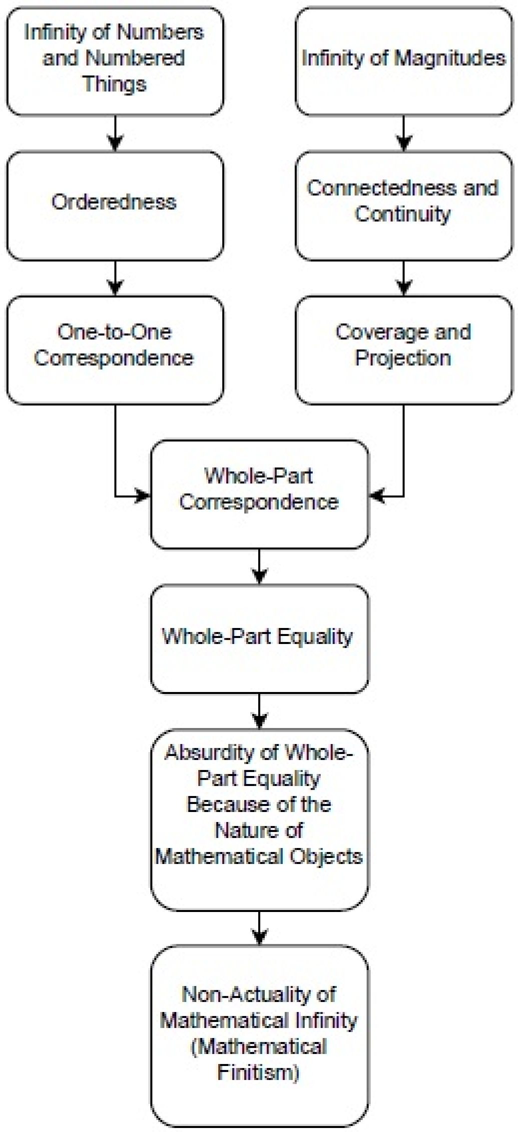

Fig. 5

4. Epistemology of Mathematics

I argue that Avicenna's epistemology of mathematics is composed of two principal elements: concept empiricism *and* judgment rationalism*. He believes that we cannot grasp any mathematical concepts unless we first have some specific perceptual experiences. It is only through the ineliminable and irreplaceable operation of the faculties of estimation and imagination upon some sensible data that we can grasp mathematical concepts. By contrast, after grasping the required mathematical concepts, independently from all other faculties, the intellect alone can prove mathematical theorems. Other faculties, and in particular the cogitative faculty, can assist the intellect in this regard; but the interference of such faculties is merely facilitative and by no means necessary.*

4.1. Introduction

The philosophy of mathematics, in general, aims to answer two fundamental questions. The *ontological* question concerns the nature of the things studied by mathematics. The ontology of mathematics investigates the metaphysical status of mathematical objects (e.g., numbers and geometrical shapes). The *epistemological* question, on the other hand, is about how we can grasp mathematical knowledge. The epistemology of mathematics explores how—by which cognitive faculties, for example—we can know mathematical objects, their properties, and their relations. It examines the role of sense perception in the formation of our mathematical knowledge and determines the extent to which our knowledge of mathematics is *a priori*, certain, or necessary. In this chapter, I discuss Avicenna's answer to the epistemological question about mathematics.

Compared to Avicenna's general theory of knowledge and his epistemology of what we call today *experimental sciences* (e.g., medicine), which have been widely discussed in modern scholarship, his epistemology of mathematics has been largely neglected.[1] This negligence is surely not due to the absence of discussion of mathematical knowledge in Avicenna's oeuvre. Quite the contrary, his epistemological discussions include numerous allusions to how we obtain mathematical knowledge. For instance, his descriptions of the functions of our cognitive faculties, his analysis of the foundational propositions of theoretical sciences, and his development of a general theory of demonstration are accompanied by many illustrative mathematical examples which clarify the mechanisms through which mathematical knowledge can be attained. A careful analysis of these references to mathematics can reveal the core elements of an ingenious epistemology of mathematics to which Avicenna is committed. This study is an attempt to provide such an analysis.

All instances of knowledge for Avicenna are either concepts or propositions. Notoriously, Avicenna believes that acquiring knowledge is either conceiving a concept (*taṣawwur*) or

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[1] For studies on Avicenna's general epistemology see, among others, Nuseibeh (1989), McGinnis (2008), Gutas (2012), Black (2013a), and Strobino (2015). For studies on his epistemology of experimental sciences, and of medical sciences in particular, see Nuseibeh (1981), Gutas (2003), and Pormann (2013).

assenting to the truth of a proposition (*taṣdīq*).[1] Given this general understanding of knowledge, we can conclude that for Avicenna acquiring mathematical knowledge is either forming mathematical concepts (e.g., the concept FOUR or the concept TRIANGLE) or assenting to the truth of mathematical theorems (e.g., the theorem that four is even or the theorem that the sum of the three interior angles of the triangle equals two right angles). Accordingly, the question of how we grasp mathematical knowledge can be reduced to two more specific questions: (1) how do we grasp mathematical concepts?, and (2) how do we make mathematical judgments and assent to the truth of mathematical propositions?[2] Not surprisingly, the answer to the latter question depends partly on the answer to the former. We cannot know a proposition unless we know the concepts the proposition is constituted from. Knowing the conceptual components of a proposition is a necessary—though not sufficient—condition for knowing that proposition. For instance, without first acquiring the concepts FOUR and EVEN, no one can know that every four is even. So (1) should be addressed before (2).

Avicenna's answer to these two questions hinges heavily on his ontology of mathematics, on the one hand, and his general psychology, on the other. The mechanisms through which we grasp (either conceptual or propositional) knowledge of an object and the cognitive faculties we employ for this purpose depend, at least partly, on the nature of the object. For instance, it seems quite plausible to think that our knowledge of fully separate (*mufāriq*) entities cannot be grasped through the same mechanisms by which we perceive sensible (*maḥsūs*)

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[1] These two notions are discussed in various places in the Avicennan corpus. See, among others, his treatment in *The Demonstration* part of *The Healing* (1956, Chapter I.1, 51-53), *The Salvation* (1985, pp. 7, 112–113), and *The Logic* part of *ʿAlāʾī Encyclopedia* (2004, pp. 5–6). Sabra (1980) has discussed Avicenna's understanding of these notions and clarified their connections to some similar notions in Aristotle. As pointed out by Strobino (2015, p. 33), using the terminology of '*taṣawwur*' and '*taṣdīq*' has become "mainstream in the Arabic tradition after Al-Fārābī." For a discussion of these concepts in Avicenna and Al-Fārābī, see Black (1990, pp. 71–78). Wolfson (1943) has reviewed the history of these concepts (and their counterparts) in different philosophical traditions.

[2] Indeed, the general question of knowledge acquisition can be reduced to the more specific questions of how we can make *taṣawwur* and *taṣdīq*. Perhaps that is why Black (1990, p. 71) believes that these notions "can be viewed as the cornerstones of medieval Arabic epistemology."

things. It seems reasonable to consider these two groups of entities as the objects of different cognitive faculties. It means that to arrive at a comprehensive understanding of Avicenna's epistemology of mathematics, we need a background knowledge of his views on the nature of mathematical objects and on human psychology. Therefore, in the following section, I discuss these preliminary issues.

In section 4.3, I investigate the roles Avicenna attributes to different cognitive faculties in the process of the formation of mathematical concepts. The faculty of estimation (*wahm*), as will be illustrated, is the protagonist of his scenario on the epistemology of mathematical concepts. The primary function of this faculty, however, hinges on what is perceived through the external senses. A careful consideration of a thought experiment proposed by Avicenna shows that the formation of mathematical concepts cannot be independent from the perception of sensible objects of the extramental world, or so Avicenna argues. He therefore seems to endorse some sort of *concept empiricism* about mathematics, albeit in a very specific sense which will be delineated. Mathematical concepts are formed through an abstraction (*tajrīd*) process which begins from experiencing some physical objects and proceeds under the heavy influence of estimation. I also discuss, in the same section, how the faculty of imagination enables us to form and grasp conceptions of mathematical objects that have no correlate in the sensible world.

In section 4.4, I turn to a more specific problem about the formation of mathematical concepts. Mathematical objects (e.g., circles), as we conceptualize them, are perfect and exact in a way that physical objects (e.g., circular material things) at least apparently cannot be. For instance, there seems to be no perfectly circular plane in the material world such that all points on its boundary (i.e., its circumference) are at exactly the same distance from a fixed point (i.e., its center). Sensible objects of the extramental world are at best imperfect approximations of ideal mathematical objects that we conceptualize in the mind. So one might wonder how, according to Avicenna, we can proceed from the perception of imperfect physical objects to the formation of perfect mathematical concepts. The first explanation which might come to mind is that for Avicenna abstraction is a machinery for constructing perfect mathematical entities which cannot exist in the extramental world. I will argue, however, that this suggestion is untenable, and that there is convincing evidence that

Avicenna accepts that perfect mathematical objects can in principle exist in the physical world. These entities, like many other objects of estimation, might not be sensible but can actually exist in the extramental realm and be perceived by this multifunctional cognitive faculty.

As I mentioned, grasping the concepts from which a proposition is constituted is not sufficient for assenting to the truth of the proposition. One who has poor mathematical skills might be unable to know that 'the largest prime number less than twenty is nineteen' even if he knows all the conceptual components of this proposition (e.g., the concept PRIME NUMBER, the concept LARGENESS, and the numerical concepts TWENTY and NINETEEN). So the next stage of our discussion should be devoted to how we can form a mathematical proposition and assent to its truth after grasping its conceptual components. Avicenna's position on these issues can be extracted from his discussions of the functions of the cogitative faculty (*quwwa mutafakkira*) and from his analysis of the foundational and basic propositions of mathematics. All other mathematical propositions can be syllogistically inferred from these principles, which cannot themselves be demonstrated on the basis of other mathematical propositions. So to complete our picture of the Avicennan epistemology of mathematics we must further know (1) how we can form a proposition as an ordered structure of concepts, (2) how we can assent to the truth of the foundational propositions of mathematics, and (3) how we can prove more complicated mathematical theorems based on these simple principles. Problems (1) and (3) are approached in section 4.5. That section also includes an investigation of the facilitating role Avicenna considers for geometrical diagrams in grasping mathematical knowledge. Problem (2) is explored in section 4.6. Avicenna's view regarding the certainty of mathematics, in comparison to the other sciences, is briefly reviewed in the same section. I close in section 4.7 by providing some concluding remarks.

4.2. Preliminaries

What are the objects that mathematical theorems are about (or true of)? What is the nature of mathematical objects for Avicenna? [1] Roughly speaking, Avicenna believes that mathematical objects are not ideal Platonic forms. He denies that mathematical objects are fully separate from matter both in the mind and in the extramental world. Moreover, he does not believe that mathematical objects are perfect mental constructions that have no counterpart in the imperfect extramental reality.[2] Mathematical objects for Avicenna are specific properties of physical objects.[3] They can, and many of them do, actually exist in the extramental realm. However, they are not independent immaterial entities and their existence depends on the existence of the physical objects of which they are properties. Since every object of the physical world is constituted of a certain kind of matter (e.g., wood and gold), mathematical objects, in the extramental reality, are necessarily associated with particular kinds of matter. Stated another way, since there is no material object that is not of a determinate kind of matter, every mathematical entity—being a specific property of a material object—is attached to some particular kind of matter. Therefore, there can be no triangle, for example, in the physical world that is neither wooden, nor golden, nor of any other particular species of matter. Triangles do, or at least can, exist in the extramental world but only in association with some determinate kinds of matter, or so Avicenna claims. In the mind, mathematical objects can in principle be stripped of all special kinds of matter they are attached to in the physical world. Nonetheless, inasmuch as they are subject to mathematical studies, even in the mind they should be considered as properties following upon matter. So, in the mind, they are still associated with materiality, even though not with any specific kind of matter. Avicenna explicitly mentions that if we do not consider number

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[1] Here I provide a breief review of what I presented in the previous chapters of his theory of mathematical objects.

[2] Ardeshir (2008), McGinnis (2006, 2017), and Tahiri (2016) argue that mathematical objects for Avicenna are, in one way or another, mental objects. However, they offer different recipes for the construction of such objects.

[3] Numbers (*aʿdād*) are the objects of arithmetic, and magnitudes (*maqādīr*) are (the most general representatives of) the objects of geometry. In *The Metaphysics* of *the Healing* (2005, Chapter III.3-4), Avicenna argues that both numbers and magnitudes are accident (*ʿaraḍ*).

as a specific property of material things (i.e., if we consider it as an entity fully separate from matter), then it would not be receptive to any increase or decrease, and, consequently, it cannot be subject to mathematical studies. It must in that case be studied by metaphysics.[1] Similarly, he argues that although we can detach geometrical shapes from all specific kinds of matter accompanying them in the physical world, they cannot be fully separate from materiality in general. Even in the mind they cannot be perceived except as material entities.[2] In this respect, mathematical objects lie between (1) the category of the objects that are fully separate from matter and materiality either in the mind or in the extramental world (e.g., God), and (2) the category of the objects that are attached to specific kinds of matter either in the mind or in the extramental world (e.g., human).[3] The former objects must be studied by metaphysics and the latter by physics (or natural science). So we have a hierarchy of objects whose degrees of association with (or dependency on) matter vary. According to a well-known Aristotelian principle which Avicenna endorses, different types of objects must be perceived by distinct cognitive faculties (*quwwa* in Arabic or *dunamis* in Greek) or senses.[4] As a result, the cognitive faculty which plays the pivotal role in the apprehension of

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[1] Avicenna (2005, Chapter I.3, 18-19).

[2] Moreover, Avicenna (2005, Chapter VII.2, 249) makes the even stronger claim that "the *definitions* of geometrical [figures] among mathematical [objects] absolutely do not dispense with matter, even though they can do with some kind of matter" (Marmura's translation, my emphasis). So the dependency of geometrical objects on matter is much stronger than that of numbers. Geometrical objects are associated with matter (though not a specific kind of matter) even in definition. In contrast, numbers can in principle be separate from matter. However, they can be studied by mathematics only when they are considered as being material accidents; otherwise, they are subject to metaphysical studies. As I argued in the second chapter, by contrast with geometrical objects which have an *ontological* dependency on materiality, numbers inasmuch as they are objects of mathematics have an *epistemological* dependency on matter.

[3] Humanness, in either the mental or the extramental realms, is attached to a specific kind of matter, i.e., flesh and blood. In other words, humanness separated from flesh and blood does not exist either extramentally or even mentally, or so Avicenna seems to believe.

[4] This principle, which is in fact a rehabilitation of a Platonic principle, is widely employed by Aristotle in his epistemology. For instance, Aristotle's demarcation of the five external senses, as Sorabji (1971) observes, is explicitly based on this principle. For studies on the reception of the Aristotle's views on the cognitive faculties in the Arabic tradition, see Hasse (2014) and Kukkonen (2015).

mathematical objects differs from the faculties by which we know the two other aforementioned types of objects. To see better the contrast between the functions of these distinct faculties, we should first sketch Avicenna's cognitive psychology.[1]

Following Aristotle, Avicenna divides the faculties of the human soul into three classes: vegetative, animal, and rational.[2] All cognitive faculties belong to either the animal or the rational soul.[3] In addition to the five familiar external (*ẓāhirī*) faculties (or senses) by which we grasp the forms (*ṣuwar*) of the particulars existing outside of us, Avicenna recognizes five internal (*bāṭinī*) faculties for the animal soul.[4] These faculties are all bodily and located in the various parts of the brain. So their objects cannot be fully separate from matter. Immaterial objects (or, more precisely, objects that are completely disassociated from matter, e.g., God) can be perceived solely by the rational soul. The most obvious example of

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[1] Avicenna's cognitive psychology is set out in numerous works from almost all the periods of his career. These works include (but are not limited to): (1) The *Compendium on the Soul* (*Maqāla fī al-nafs 'alā sunna al-ikhtiṣār*), which is probably Avicenna's first philosophical writing. The original Arabic text of this work and its German translation can be found in Landauer (1875). (2) *Book on the Soul* (*Kitāb al-Nafs*) of *The Healing* (1959). Avicenna's most extensive investigation of cognitive psychology is offered in this work. (3) *Book on the Soul* of *The Salvation* (1985, pp. 318–396). Its English translation can be found in Avicenna (1952b). (4) The psychological part of *The Pointers and Reminders* (1957, Chapter II.3). For an English translation of this work see Avicenna (2014, pp. 94–115). (5) *On the Rational Soul* (*Fì al-nafs al-nāṭiqa*), which is probably Avicenna's last philosophical writing. The original Arabic text of this essay can be found in Avicenna (1952a, pp. 195–199). Gutas (2014, pp. 67–75) has translated the essay into English. Setting aside some minor variations regarding the precise functions of the internal faculties of the animal soul (which I shortly introduce), Avicenna's treatment of the faculties of the soul by and large remains consistent over time. The main tenets of his psychological theory are explained in, among others, Hall (2004), and McGinnis (2010a, Chapters 4–5).

[2] Avicenna (1959, Prefeace, 1-3). This does not however mean that the human soul is not simple (*basīṭ*). The different faculties of the soul should not be considered as its mereological parts. They are different manifestations, powers or potentialities of a simple unity. For a meticulous analysis of Avicenna's ontology of the human soul, see Mousavian and Saadat Mostafavi (2017).

[3] The animal soul has both cognitive and non-cognitive powers. For example, *volitional motion* is one of the non-cognitive powers of the animal soul.

[4] Avicenna (1959, Chapter IV.1). In this context, Avicenna uses 'faculty' (*quwwa*) and 'sense' (*ḥiss*) interchangeably.

the objects perceptible to the animal soul are sensible forms. They are the objects of the external faculties in a direct sense. But they can also, in an indirect sense, be the objects of some internal faculties. Put otherwise, the sensible forms perceived by the external faculties can be conveyed to some internal faculties so as to be subject to some further cognitive processes. So sensible forms are accessible to the internal faculties indirectly and through the mediation of the external faculties. This does not imply, however, that the domain of the objects of the internal faculties is restricted to the sensible forms. There are some non-sensible attributes or properties of particular physical objects existing in the extramental world to which the external faculties have no access, or so Avicenna argues. He calls such attributes 'connotational attributes' (*maʿnā*).[1] For example, the hostility of a wolf is a non-sensible connotational attribute that actually exists in the wolf but cannot be known by a sheep (or even us) merely through sense perception and without the aid of any internal faculty. Undoubtedly, perceiving at least some sensible properties of the wolf (e.g., its color, smell, howl, etc.) is necessary for perceiving its hostility. But the knowledge of hostility cannot be automatically derived from the mere knowledge of such sensible properties. It is the operation of an internal faculty upon the sensible data that makes hostility available/perceptible to the animal soul. Generally, the internal faculties grasp connotational attributes of physical objects by operating upon the information they receive from the external faculties. In sum, connotational attributes are exclusive objects of the internal faculties; but they cannot be known without the effective operation of the external faculties. As Avicenna says:

[1] '*Maʿnā*' is sometimes translated as 'intention' in the contemporary scholarship. See, among others, Black (1993, 2010), Banchetti-Robino (2004), Kaukua (2007), Kukkonen (2015), and McGinnis (2017). This translation is inspired by the Latin tradition in which '*maʿnā*' is usually translated as '*intentio*'. Hasse (2000, sec. II.4) has argued, convincingly in my opinion, that for Avicenna *maʿnā* is the object of perception; so it cannot be *in* the perceiver. But intention is something that belongs to the perceiver. Thus 'intention' is not an appropriate translation for '*maʿnā*'. He then suggests (2000, p. 132) 'connotational attribute' as a more plausible alternative translation. It is not clear, however, why Hasse (2014) decided to resume using 'intention' as the translation of '*maʿnā*'. For another argument on why 'intention' is a misleading translation in this context, see Appendix (3) of Gutas (2012). In this chapter, I consider 'meaning' and 'connotational attribute' as the equivalents of '*maʿnā*'.

> **TEXT # 4.1.** There is difference between the perception of form and the perception of connotational attribute. Form is something that is perceived by both the internal sense and the external sense. But the external sense perceives it first and then delivers it to the internal sense. This is like the case of the sheep's perception of the form of the wolf—i.e., of its shape, disposition (*hayʾa*), and color. The internal sense of the sheep perceives it [i.e., the form of the wolf], but it is first perceived by its [i.e., the sheep's] external sense. By contrast, the connotational attribute is something that the soul perceives from the sensible (*al-maḥsūs*) without the external sense first perceiving it. This is like the case of the sheep's perception of the connotational attribute of enmity [existing] in the wolf or of the connotational attribute that causes the sheep to fear and escape from the wolf. [The sheep perceives these things] without the [external] sense first perceiving them.[1]

According to Avicenna, connotational attributes really exist in the extramental world. They are not mere productions of the mind by a mechanism like abstraction (*tajrīd*). They are not purely mental constructions. They have actual existence in the extramental world as properties and attributes attached to material objects. Since these attributes are non-sensible, they cannot be perceived by the external faculties. They are objects of the internal faculties. These faculties, however, cannot perceive the connotational attributes of a particular object unless they have already perceived (at least) some sensible attributes of the object. So it seems plausible to say, albeit metaphorically, that the internal faculties perceive connotational attributes through the lens of the external faculties.[2] How is this possible?

[1] Avicenna (1959, Chapter I.5, 43). Unless otherwise specified, all translations of this chapter are mine. As we see in the foregoing passage, Avicenna sometimes uses the single form 'sense' or 'faculty' to refer to a plurality of (either internal or external) faculties or senses.

[2] As we will see in the next section, the aforementioned construal of (1) the ontological status of connotational attributes and (2) the epistemic channels via which we can know them is a key to unravel the mechanism of forming mathematical concepts in Avicenna's epistemology.

To answer the above question Avicenna develops an extensive theory of five internal faculties.[1] The first of these faculties is the *common sense* (*ḥiss mushtarak*) which is a receptive faculty placed in the anterior ventricle of the brain. This faculty receives the forms of the sensible particulars from the external senses and processes these inputs to produce phenomenally unified perceptual experiences. In other words, the common sense is the main faculty responsible for sense-perception. The storehouse of the forms and images perceived by the common sense is the second internal faculty: the *form-bearing* (*muṣṣawira*) or *imagery* (*khayāl*) faculty. This retentive faculty is located behind the anterior ventricle of the brain. In addition to the images and forms perceived by the common sense, the imagery faculty stores the images and forms constructed or created by the operation of another internal faculty called *imagination* (*mutakhayyila*). This faculty is located at the medial ventricle of the brain and its main function is to operate on forms and connotational attributes by shuffling, separating, and recombining them to create new mental entities (i.e., images or forms) that (at least some of them) have no counterpart in the extramental world. So fictional beings, e.g., a phoenix, are constructed by the faculty of imagination.[2] By contrast with the imagery faculty which has only a passive storage function, imagination can actively engage with forms, images and connotational attributes received from other internal faculties.[3] The intellect can employ imagination and control its function to serve the mechanism of thinking. It is exactly because of this application that imagination is also called the *cogitative* (*mufakkira*) faculty.[4] The fourth internal faculty is *estimation* (*wahm*), which is the chief perceiver of connotational attributes in the animal soul. The physical position of this faculty

[1] For a discussion of the epistemological roles of the internal faculties, see Gutas (2006). Wolfson (1935) offers a comprehensive study on the internal faculties in the Latin, Arabic, and Hebrew traditions.

[2] Avicenna's sophisticated treatment of the ontology and the epistemology of fictional beings is studied by Black (1997) and Druart (2012). Considering the theories propounded between the eleventh and thirteenth centuries in the Islamic tradition, Benveich (2018) has discussed the problem of non-existent objects of thought in a broader historical context.

[3] As illustrated later in the chapter, there is an ongoing debate among scholars regarding whether or not imagination can perform upon universal concepts.

[4] This faculty also plays a remarkable role in the mechanism of revelation. See Black (2000, 2013b) and Gutas (2001, 2006) for a detailed analyses of different functions of the cogitative faculty.

in the brain is the back of medial ventricle. Estimation is a multifunctional faculty that can also contribute in making certain judgments and causing certain actions. In all animals other than the human estimation is the most superior cognitive faculty. It governs and guides all the cognitive faculties of animals which lack a rational soul and is the source of almost all of their actions.[1] It is estimation that enables us to perform thought experiments and to mentally implement scenarios that are unrealizable in the actual world.[2] The fifth and final internal faculty is memory (*ḥāfiẓa*). This faculty, located at the posterior ventricle of the brain, retains what is perceived or judged by the estimative faculty. So all perceived connotational attributes and all estimative judgments are stored in memory. These five internal faculties intercommunicate with each other by sending and receiving images, forms, connotational attributes, and some specific propositions that are judged to be true.[3] The harmonic performance of the internal faculties prepares the rational soul (or, more precisely, the intellect) to grasp universal concepts (or intelligibles) and to assent to the truth of universal propositions. The internal faculties bridge the gap between the material world and the immaterial intellect, and make the former apprehensible to the latter, albeit indirectly and through a step-by-step abstraction procedure.[4]

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[1] Avicenna (1959, Chapter IV.1, 167). For example, estimation is responsible not only for the sheep's perception of the wolf's hostility, but also for determining the sheep to flee from the wolf's potential threat. Estimation can in principle contribute to making some judgments, but it is not always reliable and some of its judgments are false. That every existence must occupy space is an example of the false judgments of estimation, or so Avicenna (1985, p. 116) contends. See Black (1993, 2000), Hall (2006), Kaukua (2007, Chapter 3) for studies on various aspects of the role of estimation in Avicenna's psychology.

[2] The functions and applications of thought experiments in Avicenna's philosophical system, and the cognitive capacities we must have to be able to carry out such experiments, are studied by Kukkonen (2014) and McGinnis (2017).

[3] This brief report is mainly extracted from *Book on the Soul* of *The Healing* (1959, Chapter IV.1).

[4] My discussion in this chapter is neutral with respect to different readings of Avicenna's theory of abstraction. Some scholars, e.g., Nuseibeh (1989), Davidson (1992, Chapter 4), and Goodman (2006), defend an *emanationist* reading according to which universal knowledge is, in the end, emanated from the Active intellect (*ʿaql faʿʿāl*). Some others, e.g., Hasse (2001) and Gutas (Gutas, 2012), support a strongly *abstractionist* view according to which the epistemic role of the Active intellect is downgraded and its function is limited to being

Depending on the four stages of the formation (or, more precisely, perfection) of the intellect, the cognitive power of the rational soul can be manifested in different degrees. These stages are respectively called: (1) 'the material intellect'[1] (*ʿaql hayūlānī*) or 'the potential intellect' (*ʿaql bi-l-quwwa*), (2) 'the dispositional intellect' (*ʿaql bi-l-malaka*), (3) 'the actual intellect' (*'aql bi-l-fi'l*), and (4) 'the acquired intellect' (*'aql mustafād*). Although the material intellect has absolute potentiality to be impressed by any intelligibles, it has no actual cognitive content; it has not yet perceived anything. The first instances of knowledge impressed upon the intellect are certain primary intelligibles (*maʿqulāt ʾūlā*) our knowledge of which is not grounded in the acquisition of other concepts and propositions. The intellect inasmuch as it has perceived only the most fundamental instances of universal knowledge is called 'the dispositional intellect'. That the whole is greater than the part is an example of such basic propositions whose truth is assented to by the dispositional intellect. More sophisticated sorts of knowledge that are not restricted to some logical tautologies and actually inform us about the substantial facts of the world will be obtained in the next step of the actualization of the intellect. Almost all kinds of universal knowledge (either conceptual or propositional) that we can in principle obtain are present (and, in a sense, stored) in the actual intellect. At this stage the intellect has the potentiality of consciously entertaining all of these instances of knowledge. But this potentiality is activated only in the fourth and final stage of the perfection of the rational soul where the acquired intellect consciously considers and entertains the intelligibles that are possessed by the dispositional and the actual intellects. Avicenna contends that, at this stage, the intellect has even a second-order consciousness of

merely the ontological reservoir of intelligible concepts and propositions. McGinnis (2007c) and D'Ancona (2008) have propounded thought-provoking syntheses of these two antithetic approaches. Compared to D'Ancona, McGinnis is more sympathetic to the abstractionist camp.

[1] This labelling highlights an analogy between the prime matter and the intellect in its first stage. Like the prime matter that is pure potentiality and has yet to be impressed by the material forms, the material intellect is pure potentiality to receive intelligibles (or universal forms) and has no content of its own. Describing the intellect as being *material* does not mean that it is constituted from matter.

what it is doing. Not only does it consciously engage with the intelligibles, but it is also conscious of doing so.[1]

With these undetailed portraits of Avicenna's ontology of mathematics, on the one hand, and of his cognitive psychology, on the other, we are well equipped to tackle his epistemology of mathematics.

4.3. Forming Mathematical Concepts

As I clarified in the previous section, Avicenna believes that mathematical objects are specific properties of physical particular objects that can—and many of them actually do—exist in the extramental world. It is not surprising, therefore, that he denies the possibility of grasping mathematical concepts without appealing to any perceptual experiences. In his criticisms of mathematical Platonism in *The Metaphysics* of *The Healing*, he says:

> **TEXT # 4.2.** If among mathematical things there is a mathematical object separate from the sensible mathematical object (*al-taʿlīmī al-maḥsūs*) at all, then in the sensible thing either there would be no mathematical object or there would be [a mathematical object]. If in the sensible thing (*fī al-maḥsūs*) there is no mathematical object (*taʿlīmī*), then it necessarily follows that there is no quadrilateral, circular, or numbered (*maʿdūd*) sensible thing. If none of [these things] is sensible, then what way is there to establish their existence [or], indeed, [even] to imagine them? For the principle of their being imagined is likewise [derived] from sensible existence—so much so that, if we suppose, through our estimative faculty, an individual who has apprehended none of [these] by the senses, we will judge that he does not imagine,

[1] This brief description of the stages of the perfection of the intellect is extracted from *Book on the Soul* of *The Healing* (1959, Chapter I.5, 48-50). See also Avicenna (1952a, pp. 195–196).

> nor, indeed, intellectually apprehend any of them. However, we have established the existence of many (*kathīr*) of them in what is sensible.[1]

Here Avicenna is proposing a thought experiment that is structurally very similar to the Flying Man argument.[2] Indeed, it can be considered as a brief version of the Flying Man argument restricted to the context of the epistemology of mathematics. The experiment goes as follows: Suppose an individual who, for some reason, has no apprehension of the mathematical objects existing in the sensible world. Such an individual would have neither any imagination (*takhayyul*) nor any intellectual apprehension (*ta'aqqul*) of mathematical objects (e.g., quadrilaterals and circles), or so Avicenna claims. But almost all individuals can imagine and intellectually apprehend some mathematical objects. This indicates, Avicenna seems to believe, not only that (at least some) mathematical objects actually exist in the sensible world, but also that perceiving such objects is a prerequisite for having any imagination or intellectual apprehension of mathematical shapes. Avicenna here does not say anything about assenting to the truth of mathematical propositions. It seems therefore that he merely wants to highlight a crucial point specifically about the formation of mathematical concepts. More precisely, the thought experiment is apparently intended to show that grasping mathematical concepts (which are objects of the intellect and intellectual apprehension) is impossible unless we first have specific perceptual experiences. The passage affirms that Avicenna embraces a *modest* version of concept empiricism regarding mathematics. I qualified my statement with the adjective 'modest' since the above passage does not deny the possible contribution of a further rational or emanational element (e.g., the Active intellect) in the process of the formation of mathematical concepts. That forming a mathematical concept (e.g., the concept CIRCLE) depends heavily on having some perceptual experiences (e.g., seeing a circular object) does not entail that necessarily

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[1] Avicenna (2005, Chapter VII.3, sec. 1). I have slightly revised Marmura's translation by replacing 'numerable' with 'numbered' as the translation of '*ma'dūd*'. Phrases in brackets are added by him. The passage quoted here is part of an extended argument against the existence of mathematical objects that are fully separate (*mufāriq*) from matter. I discussed the subtleties of that argument in the first chapter.

[2] Among others, Marmura (1986), Alwishah (2013), and Adamson and Benevich (2018) have studied the logical structure and the philosophical consequences of the Flying Man argument.

mathematical concepts are constructed rather than emanated. Receiving data through the perceptual experiences might be only a subsidiary step for the preparation of the intellect to receive the universal concepts emanated by the Active intellect. But in any case—whether or not the emanationist account is defensible—the significant conclusion of this passage is that mathematical concepts are neither innate nor given at birth. They cannot be grasped unless we have some *a posteriori* perceptions. This reading of Avicenna's account of the formation of mathematical concepts is perfectly compatible with Gutas's general claim that for Avicenna all concepts are derived eventually from what we perceive by our external senses.[1]

It is still not clear which cognitive faculties play the pivotal role in grasping mathematical concepts. Mathematical objects are specific properties or attributes of physical objects and, as the above passage witnesses, they cannot be known unless we receive some data through our sense perceptions. Nonetheless, it is still not clear whether or not mathematical objects are themselves sensible. As we saw in the previous section, according to Avicenna, not only sensible attributes but also non-sensible connotational attributes of physical objects cannot be perceived if we lack sense perception. In fact, our knowledge of the connotational attributes of physical particulars is indirectly extracted by the faculty of estimation from the data transferred from the faculty of common sense. So the fact that the lack of sense perception results in the lack of mathematical concepts does not on its own show that mathematical objects are themselves sensible attributes. Avicenna's language regarding the exact ontological status of mathematical objects is obscure and equivocal. Although he has never denied that they can exist in the actual world as properties or attributes of sensible particular objects, it is not crystal clear whether or not he considers these properties to be sensible. In some places, he seems to be explicitly claiming that mathematical objects are sensible (*maḥsūs*). But, in other places, he uses a more conservative wording and describes these objects as existents *in* the sensible things (*fī al-maḥsūsāt*). For example, in the previous passage, Avicenna uses both of these locutions. On the one hand, he explicitly states that

[1] See Gutas (2012). His general claim about all concepts entails my analysis which is restricted to the scope of mathematical concepts; but the other way around does not hold. By contrast with Gutas, I am reluctant to surmise that Avicenna's empiricism is extensible to all concepts.

there are some *sensible mathematical objects*. On the other hand, he argues that mathematical objects must exist *in* the sensible things.[1] However, these two claims seem to conflict each other. The latter terminology is what Avicenna usually employs to refer to connotational attributes (*maʿānī*), which are, as we saw in the previous section, non-sensible. For instance, in *The Salvation* Avicenna describes estimation as a faculty which perceives "non-sensible connotational attributes existing *in the sensible* particulars" (*al-maʿānī al-ghayr al-maḥsūsa al-mawjūda fī al-maḥsūsāt al-juzʾīya*).[2] Accordingly, it seems plausible to think that mathematical objects—which are described as existing in the sensible things—are non-sensible connotational attributes of those things. So it seems that in the above passage mathematical objects are claimed to be both sensible and non-sensible.

There is however other evidence that mathematical objects should be considered as non-sensible connotational attributes. This construal of the ontological status of mathematical objects is reinforced by Avicenna's frequent references to the role of estimation in the apprehension of mathematical concepts. For example, in his discussions of the ontological status of the objects of different scientific disciplines in *The Introduction* (or *Isagoge*) of *The Healing*, Avicenna says:

> **TEXT # 4.3.** The things that mix with motion are of two kinds. They are either such that they have no existence unless they undergo admixture with motion, as for example, humanness, squareness and the like; or they have existence without this condition. The existents that have no existence unless undergoing admixture with motion are of two divisions. They are either such that, neither in subsistence nor in the *estimation* (*wahm*) would it be true for them to be separated (*tujarradu*) from some specific matter as for example, the form of humanness and horseness; or else, this would be true for them in the *estimation* but not in subsistence, as for example, squareness. For, in the case of the latter, *conceiving it* (*taṣawwuruhū*) does not require

[1] In particular, see the last sentence of TEXT # 4.2.

[2] Avicenna (1985, p. 329), my emphasis. A similar description is presented in *The Pointers and Reminders* (1957, Chapter II.3, sec. 9 , 379). See also TEXT # 4.1 in which Avicenna says that the connotational attribute of enmity exists "*in* the wolf".

that it should be given a specific kind of matter or that one should pay attention to some state of motion.[1]

It seems quite defensible to think that Avicenna's description of the ontological and epistemological status of squareness is uncontroversially generalizable to all geometrical objects. If so, geometrical objects actually exist *in* the sensible world in association with some specific kinds of matter. By the act of the faculty of estimation (*bi-l-tawahhum*) we can separate them from those specific kinds of matter they are attached to in the extramental realm.[2] This enables us to conceive the concepts of geometrical objects free from any particularized association with matter, in exactly the same way as we entertain geometrical concepts in practicing pure geometry.[3] Estimation plays a somewhat analogous role in conceiving numbers. In his analysis of numbers in *The Introduction* of *The Healing* Avicenna says:

> **TEXT # 4.4.** [An] accident, even though it cannot occur except in relation to matter and mixed with motion, [might be] such that its state can be apprehended by the *estimation* and discerned without looking at the specific matter and motion [it is attached to]. An example of this would be addition and subtraction, multiplication and division, determining the square root and cubing, and the rest of the things that append (*talḥaqu*) to number. For all this attaches to number either in men's faculty of *estimation*, or *in* the existents that are subject of motion, division, subtraction and

[1] Avicenna (1952c, Chapter I.2, 12-13). I have borrowed this translation, with some modification, from Marmura (1980). The emphases are mine.

[2] See Avicenna (2005, Chapter III.4, sec. 2). There Avicenna says that magnitude (*miqdār*), which is the subject matter of geometry and can be interpreted as the most general representative of geometrical objects, "does not separate from matter except in the act of estimation" (Marmura's translation).

[3] This does not however mean that we can conceive geometrical shapes as entities independent from matter and materiality in general. Avicenna (2005, Chapter VII.2, sec. 21) insists that even the definitions (*ḥudūd*) of geometrical objects "do not utterly dispense with matter, even though they can do with any given species of matter" (Marmura's translation, modified). So even in the faculty of estimation geometrical shapes are considered as properties of material entities.

addition. *Conceiving its concept*, however, involves a degree of abstraction that does not require the specifying of matters of certain species.[1]

Another passage of the same spirit can be found in *The Metaphysics* of *The Healing*:

> **TEXT # 4.5.** The science of arithmetic, inasmuch as it considers number (*yanẓuru fi al-ʿadad*), considers it only after [number] has acquired that aspect possessed by it when it exists *in* nature (*fī al-ṭabīʿa*). And it seems that the first consideration [or theoretical study] of [number that the science of arithmetic undertakes] is when it is in the *estimative* faculty having the description [mentioned] above; for this is an *estimation* [of number] taken from natural states subject to addition and subtraction and unification and division.[2]

Number can be studied by arithmetic only if it is subject to addition, subtraction, etc. But number is subject to these accidents only when it is *in* the nature (i.e., in the sensible world).[3] In other word, number inasmuch as it is the subject matter of arithmetic should be attached to matter. So the object of arithmetic exists *in* the sensible world. However, from a purely mathematical point of view, the specific kind of matter with which number is mixed in the sensible world has no significance. Mathematicians perceive number as something attached to matter but without any particularized association with any specific kind of matter. To conceive the concepts of numbers, mathematicians look at numbered things (*ma'dūdāt*) in the sensible world and overlook all of their particularized (and mathematically unimportant) features. It is only the faculty of estimation which bestows this ability to human beings, or so Avicenna seems to believe.

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[1] Avicenna (1952c, Chapter 1.2, 13-14). I have taken this translation, with some modification, from Marmura (1980).

[2] Avicenna (2005, Chapter I.3, sec. 18). I have revised Marmura's translation.

[3] By contrast with geometrical objects, number can in principle be fully separate from matter. However, fully immaterial number is not subject to mathematical accidents and should be studied by metaphysics, rather than arithmetic. Thus even in the faculty of estimation, numbers—inasmuch as they are subject to mathematical studies—should be considered as properties of material entities. In this latter respect, there is no difference between numbers and geometrical shapes. See Avicenna (1952c, Chapter I.2, 14, 2005, Chapter I.3, sec. 20).

The mechanism of forming mathematical concepts is usefully comparable to that of the concept of hostility. Hostility is a connotational attribute of certain sensible existents like wolves. Although it actually exists *in* the sensible world, it is not itself sensible. To perceive hostility we should first have a sense perception of a sensible thing in which this attribute exists. For instance, we should see a wolf (hopefully from a safe distance!) and perceive its sensible attributes (e.g., its color, smell, howl, etc.). The data collected by our external senses will be transferred to the faculty of estimation (through the faculty of common sense). Finally, we can perceive the hostility of the wolf by our estimation. In fact, estimation can extract something from the received data that is not perceivable to the common sense. It is true that our apprehension of hostility has been formed through having a sense perception of a wolf. But our estimation enables us easily to overlook all other characteristics of the experience we have had, and consider the property of being hostile in itself and independently of all other particularized properties of the wolf we have seen. Hostility perceived as such is something that no immaterial existent (indeed, no existent other than in animals) can possess. Moreover, it is something that actually exists in the sensible world and is perceived by estimation; it is not merely imagined or mentally constructed. Similarly, mathematical objects really exist in the sensible world. We first perceive the sensible attributes of the particular physical things *in* which mathematical objects exist. For instance, we perceive a bronze circular plane or a set of two wooden chairs. What we grasp through these perceptual experiences will be given, by the mediation of the common sense, to estimation. Our estimation neglects all mathematically unimportant features of the received data and eventually perceives the circle and the number two that exist in those sensible things. The moral of this comparison is that (at least some) mathematical objects actually exist in the sensible world and can be perceived by estimation. They are not merely imagined or mentally constructed. Mathematical objects perceived as such have no existence but in material things.[1] Given this interpretation, mathematical objects are some specific non-sensible connotational attributes of sensible objects. Accordingly, when Avicenna talks about sensible mathematical objects, what he has in mind is the sensible objects in which mathematical objects exist (or, equivalently, of which mathematical objects are

[1] See Avicenna (2005, Chapter VI.5, sec. 52).

connotational attributes). He does not really mean that mathematical objects, inasmuch as they are considered by pure mathematicians, are sensible.

It is noteworthy that the objects of estimation are not universal concepts. Estimation can perceive the twoness of many different sets of two objects, in exactly the same way that it can perceive the hostility of many different animals. Indeed, it is the repetitious actions of estimation in perceiving various instances of twoness and hostility that prepare the intellect to grasp the universal concepts TWO and HOSTILITY.[1]

So far, I have shown that, according to Avicenna, many mathematical objects exist in the sensible world (as the last sentence of TEXT # 4.2 assures us) and can be perceived by estimation (as is supported by TEXT # 4.3 and TEXT # 4.4).[2] But many of the complicated geometrical objects that mathematicians work with in Euclidean geometry have no counterpart in the sensible world. They cannot therefore be either sensible or non-sensible attributes of particular material objects. Accordingly, they are objects of neither sense perception nor estimation. So it is necessary to clarify how Avicenna's epistemology accommodates the possibility of engaging with non-existent geometrical shapes. Avicenna himself concedes that it is impossible for many geometrical objects to exist in the extramental world. In an abstruse passage of the physics part of *The Pointers and Reminders* he says:

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[1] In this chapter, I do not touch on the role of the Active intellect in the formation of mathematical concepts. This is because, to the best of my knowledge, there is no textual evidence that the role of the Active intellect in the formation of mathematical concepts differs from its role in the formation of other kinds of concepts. So any defensible account of the role of the Active intellect in the formation of other concepts is extendable to the context of the epistemology of mathematics.

[2] There are still other allusions to the role of estimation in grasping mathematical concepts in Avicenna's oeuvre. For example, in *The Metaphysics* of *The Healing* (2005, Chapter III.4), he argues that although in the extramental realm, point cannot be separated from line, and line cannot be separated from plane, and plane cannot be separated from body, estimation enables us to perform such separations mentally and consider point, line and plane as separate mathematical objects. As another example in the same book, see his reference to the role of estimation in making infinite magnitudes conceivable to the human mind (2005, Chapter III.4, sec. 2).

TEXT # 4.6. To perceive (*ʾidrāk*) a thing is that its reality (*ḥaqīqa*) is represented (*mutamaththal*) to the perceiver, such that the perceiver observes it [i.e., the reality of the perceived thing] by that with which he perceives. It is either that that reality is exactly the reality existing externally to the perceiver when he perceives. [But this cannot be the case because] there might be a reality that has no actual existence in the extramental world; e.g., many (*kathīr*) of the geometrical shapes or many of the impossible things that are posited in geometry but cannot be realized at all. Or a representation (*mithāl*) of its reality [rather than the reality itself] is impressed (*murtasam*) on the essence of the perceiver, [a representation that is] not separated from [or external to] him.[1]

This passage is not exclusively related to the epistemology of mathematics. Here Avicenna is arguing that perception cannot be the presence of a concrete object to the perceiver. This is because we are able to perceive non-existent things which have no concrete existence to be presented to other things (e.g., a perceiving agent). Therefore, what is presented to the perceiver should be a representation of the concrete object we perceive, rather than the object itself. What is relevant to our discussion here is that the passage leaves no doubt that many conceivable geometrical objects are not actually exemplified in the sensible world. If those objects existed, they would have been attributes of sensible objects. But they do not exist in the sensible world at all. Consequently, they cannot be perceived by either the external senses or the faculty of estimation. So an immediate natural question raises: how can we conceive such objects?

Before discussing Avicenna's response to this question, I would like to highlight two remarkable points we can extract from this passage. First, Avicenna does not claim that *all* or *most* geometrical objects do not exist in the extramental world. He merely says that *many* (*kathīr*) of them have no concrete existence. It reassures us that any understanding of Avicenna's philosophy of mathematics that renders all geometrical objects as purely mental constructions is wrong. Second, as we saw in TEXT # 4.2, Avicenna believes that the existence of many geometrical objects in the sensible world is undeniable. Given the fact that he

[1] Avicenna (1957, Chapter II.3, sec. 7).

mentions circle, triangle and square as examples of geometrical objects that actually exist in the sensible world, we would expect that he considers some more peculiar geometrical objects as examples of the things that are conceivable but do not exist in the extramental world. This expectation is well supported by Al-Ṭūsī's analysis of this passage. In his commentary on *The Pointers and Reminders*, Al-Ṭūsī points out that the geometrical shapes to which Avicenna refers are either (1) things that although they do not currently exist in the extramental world, it is in principle possible for them to come to exist, or (2) things that neither do nor can ever exist in the extramental world. Examples of the latter group are, Al-Ṭūsī suggests, impossible things we posit for the sake of a *reductio ad absurdum*. We can conceive such entities despite the logical impossibility of their actual existence. As an example of the former group, Al-Ṭūsī mentions "a sphere in which a pentagonal dodecahedron is inscribed".[1] He has apparently borrowed this example from Avicenna's discussion of universal and particular terms in the logic part of *The Pointers and Reminders*.[2] There Avicenna introduces the term 'spherical shape in which a pentagonal dodecahedron is inscribed' as a universal that does not exist in anything at all, but which could exist in many.[3] So Avicenna seems to believe that although many basic mathematical objects (e.g., some numbers, circle, triangle, and square) actually exist in the sensible world and can be perceived by the faculty of estimation, there are also many more complicated mathematical objects that have no extramental existence.[4] This observation about the overall complexity of non-existent objects of mathematics takes us one step closer to knowing how we conceive such objects.

The complex geometrical shapes that do not actually exist in the sensible world can be decomposed into simpler parts each of which is either a geometrical shape that exists in the sensible world (e.g., a circle or a triangle) or a part of such a shape (e.g., an arc of a circle). So

[1] Avicenna (1957, Chapter II.3, sec. 7, 361).

[2] Avicenna (1957, Chapter I.1, sec. 8, 149).

[3] Avicenna's motivations for replacing the example of *ʿanqāʾ* with this example have been illustrated by Druart (2012).

[4] As we saw in the previous chapter, Avicenna believes that only a finite number of numbers actually exist in the extramental world.

it is quite natural to think that we can conceive the complex geometrical shapes that have no counterpart in the material world through separating, dividing and combining what we have previously perceived from simpler mathematical objects that actually exist in the sensible world. As we saw in the previous section, Avicenna nominates a specific cognitive faculty for accomplishing such a mission: imagination. The following passage from *Book on the Soul* of *The Healing* summarizes how this faculty enables us to think about non-existent things:

> **TEXT # 4.7.** We certainly know that it is in our nature to combine part of sensible things with another part and to separate part of them from another part, not according to the form of them we found in the extramental world, nor even with affirming the existence or non-existence of any of them. Thus, there must be a faculty in us by which we perform that. This is what is called the 'cogitative' [faculty] when employed by the intellect and 'imagination' when employed by the animal faculty.[1]

It is through the act of imagination that we can take what is grasped and stored by other internal faculties to construct some mental artifacts regardless of whether or not they would have any counterpart in the sensible world. By separating and combining different elements of sensible and non-sensible attributes that we have perceived from the material objects existing in the extramental world, we can conceive objects that do not exist but could have existed (e.g., a sphere in which a pentagonal dodecahedron is inscribed or a heptagonal house) or even objects that can never come to exist (e.g., a phoenix).[2] Definition (*taʿrīf* or

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[1] Avicenna (1959, Chapter IV.1, 165-166).

[2] The example of the heptagonal house is taken from *The Metaphysics* of *The Healing* (2005, Chapter V.1, sec 2). Avicenna dealt with the epistemology of non-existent objects in *The Letter on the Disappearance of the Vain Intelligible Forms after Death*. The original Arabic of this letter and its French translation can be found in Michot (1987). Michot (1985) provides an English translation of this letter. As we saw, in his commentary on TEXT # 4.6, Al-Ṭūsī refers to impossible hypothetical geometrical shapes posited for the sake of a *reductio ad absurdum* and considers them as another example of the non-existent objects of thought. Nonetheless, it seems to be dubious that imagination or the cogitative faculty can conceive such objects. This is because there cannot be any consistent image of them, by contrast with other non-existent objects like phoenix. So my suggestion is that such impossible objects are directly posited by the intellect as a collection of universal concepts. See section 4.5.

ḥadd) of a complex geometrical object can be considered as a recipe for imagination to build that object in the mind by combining different elements of the simpler objects mentioned in the definition. Finally, by the intervention of the Active intellect, these mental artifacts are turned into universal concepts and become graspable by the human intellect.

In sum, every mathematical object is either (1) a non-sensible attribute of a physical object that actually exists in the sensible world and can be perceived by the faculty of estimation, or (2) a mental artifact that is built by the faculty of imagination through separating and combining parts of mathematical objects that are previously perceived by estimation. In any case, even in the mind, mathematical objects should be considered as properties of material entities. They can never be grasped as fully immaterial entities.[1]

There still remains an important worry about the formation of mathematical concepts. One might say that mathematical objects as considered by mathematicians are so perfect and idealized that they cannot be found in the extramental world. There is no perfect circle in the sensible world such that all points on its circumference are of exactly the same distance from its center. Even if this is not visible to the naked eye, there is no doubt, the complaint might continue, that the circumference of any seemingly circular material object is serrated. There is no real circle in the sensible world. Similarly, there is no perfectly straight line and, consequently, no real square or triangle in the extramental reality. If so, perfect mathematical objects as considered by mathematicians cannot actually exist in the sensible things. Accordingly, they cannot be objects of estimation. So estimation plays, if any, an ancillary role in the formation of mathematical concepts. Since, according to this caveat, all perfect objects have no existence in extramental reality, they should be constructed by imagination. Therefore, it is imagination, rather than estimation, which plays the protagonist in the epistemology of mathematical concepts. This argument, if sound, shows that the core of Avicenna's ontology of mathematics is better captured by a *fictionalist-abstractionist*

[1] Universal mathematical concepts, just like all other concepts, are fully immaterial intelligibles. But this does not mean that mathematical objects can be conceived as immaterial entities. The concept HUMAN is itself an immaterial intelligible; nonetheless, a human being cannot be conceived as an immaterial entity.

account, rather than a *literalist* one. However, as we will see in the next section, Avicenna denounces this line of argument.

4.4. Perceiving Perfect Mathematical Objects

As mentioned in the last sentence of TEXT # 4.2, Avicenna puts forward arguments to establish the existence of some geometrical figures in the sensible world. For example, in *The Metaphysics* of *The Healing*, he propounds a complicated argument to show that, contrary to the atomists' view, perfect circles truly exist in the extramental world.[1] Avicenna's report of the atomists' position regarding the existence of the circles goes as follows:

> **TEXT # 4.8.** If one assumes a circle as a sensible thing (*ʿalā al-naḥw al-maḥsūs*), it being, as they state, not a real circle but [a figure whose] circumference is serrated (*muḍarraṣ*).[2]

But Avicenna denounces the atomists and claims:

> **TEXT # 4.9**. As for the doctrine of the person who constructs magnitudes from atoms, it would also be possible to prove the existence of the circle against him from his [own] principles. With the existence of the circle, one would then repudiate the existence of atom [to which he subscribes].[3]

As these two passage explain, Avicenna believes that (1) the atomists deny that a perfect circle can in principle exist in the sensible world, (2) the existence of such a geometrical object can be derived from the principles they endorse, therefore, (3) the atomists' position is self-refuting.[4] It is worth noting that the existence of a quasi-circular object that looks like a perfect circle but has some microscopic deviations in its circumference does not

[1] Avicenna (2005, Chapter III.9).

[2] Avicenna (2005, Chapter III.9, sec. 6). I have revised Marmura's translation.

[3] Avicenna (2005, Chapter III.9, sec. 5).

[4] (1) is true because, according to the atomists, there is no possible non-serrated arrangement of atoms on the circumference of a circle.

necessarily contradict atomism. An atomist can self-consistently say that the imperfectness of many geometrical shapes appears only at the atomic level and is not easily detectable by our sense organs. Atomism is an ontological doctrine which cannot be disproved by the mere epistemic fact that some objects look like perfect circles whose circumferences are not jagged or bumpy. So the aim of Avicenna's argument is exactly to establish the existence of perfect circles in the sensible world, rather than merely to confirm that some sensible objects look like a perfect circle. This much suffices to convince us that Avicenna actually believes that perfect mathematical objects *literally* exist in the sensible world. However, it is worth scrutinizing another passage in which, without providing any argument, Avicenna simply presupposes the existence of certain perfect geometrical shapes in the sensible world to cast doubt on the soundness of atomism. In *The Physics* of *The Healing*, he argues:

> **TEXT # 4.10.** In fact, the existence of atoms would necessarily entail that there be no circles, right triangles, or many other [geometrical] figures [...]. When two sides of a right triangle are each ten units, then the hypotenuse is the square root of two hundred, which [according to the present view] would either be an absurdity that does not exist, or it is true, but atoms would be broken up, which [according to the present view] they are not.[1]

Here Avicenna does not provide any argument for the existence of perfectly right triangles. He simply takes it for granted that such triangles could possibly exist. Then, based on this assumption, he advances an argument against atomism which can be formalized as follows:

(1) According to atomism every finite magnitude is constituted of a finite number of indivisible atoms of equal length. Equivalently, for every finite magnitude, there should be a natural number *n* such that the length of the magnitude is equal to the length of *n* atoms.
(2) There could possibly exist a right triangle that the length of each side of its right angle is equal to the length of 10 atoms.

[1] Avicenna (2009b, Chapter III.4, sec. 5). Apart from the replacement of 'parts' by 'atoms' in the last sentence, I have been faithful to McGinnis's translation.

(3) According to The Pythagorean Theorem, the length of the hypotenuse of such a right triangle is equal to the square root of the length of 200 atoms.

(4) The square root of 200 is not a natural (or even rational) number ($\sqrt{200} = 14.142135\ldots$).

(5) There is no natural number *n* such that the length of the hypotenuse is equal to the length of *n* atoms.

(6) Either there is no hypotenuse of a right triangle whose length is equal to the square root of the length of 200 atoms or there is such a hypotenuse but it is constituted of 14 complete atoms and a broken atom.

(7) If there is no hypotenuse of a right triangle whose length is equal to the square root of the length of 200 atoms, then (2) is false.

(8) If there is such a hypotenuse but it is constituted of 14 complete atoms and a broken atom, then (1) is false.

(9) Either (1) or (2) is false.

(10) (2) is obviously true.

Therefore,

(11) (1) is false, and atomism is refuted.[1]

As we see, this argument works only if we accept that perfect right triangles could possibly exist in the sensible world. The atomist can easily rebut this argument by insisting that all perfect mathematical objects are purely mental constructions that have no counterpart in the sensible world. There possibly exist some triangular objects in the sensible world that look approximately like perfect right triangles; but since their imperfectness appears only at the atomic level—that is perhaps undetectable by our sense organ—we cannot distinguish

[1] In a footnote to his translation of this passage McGinnis has provided a brilliant reconstruction of this argument that slightly differs from mine. See Avicenna (2009b, Chapter III.4, sec. 5, 285, n. 9). An advantage of my analysis over that of McGinnis's is that, according to my reconstruction, the soundness of Avicenna's argument does not depend on the geometrical configuration of atoms. By contrast, McGinnis's reconstruction is built upon the *mutakallimūn*'s assumption that atoms are cuboidal. If his analysis is valid, so is mine. But the other way around does not necessarily hold.

such triangular objects from perfect right triangles. Moreover, the Pythagorean Theorem is not precisely applicable to imperfect objects that look very similar to perfectly right triangles. In other words, the second premise of the above argument is false and there cannot exist in the sensible world any perfectly right triangle with the aforementioned descriptions to which the Pythagorean Theorem is precisely applicable. Accordingly, the above argument fails. Thus, if we construe Avicenna as holding a purely abstractionist-fictionalist view about mathematical objects, we cannot explain how Avicenna might have thought that his argument could reject atomism. The argument is sound only if literalism is true. This can be considered as another justification for why Avicenna should be interpreted as a literalist.

My literalist reading of Avicenna's philosophy of mathematics might be better understood in contrast with McGinnis's well-developed abstractionist alternative. He says:

> [T]he estimative faculty is what allows the mathematician to consider perfect geometrical figures or numbers in the abstract *even though these are never instantiated physically*; it is the power that allows the physicists to imagine perfectly frictionless planes or a sphere touching a two-dimensional surface at a single point, *even though again in the nitty-gritty world around us none of these exists*. These mathematical abstracta, Avicenna says, exist by supposition (*bi-l-farḍ*), usually a supposition imagined by the estimative faculty. That is to say, while mathematical abstracta exist in a mental act of conceptualization (*taṣawwur*), they do not exist, at least not in the exact way that the mathematician investigates them, in the concrete material particulars that populate the world. It is the estimative faculty, then, that provides mathematicians and (theoretical) physicists with an idealized picture of the world.[1]

[1] McGinnis (2017, p. 80), my emphasis. The textual ground of McGinnis's analysis is chapter I.2 of *The Introduction* of *The Healing* (1952c, Chapter I.2, 12-13) from which I have quoted TEXT # 4.3 and TEXT # 4.4. As is clear from his analysis, we defend different readings of these passages. There is a parallel debate in the context of Aristotle's philosophy of mathematics. My reading of Avicenna is comparable to the literalist interpretations of Aristotle's ontology of mathematics as defended by, among others, Mueller (1970, 1990). By contrast, McGinnis's reading of Avicenna is analogous to the abstractionist-fictionalist interpretations of Aristotle as supported by, among others, Lear (1982) and Hussey (1991).

I agree with McGinnis that estimation plays a pivotal role in the mechanism of forming mathematical concepts. However, by contrast with him, I argued that perfect mathematical objects could possibly exist in the extramental world. Moreover, in the present context, the main role of estimation is *perceiving* what actually exists in the sensible world, rather than *constructing* a purely mental abstractum. Mathematical objects are, in the first place, perceived rather than produced, or so Avicenna seems to believe. The fact that by our estimation we separate mathematical objects from the specific matters they are attached to in the sensible world, does not imply that those objects do not exist in the sensible world.[1] This resembles the mechanism through which we perceive hostility. The fact that by our estimation we can separate hostility from the animal through seeing which we have perceived its hostility, does not imply that hostility does not exist in the sensible world. Mathematical objects are in this sense analogous to hostility and other attributes similar to it. They are all non-sensible connotational attributes that actually exist in the sensible world and can be perceived by estimation. They are not purely mental products.[2]

[1] Avicenna sometimes says that mathematical objects are *mujarrad*. See, among others, Avicenna (2009a, Chapter I.8, sec. 2 & 6). This fact can be considered as a justification of describing mathematical objects as mental *abstracta*. However, Avicenna's understanding of abstraction in this context is merely considering mathematical objects in separation (or isolation) from the specific species of matter they are accompanied with in the sensible world. In this sense, abstraction is not constructing a new entity. Rather, it is considering some specific features of some objects existing in the sensible world while overlooking their other features. In other words, abstraction in this context has primarily an epistemological—rather than ontological—function. Given this significant qualification, describing mathematical objects as *abstracta* is unproblematic.

[2] The essence or quiddity (*māhīya*) of an imperfect geometrical shape existing in the sensible world differs from the essence of its perfect counterpart existing as an abstractum in the mind. For example, the essence of a perfectly right triangle to which the Pythagorean Theorem is applicable differs from the essence of an imperfect object to which the theorem is not precisely applicable. The former object has some essential properties that the latter lacks. So they do not share the same essence. Accordingly, we should provide an explanation of *how* we can grasp the quiddity of the objects that neither have a counterpart in the sensible world nor are composed of elements each of which has a counterpart in the sensible world. Avicenna's concept empiricism seems to suggest that there is no such explanation. This argument can be considered as a serious epistemological challenge against any purely abstractionist interpretation which does not compromise Avicenna's concept

Admittedly, there are still many other mathematical objects that are studied by mathematicians although they have no counterpart in the sensible world. Our conception of such objects will be formed, more than anything else, under the influence of imagination.[1] By combining different parts of objects previously perceived by estimation, imagination can mentally build some new mathematical objects.[2] But even these mental products, inasmuch as they are subject to mathematical studies, should be considered as attributes of material entities. In other words, they should be treated by mathematicians *as if* they are non-sensible properties of some actually existing sensible objects. These items grasped or produced respectively by estimation and imagination will be delivered to the intellect where the intervention of the Active intellect turns them into purely immaterial universal concepts. So far, I have illustrated how we form mathematical concepts. It is now time to investigate how we assent to the truth of mathematical propositions.

4.5. Knowledge of Mathematical Propositions I: How Imagination Contributes

A proposition is an ordered structure constituted from concepts. So knowing the conceptual components of a proposition is necessary for knowing the proposition or, more precisely, assenting to its truth. However, knowledge of the conceptual components of a proposition does not automatically imply knowledge of that proposition. Due to lack of mathematical skill, one might be unable to assent to the truth of, for example, the proposition that 'the

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empiricism. As we saw in the first chapter, Avicenna (2005, Chapter VII.3, sec 2-3) puts forward a very similar challenge to argue against mathematical Platonism.

[1] This act of imagination is the source of a potentiality for widening the domain of objects that can be studied by mathematicians. Ergo, Avicenna endorses some sort of *literalism*, on the one hand, and some sort of *potentialism*, on the other. See also my discussion of Avicenna's potentialism in the second chapter.

[2] In this respect there seems to be no disagreement between me and McGinnis. I cannot agree more with the description of the cognitive roles of imagination as it is presented in McGinnis's book on Avicenna (2010a, pp. 114–115).

largest prime number less than twenty is nineteen' even if he knows all of its conceptual components (e.g., the concept PRIME NUMBER, the concept LARGENESS, and the numerical concepts TWENTY and NINETEEN). Thus we need to clarify the intermediary steps between acquiring the concepts a proposition is constituted from and assenting to the truth of that proposition. Roughly speaking, Avicenna, like Aristotle, defends a *foundationalist* theory of knowledge according to which all instances of knowledge rest ultimately on the foundations (*mabādiʾ*) of basic concepts and propositions which can be known in a direct and immediate way.[1] Non-basic concepts and propositions can be derived, respectively, by definitions (*taʿārīf* or *ḥudūd*) and syllogisms (*qiyāsāt*). So there seems to be three steps to take for assenting to the truth of a mathematical proposition after grasping its conceptual components: (1) forming and considering that proposition as a unity structured from concepts, (2) assenting to the truth of the foundational propositions of mathematics, and (3) drawing the proposition under discussion by some syllogisms from these foundational propositions. In the present section, I engage with the first and third issues. Then, in the next section, I turn to the second one.

According to Avicenna, imagination performs a set of crucial functions in both forming propositions and deriving them from basic propositional principles. As I explained, imagination has the potentiality to search through the items grasped by the soul and to consider different combinations of these items (or their parts) by arranging them in various ways. Thus imagination seems to be, at least at first glance, the most bona fide faculty for considering different possible structures of concepts and forming meaningful propositions that are liable to truth and falsity. Moreover, proving a mathematical theorem is nothing but

[1] See Avicenna (Avicenna, 1985, pp. 112–113). The application of the modern terminology of 'foundationalism' to this context is inspired by Black (2013a). Her description of this notion is neutral with respect to the *a priority* (or *a posteriority*) of the foundations from which all other instances of knowledge should be derived. McGinnis (2008) does not deny that Avicenna is a foundationalist in this minimal sense but insists that Avicenna's foundationalism should not be confused with contemporary foundationalist accounts of justification according to which "the justification or verification of a body of beliefs must ultimately be based on what contemporary philosophers have variously termed '*a priori* truths', 'self-evident truths', 'self-presenting truths', and 'the given'."

arranging a chain of appropriate syllogisms which lead us from the propositional principles of mathematics to that theorem. Making a good syllogism to establish a proposition is itself looking for and finding an appropriate middle term.[1] So in thinking about mathematical propositions and providing proofs for them we are in principle dealing with a *search* process through the items that are previously grasped by the soul; and this is exactly what we expect to be done perfectly by the imagination (or its rational manifestation, i.e., the cogitative (*mufakkira*) faculty). Therefore, it seems quite reasonable to consider imagination as the faculty in charge of (1) and (3). This construal, however, is vulnerable to an inevitable objection.

As I said, imagination is usually known as a bodily faculty which is placed in a particular place of the brain. Given its material nature, many scholars believe that imagination cannot directly engage with fully immaterial entities. If so, imagination has no direct access to universal concepts and propositions (which are themselves specific structures of concepts) and cannot carry out any immediate operation upon them. So forming propositions and looking for their proofs cannot be conducted by imagination. To overcome this impediment, Gutas argues that for Avicenna thinking occurs at two parallel levels, one conducted by imagination and the other by the intellect. He says:

> [Avicenna sets up] *two parallel processes of thinking*, one in the rational soul and the other in the animal. The function of the former is to combine universal propositions or terms to form syllogisms and reach conclusions—essentially, what we call plain reasoning—only that it takes place necessarily in the intellect because of the immaterial nature of the concepts involved, the intelligibles. The function of the second process in the animal soul, that of the Cogitative faculty, is to combine conceptual images of particulars *in imitation of* (*muḥākāt*) the process in the intellect for the purpose of aiding it. Particulars and their images are always at hand, whereas

[1] The middle term can also come to the mind instantaneously and without any search by way of *intuition* or *guessing correctly* (*ḥads*). However, for the sake of brevity, I put aside all the debates around the notion of intuition and confine my discussion to non-intuitive thinking. For a detailed discussion on various aspects of the notion of intuition, see Gutas (2001).

> the intelligibles are not; one therefore starts with what is available—like the geometer drawing his diagrams—in order to proceed to correct conceptualization and reach the abstract solution. The function of cogitative faculty thus is useful, as Avicenna says, like the geometrical diagrams, but it is imitative and hence derivative; the real thinking with the real intelligibles takes place in the rational soul.[1]

Gutas suggests that the cogitative faculty and, *a fortiori*, imagination play only a preparatory and secondary role in the mechanism of thinking. The cogitative faculty facilitates the procedure of reasoning by constructing an imaginative *model* (or imitation) of what should be actually going on in the intellect. This model mirrors what is happening in the intellect; but it is not itself the *real* mechanism of thinking (or even part of it). For instance, geometrical diagrams can shed light on the path that should be followed by the intellect to establish the truth of a geometrical theorem. But these diagrams cannot be themselves subject to the operation of the intellect. Otherwise put, because of its corporeal nature, the cogitative faculty cannot engage with universal concepts that are purely immaterial entities. Nonetheless, for many (if not all) of these concepts there are counterpart images that are *particular* entities accessible to the cogitative faculty.[2] By working on these particular images, instead of universal concepts, the cogitative faculty can carry out a reasoning mechanism that looks like what occurs in the intellect. However, the real mechanism of thinking proceeds only with incorporeal universal intelligibles and occurs exclusively in the intellect.

There is however a rival reading of Avicenna's view about thinking. Criticizing Gutas's account, Black considers a *hybrid* nature for the cogitative faculty because of which it can work with not only particular images but also universal intelligibles. She says:

[1] Gutas (2001, p. 22), emphasis in original.

[2] In the case of geometry, many (if not all) of these images are visual. However, Gutas does not claim that every imaginative correlate of a universal concept is necessarily visual. Indeed, in the present context, any imitation of a universal concept that is accessible to the faculty of imagination can be considered as an image, regardless of whether or not the imitation is visual. The cogitative faculty can look for these images in their storehouse, i.e., the imagery (*khayal*) faculty.

> On the one hand, as a manifestation of the compositive imagination, the cogitative power is a bodily faculty whose proper objects are sensible images and the estimative intentions associated with them. As an internal sense power, its distinguishing characteristics are its combinatory capacity and its incessant exercise of that capacity. On the other hand, the cogitative faculty is also rational by definition, and as such it has some sort of access to the universal intelligibles that are the proper objects of an immaterial intellect.[1]

Black repudiates the idea that the intellect on its own can undergo any process of thinking. The exclusive agent of thinking in the human soul is the faculty of imagination when it is functioning under the control of the intellect. That is perhaps why Avicenna has chosen the label 'cogitative' (*mufakkira*) for this specific function of imagination. According to Black, the role of the cogitative faculty in thinking and reasoning is neither merely preparatory nor substitutable with any independent action of the intellect.[2] Of course, the epistemology of mathematics is not the main focus of Black and Gutas in this debate. Nonetheless, if we investigate some of Avicenna's passages regarding the role of diagrams in geometrical reasoning—while having the central issue of the aforementioned debate in the background—we will arrive at clearer understandings about Avicenna's theory of mathematical reasoning, on the one hand, and about the more tenable position in the debate,

[1] Black (2013b). Both Black's 'compositive imagination' and Gutas's 'imagination' refer to the same thing, i.e., the faculty of '*mutakhayyila*'.

[2] Black (1997, sec. 4) and Davidson (1992, Chapter 4) defend the same position. To the contrary, Adamson (2004) and Mousavian and Ardeshir (2018) are sympathetic to Gutas's approach. It is worth mentioning that, unexpectedly, in some places Gutas's description of the function of the cogitative faculty resembles that of Black. For example, he says (2006, p. 356): "when used by the rational soul, this faculty [i.e., imagination] is also called cogitative (*al-mufakkira*), in the sense that it combines and separates concepts." Here it seems that Gutas concedes that the cogitative faculty can in principle entertain *concepts*. Accordingly, the contrast between Gutas and Black is preserved only if concepts are not understood as universal intelligibles that are entirely immaterial. Otherwise, Gutas's statement implies the core of Black's view, i.e., that the cogitative faculty has a hybrid nature because of which it has access to both particular material images and universal immaterial concepts.

on the other hand. In *The Discussions*, replying to a question raised by Al-Kirmānī, Avicenna says:

> **TEXT # 4.11.** It is necessary to know that combining the universal terms [to make a syllogism] is not something that can be done by bodily faculties and organs; even if the compliance (*idhʿān*) of those faculties and their imitation of that [combining mechanism] by means of particular images—as the geometer does with his board and stylus—is beneficial (*nāfiʿ*).[1]

There are three remarkable points about this passage. First, it explicitly affirms that the mechanisms of thinking and making syllogisms, due to their engagement with universal intelligibles, cannot be carried out by the bodily faculties. Such faculties can entertain only particular items which have not entirely lost their association with matter. Second, the passage acknowledges the possibility that the real mechanism of thinking can be modeled by bodily faculties through the employment of some particular imitative images.[2] Furthermore, drawing geometrical diagrams is introduced as an example of such models and imitations. Third, the passage does not claim that the aforementioned function of the bodily faculties is necessary and ineliminable for the mechanism of thinking (albeit *after* grasping the required universal concepts). Such a function is beneficial but not indispensable. In several other places, Avicenna has emphasized the secondary role of geometrical diagrams in proving geometrical theorems. For example, in *The Demonstrations*, Avicenna discusses and affirms Aristotle's view on how geometrical diagrams, despite the fact that they might be not mathematically ideal, can help us to grasp the truth of geometrical theorems. He reports:

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[1] Avicenna (1992, sec. 112). This passage is Gutas's main evidence for the view I quoted from him.

[2] For another passage from *The Discussions* in which Avicenna suggests that thinking can be carried out by both the faculty of imagination and the intellect, see Avicenna (1992, sec. 255). There he contends that the referent of the 'thinking faculty' (*al-quwwa al-fikrīya*) is determined by the nature of the objects upon which this faculty is operating and by the exact characteristics of the cognitive action it is performing. If by 'thinking' we mean conducting a search directly through intelligibles, then the 'thinking faculty' refers to the rational soul, rather than the faculty of imagination.

> **TEXT # 4.12.** It is said [by Aristotle] that the drawn line and the drawn triangle are not drawn because the demonstration needs them. The demonstration [of a geometric theorem] is [demonstrated] on a line which is really [i.e., perfectly] straight and width-less; and [it is demonstrated] on a triangle which has really [i.e., perfectly] straight sides with the same length. This triangle and that line [drawn on the paper] are rather for preparation of the mind to imagine. Demonstration is [demonstrated] on the intelligible, not sensible or imaginable (*mutakhayyal*). If it was not difficult to conceive the demonstration abstracted from imagination, there would be no need at all to draw geometrical figures.[1]

Here Avicenna argues that geometrical diagrams that we usually draw to provide demonstrations for geometrical theorems do not represent ideal mathematical objects. Therefore, such diagrams cannot be the real objects of our demonstrations.[2] One might count this passage as evidence against my literalist interpretation of Avicenna's ontology. However, the passage, as I understand it, does not deny the existence of perfect mathematical objects in the material world in general. Nor does it conflict with the idea that mathematical objects are—or, at least, must be considered as—properties of sensible objects existing in the extramental world. The passage merely claims that a very specific kind of objects, i.e., *sensible* figures drawn on a piece of paper, are not perfect mathematical objects. I contend that the emphasis of the passage is on the intelligibility, immateriality and universality of the objects of demonstrations, rather than on the perfectness of those objects. If the emphasis was on the latter issue, he could not rule out the eligibility of imaginable things for being the objects of geometrical demonstration. This is because perfect mathematical objects are easily imaginable, even if not sensible. Demonstrating a geometrical theorem is a procedure in which we should primarily engage with universal intelligibles, rather than particular

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[1] Avicenna (1956, sec. II.10, 186). He is probably referring to Aristotle's *Prior Analytics* (49b32–50a4).

[2] There is a passage in *The Discussions* which puts forward a similar line of argument. See Avicenna (1992, sec. 580).

sensible things or even their counterpart mental images.[1] The last sentence of the passage suggests that, though very difficult, it is in principle possible to demonstrate a geometrical theorem without appealing to either geometrical diagrams drawn on paper or even their mental images.[2]

These passages show that, at least in the context of mathematical reasoning, Gutas is right. Universal concepts are not accessible to the bodily faculties. Therefore, it is the intellect that analyzes and synthesizes different combinations of concepts to form structured propositions and to find the appropriate middle terms for making conclusive syllogisms. In other words, it is primarily the intellect that is responsible for mathematical reasoning. However, this intellectual mechanism can be accompanied by a more terrestrial imitative mechanism that is conducted by the bodily faculties and particularly by the rational manifestation of imagination, i.e., the cogitative faculty. Compared to the universal concepts, the images that the cogitative faculty works with are less abstract and more easily accessible to the human soul. Therefore, the compositive faculty can make an imaginative maquette of what is (or should be) going on in the intellect to prove a mathematical theorem. So, for example, a geometrical diagram is like a map that guides the intellect to find the right path for establishing the truth of a mathematical theorem.

[1] Avicenna (1956, sec. II.10, 187) says: "The objects of demonstrations are intelligible forms that are abstracted from matter. They are neither sensible nor vulnerable to corruptibility."

[2] That, in the procedure of mathematical reasoning, the assistance of sensible or imaginable diagrams is useful but not necessary is highlighted also in a passage from *The Discussions*. See Avicenna (1992, secs. 151–152).

This general picture seems quite tenable. However, it is still not clear how the objects of the cogitative faculties and those of the intellect correspond to each other. Avicenna does not provide a general answer for this question. Nonetheless, in his discussions of the dependency of the function of the cognitive faculties (other than the intellect) on their material nature, he elaborates how the consideration of geometrical shapes by the bodily faculties (including the cogitative faculty) differs from their consideration by the intellect. A careful investigation of his view in this regard can reveal how particular images and universal concepts of geometrical shapes correspond with each other. Consider the following shape:

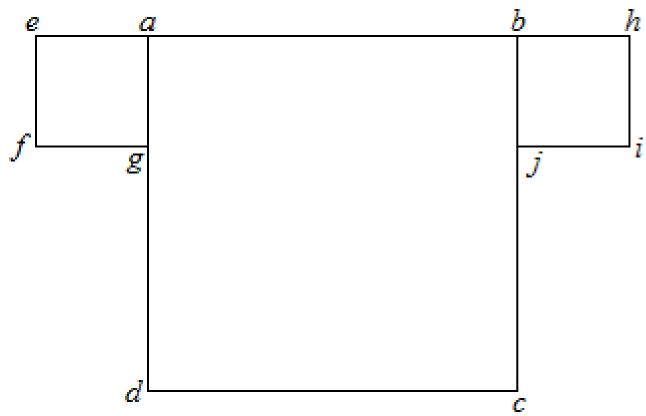

Fig. 1

In this figure, *eagf* and *bhij* are two numerically distinct squares that are exactly similar to each other in all respects except their positions; they respectively lie on the left and right sides of the square *abcd*.[1] Avicenna argues that the above shape comes to our imagination only when it is literally imprinted on a part of our brain.[2] It is only because of the different physical locations on which *eagf* and *bhij* are imprinted that we can distinguish these two squares. Avicenna puts forward an ingenious argument to show that if we take the above image as a fully immaterial entity, it does not make any sense to talk about right and left sides. Left and right can be defined only in relation to a physical position. Fully immaterial particulars lack such relational properties, and therefore there remains nothing by appealing to which we can distinguish those small squares. They are of the same essence and apart

[1] Obviously, in the mirror image the positions of these squares are reversed.

[2] Avicenna has presented this argument, with ignorable differences, in the psychology parts of both *The Healing* (1959, Chapter IV.3, 189-93) and *The Salvation* (1985, pp. 351–355).

from their material accidents, if any, all of their characteristics are exactly similar. So if we insist that this image is a fully immaterial but yet particular entity, then we should accept that *eagf* and *bhij* are indiscernible and, accordingly, identical. To reinforce the main point of Avicenna's argument we can remove the big square in the middle of Fig. 1 and consider the two small squares on their own:[1]

Fig. 2

We can definitely imagine these two squares. But if we consider their images as fully immaterial particulars, then there is nothing based on which we can distinguish them. They are of the same essence and have the same characteristics. Therefore, they are indiscernible and identical. Since we can actually distinguish two such particular images in our imagination, they should be literally imprinted on parts of our brain. It is only because of their physical locations that they are capable of possessing certain different relational properties, or so Avicenna argues. In general, Avicenna seems to believe that all geometrical diagrams that we imagine are particular things that are literally imprinted on some specific parts of the brain. The cogitative faculty can search through them and assist the intellect in the real mechanism of thinking. Nonetheless, the intellect itself has no access to these particular things. But there are certainly some universal theorems that hold for all geometrical figures that look like what we have in Fig. 1. Now one might wonder what the objects of such theorems could be. As we saw, Avicenna believes that the real process of thinking about such theorems occurs only in the intellect. So we should clarify how the intellect can consider the universal counterparts of the two small squares presented in Fig.

[1] To make the force of Avicenna's argument more apparent I have removed the alphabetical labels of the vertexes of these squares. Objects cannot be distinct from each other only because of their names. We can choose a name for a particular object (not another) only if we can distinguish it from the other objects. So individuation has an unavoidable priority over naming.

1. In particular, we should explain how the intellect can distinguish such universal counterparts from each other. Avicenna's suggestion is that the intellect considers them as two distinct collections of universal concepts. One of them is combined from the universal concepts SQUARE and LEFT and the other from the universal concepts SQUARE and RIGHT. Such collections of universal concepts still do not refer to particular objects;[1] and that is why the bodily faculties cannot distinguish particular images in the same manner.

One might suggest that we can distinguish those two particular squares in our imagination by merely *supposing* that one of them is on the right and the other on the left. Avicenna, however, rejects this possibility. We cannot suppose a particular object to have *any* arbitrary property we would like to attribute to it. There should be a ground for supposing that a particular object is on the right side, rather than the left. Moreover, we cannot suppose that a particular square, not the other, is on the right unless we have already distinguished these squares from each other. So we return to the question we started from: how can these squares be distinct from each other? The objector might insist that the squares will be individuated only after we add, by supposition, rightness and leftness to squareness. But if we endorse this proposal, we have no choice but to accept that imagination can actually entertain universals. The combination of rightness or leftness with squareness does not form particular objects. So this proposal conflicts with Avicenna's psychology according to which bodily faculties have no access to universals. The following passage of the *Book on the Soul* of *The Salvation* summarizes the core of Avicenna's argument and explains how the objects of the intellect differ from those of the bodily faculties:

> **TEXT # 4.13.** It will be asked: How was it possible for the supposer to suppose it [i.e., one of those two squares] in this state so that it becomes distinct from the second square? And, based on what did he suppose this square in this way and that square in that way? In the case of the universal [square], it becomes possible through an item which the intellect adds to it. This item is the concepts RIGHT and LEFT. And [adding] such a concept to a universal intelligible is sound. But in the case of the particular

[1] Roughly speaking, a collection of universal concepts does not characterize a particular objects unless it finds some association with some particular chunk of matter.

> [imaginative square], one concept but not the other cannot be found in it, unless there is something because of which it deserves to have this but not the other concept. And it is not the case that imagination supposes it [i.e., the particular square] together with a condition which it [i.e., imagination] adds to it; but imagination imagines the square all at once as if it is so in itself without any supposition. So it imagines this [square] on the right and that on the left just because of some condition that is already added to this or that [square]. There [i.e., in the intellect] the concepts RIGHT and LEFT can be added to the [concept] SQUARE—and this is a [concept of] square to which no other thing pertains. So this is the attachment of a universal to another. But here [i.e., in imagination] if it [i.e., the square] has not already found a particular defined position, it will not find it afterwards by any concept [that might be added to it]. Here it is not the supposition that puts the square in that [particular] position in imagination. Rather, it is the position [of the square] in imagination that makes the supposition true of it. And, imagination has no concept at all, since every concept is universal. So how can imagination add the concept itself [to the square]?[1]

This discussion is primarily intended to establish the ontological states of the objects of the bodily faculties in contrast with the objects of the intellect. It however clarifies a significant point about the epistemological function of imagination and, in particular, the cogitative faculty. Contrary to the intellect in which we deal with a collection of universal concepts, the cogitative faculty deals with images as particular unified structures. Relying on this picture, we can—although Avicenna himself does not—provide a reasonable explanation of why and how employing sensible or geometrical figures can facilitate the procedure of mathematical reasoning. In our cogitative faculty, we entertain different images of mathematical objects with no need to consider consciously all the things because of which these objects are distinct from each other. This gives us the opportunity to focus only on those aspects of the images that mathematically matter. However, if we insist on setting all of the images aside and directly considering their purely universal counterparts, then we must deal with complicated collections that are too crowded by universal concepts some of which have no

[1] Avicenna (1985, pp. 353–354).

direct mathematical significance. Moreover, dealing with collections of concepts seems to be more complicated than dealing with a unified picture that can be considered all at once.[1] As another witness for the auxiliary role of imagination in the mechanism of mathematical reasoning, we can refer to chapter III.2 of *The Demonstration* of *The Healing* in which Avicenna explains, among other things, why imagination is usually fruitful and reliable in mathematical reasoning while in other disciplines it might be misleading. There he elaborates that the mistakes occurring in the procedures of reasoning are usually due to employing either equivocal middle terms or fallacious moods of syllogism.[2] He then argues that the latter rarely (*fī al-nudra al-nādira*) happens in mathematics, since we usually employ the first or second mood of the first figure of syllogism (i.e., Barbara and Celarent).[3] But more pertinent to our present concern, he claims that mathematics and especially geometry are usually immune to the former kind of mistakes—i.e., making a syllogism based on equivocal terms—because parallel to the intellectual apprehension of geometrical terms, we acquire a mental image of them which helps us not to confuse the meaning of those terms with other things. These parallel mechanisms are described as follows:

> **TEXT # 4.14.** The first type [of the aforementioned mistakes] does not occur in mathematics. This is because the meanings of the terms referring to the geometrical things can be known through learning; so [when we deal with such a term] nothing but the intended meaning [of it] comes to estimation. For each of these terms, there is a meaning which can be understood in accordance with what it is aimed to refer to

[1] It should be noted that the cogitative faculty cannot imitate all the functions of the intellect. For example, there is no image of a self-contradictory object that is simultaneously fully black and fully white. Therefore, the cogitative faculty cannot conceive such objects through imitative images. By contrast, the intellect can easily conceive such objects and, accordingly, can concede that they do not exist. This is because the intellect is capable to consider such objects as an aggregated collection of universals which includes the concepts BLACK and WHITE. See Avicenna (1959, Chapter IV.3, 193). As a more geometrical example, we can consider square-circles. There is no image of such objects accessible to the cogitative faculty. Nonetheless, the intellect conceives such things by considering a collection of the universal concepts SQUARE and CIRCLE.

[2] Avicenna (1956, Chapter III.2, 196).

[3] Avicenna (1956, Chapter III.2, 198).

> or in accordance with a preceded definition. The meanings of those terms are close to imagination. While the intellect apprehends the meaning of one of these terms, an image (*khayal*) would be constituted for this [term] in estimation (*wahm*). The image of this term fixates the reality of that meaning and preserves it. So the mind will not be distracted from it.[1]

Avicenna believes that such perfect correlations between the objects of the intellect and their counterpart images—as the objects of the internal faculties—do not occur in other disciplines (e.g., dialectics);[2] and that is why appealing to imagination might be misleading in those sciences:

> **TEXT # 4.15.** Imagination in the disciplines other than mathematics, in most of the cases, is misleading; but in mathematics, it has a leading and guiding role. And, that is why it is difficult to teach mathematical issues without drawing sensible figures that are labeled by the alphabet letters. Such figures assist and strengthen imagination. By contrast with the other sciences, there is no fear of it [i.e., appealing to imagination] in mathematics.[3]

Although Avicenna does not directly refer to the cogitative faculty here, I take TEXT # 4.14 as another textual witness for Gutas's two-level account of thinking (especially in the context of mathematics).[4] By contrast with other disciplines, in mathematics concepts appropriately

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[1] Avicenna (1956, Chapter III.2, 196).

[2] Another passage in which Avicenna claims that imagination is useful in geometry but not in other sciences can be found in *The Notes* (1973, pp. 83–84).

[3] Avicenna (1956, Chapter III.2, 197).

[4] In TEXT # 4.14 Avicenna says that the image of a mathematical object is formed in estimation. So he might be alluding to the perception of mathematical objects, as connotational attributes of physical objects, by the faculty of estimation in the sense I explained in section three. The other possibility, with which I am more sympathetic, is that here estimation is taken as a representative of internal faculties. If so, Avicenna is just underlining a contrast between the apprehension of mathematical objects as universal concepts in the intellect and as particular images in the internal faculties. In any case, the main faculty which can search through such images and assist the intellect in mathematical reasoning (by separating and combining these images, making new images, etc.) is the cogitative faculty. However, estimation is the only internal faculty which can make

match with their counterpart images, and this enables the cogitative faculty to benefit the intellect by running an imitative mechanism of thinking. Instead of the universal concepts with which the intellect engages, the internal faculties entertain the imaginative correlates of those concepts. In particular, by analyzing and synthesizing images, the cogitative faculty can imitate what the intellect does (or should do) (1) to form propositions as unified structures of universal concepts and (2) to advance the procedure of reasoning by finding appropriate middle terms and making conclusive syllogisms. As explicitly mentioned in the last and some other previous passages, proving mathematical theorems without appealing to the sensible or imaginative figures (and, accordingly, without the help of the cogitative faculty) is difficult but not impossible. This indicates that the role of the cogitative faculty in mathematical reasoning is auxiliary and preparatory, rather than primary and necessary.

4.6. Knowledge of Mathematical Propositions II: How We Grasp the Propositional Principles

In the following passage of the logic part of *The Salvation* Avicenna offers a dense presentation of his foundationalism:

> **TEXT # 4.16.** Syllogism has parts that are assented to and conceptualized. And definition has parts that are conceptualized. But this does not go on *ad infinitum*, in such a way that knowledge of these parts is obtained through the acquisition of [knowledge of] other parts and this [process] is so *ad infinitum*. Rather, matters end in things that are assented to and conceptualized without the mediation [of our knowledge of other concepts and propositions].[1]

Our knowledge of a complex proposition is obtained, through syllogisms, from our knowledge of simpler propositions and concepts. Our knowledge of a complex concept is

judgments about these mental images. That is why estimation plays the central role in conducting thought experiments.

[1] Avicenna (1985, p. 113).

obtained, through definitions, from simpler concepts. Simpler propositions and concepts can themselves be derived from even simpler instances of knowledge. But this procedure, Avicenna believes, cannot go on *ad infinitum* and must stop at some point. Consequently, the whole system of knowledge can be reduced to a collection of basic concepts and propositions which are known in an immediate way without being derived from other instances of knowledge. Making conclusive syllogisms and providing useful definitions are primarily functions of the intellect. However, as we saw in the previous sections, the performance of the intellect in this regard is assisted by the internal faculties and, in particular, the cogitative faculty. Moreover, I showed how our knowledge of mathematical concepts is based on and rooted in what we perceive from the sensible world through the function of the faculty of estimation. Thus the last piece of the jigsaw of Avicenna's epistemology of mathematics is his view regarding the epistemological status of the propositional principles (or basic propositions) of mathematics. In this section, I expound how, according to Avicenna, we assent to the truth of the propositional principles of mathematics.

Avicenna puts forward three different criteria for categorizing the principles (*mabādiʾ*) of knowledge. He first categorize these principles in terms of their generality and inclusiveness. Proper principles (*mabādiʾ khāṣṣa*) are exclusively for a particular science. By contrast, general principles (*mabādiʾ ʿāmma*) belong to more than one scientific discipline. A few such general principles can be considered as the foundational principles of all sciences. That 'every declaration is either true or false' is an example of such a general principle.[1] The general propositional principles that belong to all sciences do not need to be demonstrated because they have no middle term. Quite the contrary, the proper propositional principles might have middle terms. If so, they must be demonstrated either in the same science of which they are principles or in another science.[2]

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[1] Avicenna (1956, Chapter II.6, 155-6).

[2] Avicenna (1956, Chapter I.12, 110). For the ease of teaching, the teacher of a science could ask her student to accept as a proper principle one of the propositions of the science that can be proved in the advanced stages of education. Avicenna believes that the parallel principle of the Euclidean geometry is an example of such a proper principle. See (1956, Chapter I.12, 114).

The second classification is based on the mixture of a logical criterion and an epistemological one. The principles that have no middle term do not necessarily belong to all sciences. For example, that 'things which are equal to the same thing are equal to each other' has no middle term but it belongs only to mathematical sciences.[1] Following Aristotle, such a principles is called 'axiom' (*ʿilm mutaʿāraf*). Axioms are necessary self-evident propositions whose truths are assented to directly by the intellect.[2] The principles that have middle terms can be divided into two groups based on the epistemic attitude of the student who is learning them. 'Hypotheses' (*uṣūl mawḍūʿa*) are those principles that seem plausible to the student, although she has no proof for them. 'Postulates' (*muṣādarāt*), on the other hand, are those principles that seem dubious to the student. Both hypotheses and postulates need to be demonstrated because they have middle terms. So their logical status is different from the axioms which have no middle term. Since different people can in principle have different cognitive attitudes towards the same principles, one person might consider a proposition as a hypothesis while it is taken by another as a postulate.[3]

The third and the most significant criterion for classifying propositional principles of syllogism is the epistemic channels through which we accept these propositions and assent to their truth. Avicenna's classification of the principles based on this criterion appears, with some slight modifications, in many places of his oeuvre.[4] According to *The Salvation* principles of syllogism are divided into sixteen types which include propositions grasped through (or based on) (1) imaginative data (*mukhayyalāt*), (2) sense data (*maḥsūsāt*), (3) data of reflection (*iʿtibārīyāt*), (4) tested and proven data (*mujarrabāt*), (5) data provided by finding the middle term of a syllogism (*ḥadsīyāt*), (6) data provided by sequential and

[1] Avicenna (1956, Chapter II.6, 155).

[2] Avicenna (1956, Chapter I.12, 110).

[3] Avicenna (1956, Chapter I.12, 114-5). Although postulates are defined relationally to the cognitive state of the person who considers them, Avicenna contends that it is better to consider the parallel principle of the Euclidean geometry as a postulate, rather than a hypothesis.

[4] See, among others, *The Demonstration* of *The Healing* (1956, Chapter I.463-7), and the logic parts of *The Salvation* (1985, pp. 112–123) and *The Pointers and Reminders* (1957, Chapter I.6, 341-64). For two studies on this classification, see Gutas (2012) and Black (2013a).

multiple reports (*mutawātirāt*), (7) estimative data (*wahmīyāt*), (8) primary data (*awwalīyāt*), (9) data with built-in syllogisms (*qaḍāyā qiyāsātuhā maʿahā* or *muqaddamāt fiṭriyyat al-qiyās*), (10) equivocal data (*mushabbahāt*), (11) conceded or admitted data (*musallamāt* or *taqrīrīyāt*), (12) absolute endoxic data (*mashhūrāt muṭlaqa*), (13) limited endoxic data (*mashhūrāt maḥdūda*), (14) data approved on authority (*maqbūlāt*), (15) initially endoxic but unexamined data (*mashhūrāt fī bādiʾ al-raʾy al-ġayr al-mutaʿaqqab*) and (16) suppositional data (*maẓnūnāt*).[1] In what follows, I first argue that the principles of arithmetic and geometry are restricted to the propositions grasped through either primary data or data with built-in syllogisms. Investigating the main characteristics of these two groups of propositions sheds a new light on Avicenna's view about the epistemological status of mathematical propositions in general. So let me start with the following passage from *The Demonstration*:

> **TEXT # 4.17.** And, in any case, it is required to posit that the principles of the [demonstrative] sciences are definitions and premises that are necessary to be accepted (*wajib qabūluhā*) in the primary [state] of the intellect (*fī awwal al-ʿaql*), or by sense [perception] (*bi-l-ḥiss*) or by [methodic] experience (*bi-l-tajriba*) or by a self-evident syllogism in the intellect (*bi-qiyās badīhī fī al-ʿaql*). And after them there are hypotheses (*uṣūl mawḍūʿa*) which can [in principle] be doubted but the student's opinion does not contradict them and postulates (*muṣādarāt*) [which do not seem plausible to the student]. And it is not the case that hypotheses are used in every science. Rather, there are sciences, e.g., arithmetic, in which only the definitions (*al-ḥudūd*) and the primaries (*al-awwalīyāt*) are used. But in geometry all of them are used. Similarly, all of them are used in natural sciences, though [they are] indistinguishably mixed.[2]

[1] Here I use Gutas's translations for these terms. See Gutas (2012, pp. 396–398). Some of these categories overlap to some extent. For example, Avicenna (1956, Chapter I.466) explicitly mentions that although "all primary propositions (*awwalīyāt*) are endoxic (*mashhūra*) and widely accepted, the other way around does not hold."

[2] Avicenna (1956, Chapter I.12, 112).

Here Avicenna is putting forward a three-level analysis. First, he distinguishes between (1) the principles that are certain and necessary to be accepted, (2) hypotheses, and (3) postulates. It is plausible to think that the first group of these principles refer to *axioms* (recall that according to the second categorization mentioned above, principles are divided into axioms, hypotheses and postulates).[1] Second, Avicenna restricts the axioms of the demonstrative sciences (from the aforementioned sixteen) to the following four categories: (a) primary propositions (*awwalīyāt*), (b) sensible propositions (*maḥsūsāt*), (c) tested and proven propositions (*mujarrabāt*), and (d) propositions with built-in syllogisms (*qaḍāyā qiyāsātuhā maʿahā*). Third, Avicenna clarifies the difference between arithmetic, geometry and natural sciences in terms of employing different groups of these principles.

Avicenna's view regarding the latter issue can be rendered in different ways. One might say that according to Avicenna arithmetic is based merely on definitions and primary propositions—i.e., (a). However, this interpretation conflicts with his introduction of some propositions of arithmetic as examples of propositions with built-in syllogism. For instance, as we will shortly see, he considers 'every four is even' as an example of such propositions. Therefore, it is indisputable that the principles of arithmetic should include the fourth category of axioms—i.e., (d). To accommodate this observation, I suggest that the term 'primaries' (*awwalīyāt*) has been used equivocally in the forgoing passage. In a restricted sense, primary propositions are one of the four types of the axioms of demonstrative sciences. However, in a more general sense, primary propositions are *a subset of the axioms* and include not only the primary propositions in the restricted sense but *at least* also the propositions with built-in syllogisms.[2] Accordingly, I understand the above text as saying that contrary to arithmetic which is based solely on the axioms, geometry and natural sciences have hypotheses and postulates. However, contrary to geometry whose axioms,

[1] In this regard, I and Ardeshir (2008) share the same interpretation.

[2] One possibility might be that the term '*awwalīyāt*' in the above passage is employed in a non-technical sense and simply refers to the *first* group of the propositions he mentioned before hypotheses and postulates. If so, '*awwalīyāt*' is referring to axioms in general. So what Avicenna has in mind here is that by contrast to geometry which has axioms, hypotheses and postulates, arithmetic has only the first group of principles, i.e., *awwalīyāt*.

hypotheses, and postulates are explicitly distinguished from each other and listed separately (as we see in Euclid's *Elements*), the principles of natural sciences are not classified in distinct categories.[1]

The most relevant issue to our current purpose which we can extract from the above passage is that the axioms of demonstrative sciences, including arithmetic and geometry, are of one of these four types: (a) primary propositions, (b) sensible propositions, (c) tested and proven propositions, and (d) propositions with built-in syllogisms.[2] Taking into account some further facts, we can rule out (b) and (c) from the list of principles of arithmetic and geometry. The first fact is that all sensible propositions (*maḥsūsāt*) and (c) tested and proven propositions (*mujarrabāt*) are about things that are associated with particular kinds of matter. However, numbers and geometrical figures—inasmuch as they are subject to mathematical studies—should be considered as separated from all specific kinds of matter. Therefore, sensible propositions and tested and proven propositions do not express abstract enough facts to be about mathematical entities. Mathematical objects are more abstract than physical objects that should be considered in association with particular matters. Mathematical objects exist *in* the sensible things; nonetheless, they are not themselves sensible forms. They are non-sensible connotational attributes of the sensible objects. As a result, we cannot study mathematical objects through the methods we usually use for studying sensible things. Most remarkably, we cannot study mathematics by methodic experience (*tajriba*). Accordingly, sensible propositions and tested and proven propositions which express the sensible features of physical objects and are derived from conducting

[1] Why arithmetic has no hypotheses and postulates is a significant problem which deserves an independent study. I think it is because of the different ontological status of numbers and geometrical shapes. As I showed in the second chapter, compared to the former, the latter has a stronger dependency on matter.

[2] Hereafter, I use the phrase 'primary propositions' in its technical restricted sense as one of the sixteen types of the principles of syllogism.

methodic experience cannot be considered as the principles of mathematics.[1] In *The Physics* of *The Healing* Avicenna says:

> **TEXT # 4.18.** So, in the construction of demonstrations in the disciplines of arithmetic and geometry, neither discipline needs to turn to natural matter (*al-mādda al-ṭabīʿīya*) or take premises that refer to matter in any way.[2]

This passage allows us to remove (b) and (c) and to restrict the axioms of pure mathematics to (a) the primary propositions, and (d) propositions with built-in syllogisms. It is noteworthy that apart from these two types, Avicenna himself never mentions a proposition of either arithmetic or geometry as an example of any other type of the aforementioned sixteen types of the principles of syllogisms. This indicates that my approach is on the right track. The above text moreover shows that not only axioms of arithmetic and geometry but also hypotheses and postulates of geometry are eventually rooted in the primary propositions and propositions with built-in syllogisms. As I explained, hypotheses and postulates of geometry in principle have middle terms and need to be demonstrated either in geometry itself or in another science (e.g., arithmetic or metaphysics). Nonetheless, they cannot be grounded on the principles that refer to natural kinds of matter, or so the above texts seem to imply. This again means that sensible propositions and tested and proven propositions cannot be relied on. So to determine the epistemological status of the principles of mathematics it suffices to scrutinize how we grasp the primary propositions and the propositions with built-in syllogisms. In his discussions of the principles of the sciences in *The Demonstration* of *The Healing* Avicenna introduces these two kinds of propositions as the only examples of the principles whose necessity (*ḍarūra*) is internal (*bāṭinīya*) to the human soul and comes from within the intellect, rather than from the other cognitive faculties. He says:

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[1] This does not however rule out the possibility that such propositions can in principle be considered as the foundational propositions of applied mathematical sciences like astronomy and music.

[2] Avicenna (2009a, Chapter I.8, sec 8), McGinnis's translation. In contrast with *natural* matter we can consider the *estimative* matter (or, in a more Aristotelian guise, *intelligible* matter). I showed in the second chapter that pure mathematics can consider mathematical objects as entities associated with estimative matter.

TEXT # 4.19. And the internal necessity either comes from the intellect or comes from the outside of the intellect and is due to a faculty other than the intellect. That which is from the intellect comes from either the intellect alone (*mujarrad al-ʿaql*) or the intellect while it is supported by something. That which comes from the intellect alone is the primary (*al-awwalī*) [proposition] whose acceptance is necessary. An example [of a primary proposition] is our saying that 'the whole is greater than the part.' And [now we turn to] what comes from the intellect while it is supported by something. Either the support (*muʿīn*) [of such a proposition] is not inherent in the intellect (*ghayr gharīzī fī al-ʿaql*), if so, this assent (*taṣdīq*) occurs (*wāqiʿ*) [in the intellect] by an acquisition (*bi-kasb*), thus it is beyond the principles, but we are talking about the principles. Or the support is inherent in the intellect; i.e., it is present [in the intellect]. Such an assent is what is known through a syllogism whose middle term innately (*bi-l-fiṭra*) exists and is present to the mind. Thus once the desired [principle] (composed from a minor term and a major term) is present the intermediate between these two terms is represented to the intellect without any need for its acquisition. And this is like our saying that 'every four is even'. It will be represented to everyone who understands [the concepts] FOUR and EVEN that '[every] four is even'. This is because it will be instantaneously represented that it [i.e. every four] is divisible to two equals. Similarly, whenever [the concepts] FOUR and TWO are represented to the mind, it will be instantaneously represented that four is the double of two. But if we replace [our example] with [the case of] thirty six or another [big] number, the mind needs to look for the middle term [since it will be not instantaneously presented]. It is better to call this kind [of principles that come from the intellect while it is supported by something that is inherent in it] as a 'premise with innate syllogism' [or premise with built-in syllogism].[1]

[1] Avicenna (1956, Chapter I.4, 64). Here the notion of *innateness* should not be understood as *being-given-at-birth*. In this passage, *innateness* merely refers to the natural operation of the intellect. Principles with built-in syllogisms are those propositions that after conceiving their minor and major terms, their middle term, through the natural operation of the intellect, comes to mind and a syllogism will be automatically constructed. In

According to this passage, primary propositions have no middle term and their truths are assented to directly by the intellect. As soon as we consider such a proposition the intellect affirms its truth. By contrast, propositions with built-in syllogisms have middle terms and, in principle, need to be demonstrated. However, they are such that as soon as the intellect considers their subject and predicate, their middle term will automatically appear in the intellect and their truth will be instantaneously affirmed. Therefore, in order to assent to the truth of these propositions the intellect does not need to appeal to anything outside of itself. Particularly, after grasping the conceptual components of such principles we do not need to receive any further data from the extramental world to affirm that they are true. The intellect can independently assent to the truth of them. The following passage from the logic part of *The Salvation* supports this construal in a more evident manner:

> **TEXT # 4.20.** Primaries are propositions or premises that are generated in man on account of his intellective faculty with no cause that necessitates assent to them except their own essences and the meaning which makes them proposition[s], i.e., the cogitative faculty. [The latter] joins simple [elements, i.e., concepts,] by way of affirmation and negation. When the simple concepts come about in man either with the help of the senses or the faculty of imagination (*khayal*) or [in] some other way and then the cogitative faculty compounds them, the mind must assent to them from the very beginning, without [recourse to] another cause and without feeling that this is something only recently acquired. Rather, man believes that he always knew it [...]. An example of this is ‘the whole is greater than the part’. *This proposition is not acquired from a sense or induction or anything else.* True, the sense[s] may be useful for conceiving the concepts ‘WHOLE’, ‘GREATER, and ‘PART’. But assenting to the truth of this proposition is due to a primary nature (*jibilla*) [of the intellect].[1]

general, I agree with Gutas (2012) that ‘*fiṭra*’ is not itself an independent cognitive faculty and, depending on the context, can refer to the natural operation of different cognitive faculties.

[1] Avicenna (1985, pp. 121–122). The translation is taken, with slight modifications, from Ahmed (2011, p. 95). The emphasis is mine. Ahmed uses the phrase ‘the formative imagination faculty’ as the translation of ‘*khayal*’. I have changed it to ‘the faculty of imagination’ and this might seem to be in conflict with what I had suggested

Interestingly, this passage reaffirms some of my claims in the previous section. As I explained, the faculty of imagination and its intellectual manifestation, i.e., the cogitative faculty, assists the intellect in, respectively, forming concepts and forming propositions as unified structures of concepts that are capable of being true or false. But the more important point of the passage that should be emphasized is that although the conceptual components of the primary propositions might be formed through the data we receive from the sensible world, the truth of such propositions can be assented to with no need for any further experience of the sensible world. Coupling the last two passages, we can conclude that if the conceptual components of a primary proposition or a proposition with built-in syllogism are properly conceived, then their truths can easily be assented to through the natural operation of the intellect, and independently from what is going on in the sensible world.[1] Since all mathematical propositions are eventually grounded on these two kinds of principles, this indicates that the truth of the principles of mathematics is independent from sensible data and can be assented to by the intellect alone.

Moreover, in the last section, I showed that more complex mathematical propositions can be derived from the principles through making syllogisms. By looking for suitable middle terms through mental images and mathematical diagrams, the cogitative faculty assists the intellect in the process of thinking and making these syllogisms. Nonetheless, the intellect can in principle conduct the mechanism of mathematical reasoning independently from all bodily faculties and by direct engagement with universal concepts (rather than their imaginative counterparts). If this construal is sound, forming mathematical concepts and assenting to the

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in the previous sections. This is because of Avicenna's imprecise and confusing terminology. In this text, he has used the term '*khayal*' to refer to the function of the faculty of '*mutakhayyila*'. That is why I have translated '*khayal*' as 'the faculty of imagination', rather than 'the image-bearing' or 'imagery faculty'. Black (2013b) clarifies that employing such sloppy terminology is not unusual in the context of Avicenna's discussions of the faculty of imagination.

[1] This does not mean that the Active intellect has no contribution to grasping those propositions. Indeed, there are some passages in which Avicenna says that primary intelligibles are the first things that are originated from the Active intellect in the material intellect and transform it to the dispositional intellect. See Avicenna (1984, p. 99). It does not, however, violate the independence of the intellect from the sensible world in assenting to the truth of the primary propositions.

truth of mathematical propositions are of two different epistemological statuses. As we saw, Avicenna endorses some sort of concept empiricism regarding mathematical concepts. He believes that mathematical concepts (at least many of them) cannot be grasped if we have no access to the sensible world. By contrast, after grasping the required mathematical concepts we can prove mathematical theorems without appeal to any further perceptual experience. Recall the thought experiment Avicenna proposed in TEXT # 4.2. According to that experiment, if we have no external senses, we cannot know any mathematical concepts. Now we can design a thought experiment of a similar spirit for grasping mathematical theorems. We can envisage by our estimation an individual who loses all of her external senses after grasping all required mathematical concepts. If Avicenna was asked whether this individual can know mathematical propositions, he would say that she still has the opportunity to discover some amazing mathematical facts. After grasping the required mathematical concepts, we no longer need our external senses to make mathematical judgments.[1] So Avicenna's empiricism about mathematical concepts is followed by a *rationalism* about mathematical judgments. Forming mathematical concepts without appealing to external senses is impossible; by contrast, after attaining the required mathematical concepts, the intellect can assent to the truth of mathematical theorems without any reliance on the sense data.

4.7. Conclusion

According to Avicenna many mathematical objects actually exist in the sensible world. They are not however themselves sensible forms. They are specific connotational attributes of

[1] Avicenna does not discuss this thought experiment himself. However, in the logic part of *ʿAlāʾī Encyclopedia*, he considers a very similar thought experiment about assenting to the truth of the primary propositions (which are the core of mathematical principles). Consider someone who suddenly comes into this world but does not know anything except the meaning of the conceptual components of a primary proposition. Avicenna claims that such a person cannot doubt that proposition and cannot refrain from assenting to the truth of it. Mousavian and Ardeshir (2018) have labeled this thought experiment as 'The Ideal Man' thought experiment.

physical objects existing in the extramental world. We can perceive mathematical objects by the faculty of estimation. In this faculty, mathematical objects are still considered as properties of material things, but their association with the specific kinds of matter they are attached to in the sensible world is neglected. So mathematical objects are perceived as things associated with matter but not with a determinate species of it. There are of course many mathematical objects that we study in mathematics which have no counterpart in the extramental realm. According to Avicenna's epistemology, the faculty of imagination can construct such objects by analyzing, synthesizing, separating and combining different elements of the items previously perceived by and stored in the cognitive faculties. It is through the preparatory act of these two faculties that the intellect can grasp universal mathematical concepts. These faculties play necessary and ineliminable roles in perceiving mathematical concepts. I showed that, according to Avicenna's epistemology of mathematics, grasping mathematical concepts is strongly dependent on the data we receive from the sensible world. Someone who has no sense perception cannot obtain any mathematical concepts.

Things change at the propositional level. After grasping mathematical concepts, we can form mathematical propositions through the act of the cogitative faculty. Moreover, this faculty can help us to progress in mathematical reasoning. By searching through the imitative images of universal mathematical concepts, the cogitative faculty can help the intellect to find the appropriate middle terms it needs for constructing conclusive syllogisms and, in the end, establishing mathematical theorems. Geometrical diagrams are particularly useful for the proper operation of the cogitative faculty. The role of these diagrams and the whole function of the cogitative faculty in the mechanism of mathematical reasoning is however merely secondary and auxiliary. All sensible and imaginative diagrams can in principle be dispensed with. The intellect can proceed in mathematical reasoning independently from the cogitative faculty. Although it is not easy, the intellect can, on its own, draw more complex mathematical theorems out from the principles of mathematics by constructing syllogisms. Mathematical theorems are all grounded on either the primary propositions or the propositions with built-in syllogisms. I showed that according to Avicenna, after grasping the conceptual components of these propositions, the intellect can assent to their truth without relying on any further perceptual experience. This indicates that after grasping the required mathematical concepts neither assenting to the truth of the principles of mathematics nor proving more complicated theorems based on these principles depends on the data we receive through our perceptual experiences. The intellect can, in principle, carry out both of these things without the assistance of other animal cognitive faculties. On my account, Avicenna's epistemology of mathematics is a mixture of concept empiricism and judgment rationalism.

Conclusion

In this dissertation, I tried to provide a systematic and comprehensive portrait of Avicenna's philosophy of mathematics through investigating the core elements of his views regarding the ontology of mathematical objects and the epistemology of mathematical concepts and propositions.

I first explored the negative aspect of Avicenna's ontology of mathematics, which concerns the question of what mathematical objects are not. According to Avicenna, mathematical objects are not independent immaterial substances. They cannot be fully separated from matter. I showed that Avicenna's arguments in chapters VII.2-3 of *The Metaphysics of the Healing* should convince us that he rejects what is now called mathematical Platonism. However, a careful reading of the view he attributes to Plato also leaves no doubt that Avicenna's understanding of Plato's view about the nature of mathematical objects differs both from Plato's actual view and from the view that Aristotle attributes to Plato. And Avicenna never rejected his own description of Plato's view: quite the contrary, from Avicenna's point of view, Plato's theory about mathematical objects is a raw and imprecise version of Avicenna's own theory about the nature of mathematical objects.

I also discussed the positive aspect of Avicenna's ontology of mathematics, which concerns the question of what mathematical objects are. I showed that a detailed analysis of Avicenna's approach to the classification of the sciences reveals the core components of his ontology of mathematics. Avicenna believes that mathematical objects are specific properties of material objects existing in the extramental world. Mathematical objects can be separated, in mind, from all the specific kinds of matter to which they are actually attached in the extramental word. Nonetheless, inasmuch as they are subject to mathematical study, they cannot be separated from materiality itself. Even in mind they should be considered as properties of material entities. So mathematical objects have some strong sort of dependency on matter. Nonetheless, numbers and geometrical shapes have different degrees of dependency upon matter. I argued that, according to Avicenna, numbers can in principle be separated from matter; so they have no strict ontological dependency upon matter. However, number is receptive to decrease and increase (and other arithmetical ascriptions) only when it is attached to matter. So number, inasmuch as it is subject to arithmetical studies, should be considered as a property of material things. Otherwise, it is to be studied by metaphysics. In other words, number, inasmuch as it is studied by mathematicians, has some sort of epistemological dependency upon matter. By contrast, geometrical shapes have ontological dependency upon matter. Although we can separate them, in mind, from all specific kinds of matter they might be attached to in the extramental world, they cannot be separated from materiality itself, even in mind. So the dependency of numbers upon matter is in a sense weaker than that of geometrical shapes. But in any case, both of these two groups of mathematical objects should be considered as specific properties of material objects. Mathematical objects can (and many of them do) literally exist in the material world. Avicenna is a *literalist* regarding the ontology of mathematical objects.

It is based on this understanding of the nature of mathematical objects that Avicenna rejects the actual existence of mathematical infinities. He emphasizes that not only magnitudes and numbered things but also numbers themselves cannot be infinite. He does so by providing arguments that are much more sophisticated than their Aristotelian ancestors. By analyzing the structure of his main argument against the actuality of infinity, namely *The Mapping Argument*, I showed that his understanding of the notion of infinity is much more modern than we might expect. In particular, I showed that according to Avicenna a set of objects is infinite if and only if it can be put into a one-to-one correspondence with one of its proper subsets. This understanding of the notion of infinity is exactly what we find in Dedekind's definition of infinite sets. By contrast with Dedekind, however, Avicenna defends mathematical *finitism*. But this is only because of his ontology of mathematics, which is radically different from that of Dedekind.

In the last chapter, I engaged with Avicenna's epistemology of mathematics. He endorses concept empiricism and judgment rationalism regarding mathematics. He believes that we cannot grasp any mathematical concepts unless we first have some specific perceptual experiences. It is only through the ineliminable and irreplaceable operation of the faculties of estimation and imagination upon some sensible data that we can grasp mathematical concepts. I argue that for Avicenna many mathematical objects are non-sensible properties of physical objects existing in the extramental world. They are connotational attributes of those physical objects and like all other connotational attributes (e.g., the hostility of a wolf) are non-sensible and are perceived by the faculty of estimation. Admittedly, there are some peculiar geometrical shapes that have no counterpart in the physical world but can still be studied by geometers. Avicenna's proposal regarding such objects is that we can construct them through the operation of the faculty of imagination upon the data we have previously perceived through sense perception or estimation. So our conception of mathematical objects is either directly or indirectly rooted in the data we perceive through our perceptual experiences. The mediation of empirical data cannot be eliminated from the process of forming mathematical concepts. By contrast, after grasping the required mathematical concepts, independently from all other faculties, the intellect alone can prove mathematical theorems. Other faculties, and in particular the cogitative faculty, can assist the intellect in this regard; but the participation of such faculties is merely facilitative and by no means necessary. After grasping the conceptual components of a mathematical proposition we do not need any further perceptual experience to assent to the truth of the proposition.

According to this picture, mathematical abstraction is primarily a mechanism for forming conceptions of the objects that actually exist in the extramental world. It is not, at least in many ordinary cases, a mechanism for creating mental objects that have no counterparts in the material world. Avicenna is a *literalist*, rather than an *abstractionist* or a *fictionalist*. To establish this claim, I tried to rebut an objection according to which mathematical objects cannot literally exist in the extramental world because, contrary to material entities, mathematical objects are perfect. I showed that Avicenna explicitly endorses the existence of perfect mathematical objects in the extramental world, though not as sensible things (rather, as connotational attributes of sensible objects).

Combining the aforementioned elements together, we attain the first holistic picture of Avicenna's philosophy of mathematics. Admittedly, there remain aspects of the philosophy of mathematics about which Avicenna's view is not clear. For example, Avicenna's views about the nature of mathematical proofs or the nature of mathematical continuum are issues which still await independent study. More interestingly, however, I think we can draw upon some Avicennan insights in order to approach certain problems which remain important for contemporary philosophers of mathematics—in exactly the same way that some aspects of Aristotle's philosophy of mathematics have been reconstructed in the language of contemporary philosophy of mathematics in order to open up new ways to approach central problems of this field. I postpone such interesting issues to my future studies.

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