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Modular forms and $\mathrm{SL}(2,\mathbb{Z})$-covariance of type IIB superstring theory

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Abstract: The local higher-derivative interactions that enter into the low-energy expansion of the effective action of type IIB superstring theory with constant complex modulus generally violate the U(1) R-symmetry of IIB supergravity by q_U units. These interactions have coefficients that transform as non-holomorphic modular forms under $\mathrm{SL}(2,\mathbb{Z})$ transformations with holomorphic and anti-holomorphic weights $(w,-w)$, where $q_U = -2w$.

In this paper $\mathrm{SL}(2,\mathbb{Z})$-covariance and supersymmetry are used to determine first-order differential equations on moduli space that relate the modular form coefficients of classes of BPS-protected maximal U(1)-violating interactions that arise at low orders in the low-energy expansion. These are the moduli-dependent coefficients of BPS interactions of the form $d^{2p}\mathcal{P}_n$ in linearised approximation, where $\mathcal{P}_n$ is the product of n fields that has dimension $= 8$ with $q_U = 8-2n$, and $p = 0$, 2 or 3. These first-order equations imply that the coefficients satisfy $\mathrm{SL}(2,\mathbb{Z})$-covariant Laplace eigenvalue equations on moduli space with solutions that contain information concerning perturbative and non-perturbative contributions to superstring amplitudes. For $p = 3$ and $n \geq 6$ there are two independent modular forms, one of which has a vanishing tree-level contribution.

The analysis of super-amplitudes for U(1)-violating processes involving arbitrary numbers of external fluctuations of the complex modulus leads to a diagrammatic derivation of the first-order differential relations and Laplace equations satisfied by the coefficient modular forms. Combining this with a $\mathrm{SL}(2,\mathbb{Z})$-covariant soft axio-dilaton limit that relates amplitudes with different values of n determines most of the modular invariant coefficients, leaving a single undetermined constant.

Keywords: Scattering Amplitudes, String Duality, Supersymmetry and Duality

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1 Overview and outline of the paper

At low energy, or small curvature, closed string theory reduces to a version of Einstein's theory that may be described in terms of the Einstein-Hilbert action coupled to a variety of massless fields. The low-energy expansion of the effective string theory action is a power series in $p\,\ell_s$, where p is the energy-momentum scale and $\ell_s = \sqrt{\alpha'}$ is the string length scale. Successive terms in this expansion may be expressed in terms of higher-derivative interactions that generalise the Einstein-Hilbert action. Such interactions have a rich dependence on the moduli fields associated with the geometry of the target space. In the case of superstring theory, the dependence on the moduli is highly constrained by perturbative and non-perturbative dualities.

1.1 Overview

The focus of this paper is on the structure of the coefficients of higher-derivative interactions that arise in the low-energy expansion of scattering amplitudes in ten-dimensional type IIB superstring theory, which is the simplest example of a theory with a non-trivial S-duality group, namely, $\mathrm{SL}(2,\mathbb{Z})$. It contains a single complex scalar field, or modulus, $\tau = \tau_1 + i\tau_2$, which parameterises the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$, where $\mathrm{U}(1) \sim \mathrm{SO}(2)$ is the R-symmetry of classical IIB supergravity.[1]

The continuous $\mathrm{SL}(2,\mathbb{R})$ symmetry of classical supergravity does not survive in the quantum theory since it is not preserved in the string extension of type IIB supergravity. Indeed it is well known that the classical superstring is not invariant under the $\mathrm{U}(1)$ subgroup of $\mathrm{SL}(2,\mathbb{R})$ that rotates the two supercharges into each other since the two supercharges move in opposite directions on the world-sheet. The theory is only invariant under a discrete Z_4 subgroup of this $\mathrm{U}(1)$ which interchanges the two supercharges and reverses the parameter σ that labels points along the string. This Z_4 is the intersection of $\mathrm{U}(1)$ with $\mathrm{SL}(2,\mathbb{Z})$. As a result, the modulus field is subject to discrete identifications that restrict it to a single fundamental domain of moduli space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$, leading to the arithmetic S-duality group $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$.[2]

A consequence is that in a fixed background, $\tau = \tau^0$, n-particle amplitudes for the scattering of massless states (i.e. supergravity states) generally violate the continuous $\mathrm{U}(1)$ symmetry that is conserved in perturbative type IIB supergravity. However, there is a particular pattern to the non-conservation of the $\mathrm{U}(1)$ charge. As will be reviewed in section 2, an interaction that violates the $\mathrm{U}(1)$ charge by $q_U = -2w$ units contributes to n-particle amplitudes with $n \geq |w| + 4$ and its coefficients are given by non-holomorphic modular forms that transform with holomorphic and anti-holomorphic weights $(w, -w)$.[3] In this paper we will consider $w \geq 0$, i.e. $q_U \leq 0$. The $w < 0$ ($q_U > 0$) cases are complex conjugates of the $w > 0$ cases.

[1] Our conventions for parameterising the embedding of the $\mathrm{U}(1)$ R-symmetry in the supergravity coset are summarised in appendix A.

[2] This pattern of the breaking of $\mathrm{SL}(2,\mathbb{R})$ is also indicated by the presence of a one-loop chiral anomaly in ten-dimensional type IIB supergravity [1].

[3] More generally, a non-holomorphic modular form has independent weights, (w, w'), but when $w' = -w$ it transforms by a phase under the action of $\mathrm{SL}(2,\mathbb{Z})$, as described in section 2.1.

These modular forms are highly constrained, and in some cases precisely determined, by the requirement that the effective action should be invariant under $\mathrm{SL}(2,\mathbb{Z})$ as well as maximal supersymmetry. This is true for the terms that arise up to dimension 14, which preserve a fraction of the supersymmetry[4] — in that sense they are 'F-terms' that can be expressed as integrals over a subspace of the full 32-component space of Grassmann on-shell superspace coordinates.

For example, the leading higher-derivative terms are associated with 1/2-BPS interactions of the same dimension as $(\alpha')^{-1}R^4$ (where the four Riemann tensors are contracted with a well-known sixteen-index tensor). This interaction arises in the expansion of the tree-level four-graviton amplitude [2, 3] and its exact non-perturbative structure [4–6] is encoded in its coefficient, $E(\frac{3}{2},\tau)$, which is a non-holomorphic Eisenstein series. This is a $\mathrm{SL}(2,\mathbb{Z})$-invariant function of the complex scalar field $\tau=\tau_1+i\tau_2$, where $\tau_1=C^{(0)}$ is the Ramond-Ramond zero form (or "axion") and $\tau_2=e^{-\varphi}$, where φ is the dilaton field. In our consideration of scattering amplitudes the background scalar field is the complex coupling constant $\tau^0 := \chi + i/g_s$. The expansion of $E(\frac{3}{2},\tau^0)$ for weak coupling ($\tau_2^0 = 1/g_s \to \infty$) yields two power behaved terms of order $g_s^{-3/2}$ and $g_s^{1/2}$, which are identified with tree-level and one-loop terms in string perturbation theory. It also gets contributions of order $e^{-2\pi/g_s}$ from D-instantons. Properties of modular forms and Eisenstein series of relevance to this paper are reviewed in appendix B.

Many other 1/2-BPS interaction terms arise in the low-energy expansion of the type IIB string action at order $(\alpha')^{-1}$ (dimension 8) in a constant background $\tau=\tau^0$, and these generally violate the U(1) R-symmetry. There is a bound on the U(1) violation of an n-particle interaction, $|q_U| \le |8-2n|$ (where $n\ge 4$), which is saturated by the 'maximal U(1)-violating' interactions [7]. The lowest-dimension interactions that saturate the bound can be expressed in linearised approximation as monomials $\mathcal{P}_n(\{\Phi\})$, which are products of n on-shell fields and field strengths of type IIB supergravity with total U(1) charge violation equal to $q_U = -2w = 8-2n$ and with dimension 8 (as will be discussed in appendix C). Interactions of the form $d^{2p}\,\mathcal{P}_n(\{\Phi\})$ have dimension $8+2p$ — with $p=2$ they are 1/4-BPS interactions and with $p=3$ they are 1/8-BPS.[5] For now we will not specify how the derivatives act, but this will be clarified in an economical manner by the kinematic structures involving Mandelstam invariants in scattering amplitudes later in this paper. These fractional BPS interactions are known to be "protected" by supersymmetry from receiving perturbative contributions. When $p\ge 4$ (dimension ≥ 16) the interactions are non-BPS (at least, in any conventional sense) and their coefficients are not constrained by supersymmetry in any obvious manner.

We will be concerned with contributions of the "protected" maximal U(1)-violating interactions to the effective action. We may write terms of this type involving n fields in the form

$$\begin{aligned} S^{(p)}_{n\,i} &= (\alpha')^{p-1}\int d^{10}x\, e\, \tau_2^{\frac{1-p}{2}}\, F^{(p)}_{w\,i}(\tau)\, d^{2p}_{(i)}\, \mathcal{P}_n(\{\Phi\}) \\ &= \kappa^{\frac{p-1}{2}}\int d^{10}x\, e\, F^{(p)}_{w\,i}(\tau)\, d^{2p}_{(i)}\, \mathcal{P}_n(\{\Phi\})\,, \end{aligned} \tag{1.1}$$

[4]The dimension of the Einstein-Hilbert action is 2.
[5]Interactions with $p=1$ are absent.

where $n = w+4$ and e is the determinant of the zehnbein. In the second expression, in which the fields have been transformed to the Einstein frame, the gravitational coupling, κ, is related to α' by $\kappa = (\alpha')^2\, g_s$. The interaction $\mathcal{P}_n(\{\Phi\})$ carries a charge $q_U = -2w = -2n+8$ and so transforms by a phase under U(1) transformations embedded in $\mathrm{SL}(2,\mathbb{R})$, and therefore invariance of the low-energy action under $\mathrm{SL}(2,\mathbb{Z})$ implies that $F^{(0)}_{w,i}(\tau)$ must be a $(w,-w)$ modular form that transforms by a compensating phase.

The subscript i on the symbol $d^{2p}_{(i)}\,\mathcal{P}_n$ labels the independent invariants made out of the $2p$ derivatives — the independent ways in which the derivatives acting on the fields are contracted into each other, up to terms which vanish on-shell. These correspond to the independent symmetric polynomials of degree p in the Mandelstam variables for the n-particle amplitude. This degeneracy of the kinematic invariants is correlated with the number of independent moduli-dependent coefficients, $F^{(p)}_{w\,i}(\tau)$, and plays an important rôle in the following discussion. A two-fold degeneracy of these invariants first arises for $n = 4$ when $p = 6$, for $n = 5$ when $p = 4$, and for $n \geq 6$ when $p \geq 3$. The BPS interactions have $p \leq 3$ and therefore degeneracy only arises with $p = 3$ and $n \geq 6$. In other cases the "degeneracy" index i is redundant so we will generally use the notation $F^{(p)}_w(\tau)$ unless the index i is needed. Each coefficient $F^{(p)}_{w\,i}(\tau)$ is a $(w,-w)$ modular form, which transforms with a U(1) charge $q = 2w = 2n-8$ under $\mathrm{SL}(2,\mathbb{Z})$, so its transformation compensates for that of $\mathcal{P}_n(\{\Phi\})$ and the action (1.1) is invariant.

All of the known examples of $F^{(p)}_w(\tau)$, which will be reviewed in section 2, are non-degenerate. In the 1/2-BPS case ($p = 0$) the coefficients $F^{(0)}_w(\tau)$ in (1.1) are non-holomorphic modular forms of weight $(w,-w)$ that are known to be generalisations of non-holomorphic Eisenstein series that transform non-trivially under $\mathrm{SL}(2,\mathbb{Z})$. A coefficient $F^{(p)}_w(\tau)$ is related to $F^{(p)}_{w+1}(\tau)$ by first-order differential equations implied by supersymmetry, which, in turn, lead to Laplace eigenvalue equations in the upper half plane [5, 6]. These have unique solutions proportional to generalisations of non-holomorphic Eisenstein series with modular weights $(w,-w)$ that are reviewed in appendix B. The complete list of dimension-8 (1/2-BPS) linearised interactions $\mathcal{P}_n(\{\Phi\})$ can be obtained from supersymmetry considerations making use of a linearised on-shell scalar superfield introduced in [8], as will be reviewed in appendix C. Some examples of these dimension-8 interaction polynomials are:

$$R^4 \;\; (w=0), \qquad G^2R^3 \;\; (w=1), \qquad G^4R^2 \;\; (w=2), \;\; \ldots, \;\; \Lambda^{16} \;\; (w=12)\,, \tag{1.2}$$

where R is the linearised curvature, G is the complex third-rank field strength and Λ is the complex dilatino.

Similar comments apply to the 1/4-BPS interactions [6, 9], which have $p = 2$. In the cases with $p = 0$ and $p = 2$ (the 1/2-BPS and 1/4-BPS cases) there is a complete understanding of $F^{(p)}_w(\tau)$ for all U(1) charges, $0 \leq q_U \leq 24$ ($q_U = 2w = 2n-8$). Less is known in the $p = 3$ cases (which are 1/8-BPS interactions), for which only the $w = 0$ coefficient has been determined (the coefficient of d^6R^4) [10]. This satisfies an inhomogeneous Laplace eigenvalue equation on the upper-half plane and has many fascinating features. The homogeneous Laplace equations for R^4, d^4R^4 and the inhomogeneous equation for d^6R^4 can also be motivated by supersymmetric Ward identities that are implied by the structure of super-amplitudes [11].[6] The coefficient of d^6R^4 has a weak coupling expansion that reproduces results of explicit superstring perturbation theory from genus-zero to genus-three [4, 10, 17–20].

1.2 Outline of results

The generalisation of the $p = 3$, $w = 0$ Laplace equation to cases with $w > 0$ will be considered in section 3. We will show that requiring consistency with the $w = 0$ case leads to first-order differential equations relating coefficients of $F_w^{(3)}(\tau)$ with different values of w. This determines a novel inhomogeneous Laplace eigenvalue equation for the $p = 3$, $w = 1$ modular form $F_1^{(3)}(\tau)$, which is the coefficient of five-particle maximal U(1)-violating interactions. In the $p = 3$, $w = 2$ case there are two distinct forms $F_{2,i}^{(3)}(\tau)$ that are related to $F_1^{(3)}(\tau)$, and which satisfy distinct inhomogeneous Laplace eigenvalue equations. Importantly, it is also known that there are two symmetric cubic invariants for the six-particle amplitude with $p = 3$. The modular form $F_{2,1}^{(3)}(\tau)$ is related to the known modular function, $F_0^{(3)}(\tau)$, by the relation $F_{2,1}^{(3)}(\tau) \sim \mathcal{D}_1\mathcal{D}_0F_0^{(3)}(\tau)$ (where $\mathcal{D}_w$ is a modular covariant derivative that will be defined later), whereas $F_{2,2}^{(3)}(\tau)$ is qualitatively distinct. In particular, its weak coupling expansion does not contain a tree-level contribution, but starts with the genus-one term of order τ.

The preceding pattern of effective interactions has an interesting interpretation in terms of type IIB superstring scattering amplitudes in the low-energy expansion, which is the subject of sections 4, 5 and 6. Such amplitudes describe the scattering of fluctuations of the massless fields around a fixed background, which will be taken to be flat ten-dimensional Minkowski space with a constant value of the complex coupling constant, $\tau = \tau^0$. We will be particularly concerned with maximal U(1)-violating n-particle amplitudes with $n = 4 + w$, which violate the U(1) charge by $2w$ units and have no massless intermediate poles.

A subset of these are amplitudes in which the external fluctuating states correspond to the fields in $d_{(i)}^{2p}\mathcal{P}_n$. The dependence of their low-energy expansion on the coupling constant is given by $F_{w,i}^{(p)}(\tau^0)$. The amplitudes transform covariantly under $\mathrm{SL}(2,\mathbb{Z})$ in the sense that the coefficient $F_{w,i}^{(p)}(\tau^0)$ transforms as a $(w, -w)$ modular form under a $\mathrm{SL}(2,\mathbb{Z})$ transformation of the background in a manner that compensates for the transformation of the external states that correspond to the fields in $\mathcal{P}_n(\{\Phi\})$.

More generally there are maximal U(1)-violating amplitudes obtained by adding m axio-dilaton fluctuations, $\delta\tau = \tau - \tau^0$, to the n-particle states associated with the n fluctuating fields in the $d_{(i)}^{2p}\,\mathcal{P}_n(\{\Phi\})$ contact terms. Such amplitudes are obtained by an mth order Taylor expansion of $F_{w,i}^{(p)}(\tau)$. But $\delta\tau$ does not respect the U(1) symmetry of the coset and such an expansion generates $(n + m)$-particle amplitudes for which the $\mathrm{SL}(2,\mathbb{Z})$ duality is not manifest. Following the usual normal coordinate expansion for nonlinear sigma models on coset spaces, invariance will be restored by a suitable reparameterisation that replaces $\delta\tau$ by the complex field Z

$$\delta\tau \to Z := \frac{\tau - \tau^0}{\tau - \bar{\tau}^0}\,. \tag{1.3}$$

[6]Other interesting applications of such super-amplitude constraints on effective actions can be found e.g. in [12–16].

This is a $\mathrm{SL}(2,\mathbb{C})$ transformation that maps the upper-half τ plane to the unit disk in the Z plane.[7] The field Z transforms with a phase appropriate to a charge-1 field under $\mathrm{SL}(2,\mathbb{Z})$. Consequently, the mth order term in the expansion of $F^{(p)}_{n-4,i}(\tau)$ in powers of Z is proportional to the mth order modular-covariant derivative of $F^{(p)}_{n-4,i}(\tau)$. This guarantees that the scattering amplitude with m Z fields and n fields from $\mathcal{P}_n$ transforms covariantly under $\mathrm{SL}(2,\mathbb{Z})$ acting on the fluctuations and the constant modulus, τ^0. For example, the term of dimension $8+2p$ in the low energy expansion of the amplitude with m complex scalars and four gravitons, is proportional to

$$F^{(p)}_{m,1}(\tau^0)\,\langle g_1\, g_2\, g_3\, g_4\, Z_1\, Z_2 \ldots Z_m\rangle|_{8+2p}\,, \tag{1.4}$$

where $\langle\ldots\rangle|_{8+2p}$ indicates the term with dimension $(8+2p)$ in the low-energy expansion of order $d^{2p}R^4$ and

$$F^{(p)}_{m,1}(\tau^0) = 2^m \mathcal{D}_{m-1}\, \mathcal{D}_{m-2} \ldots \mathcal{D}_0\, F^{(p)}_0(\tau)\Big|_{\tau=\tau^0}\,. \tag{1.5}$$

The subscript 1 is redundant except for the cases with $p=3$ and $w=m\geq 2$. In these cases there is a separate contribution proportional to $F^{(3)}_{m,2}(\tau^0)$, which corresponds to the coefficient of another independent interaction term at this order. This interaction has a vanishing tree-level contribution and the lowest-order term in its perturbative expansion is the one-loop term.

In section 4 we will also briefly review the ten-dimensional helicity-spinor formalism, which is an efficient framework for constructing supersymmetric amplitudes. We will describe a general soft axio-dilaton limit, which is confirmed by explicit type IIB superstring tree amplitudes with $n\leq 6$ external states.

Soft axio-dilaton limits will be further considered in section 5. In particular, for the low-dimension terms in the low-energy expansion considered in this paper the soft limits determine the expansion coefficients of higher-point amplitudes completely in terms of the lower-point ones. The soft limits relate the coefficient modular functions of different weights in the manner of (1.5).

Supersymmetry constraints on the BPS terms are investigated in section 6 by using an extension of the ideas in [11], where it was shown that the well-known Laplace equations satisfied by $w=0$ functions $F^{(p)}_0(\tau)$ can also be understood from the constraints imposed by on-shell super-amplitudes. Here we will generalise this approach to obtain first-order differential equations, including the equations for the coefficient modular functions $F^{(3)}_{2,1}(\tau^0)$ and $F^{(3)}_{2,2}(\tau^0)$. This confirms the results obtained in section 3 that were based on somewhat different considerations.

The conclusions of this paper will be summarised in section 7, where we will also discuss some of their implications.

[7]The field, Z, was introduced (but called B) as the modulus field in the $\mathrm{SU}(1,1)$ formulation of type IIB supergravity in [21].

2 Effective interactions in the low-energy expansion

The effective interactions that arise in the first few orders in the low-energy expansion of the ten-dimensional type IIB superstring effective action have been determined by imposing the requirements of maximal supersymmetry together with $\mathrm{SL}(2,\mathbb{Z})$ S-duality. The coefficients of these higher-derivative interactions are functions of the complex scalar field τ that transform covariantly under the action of $\mathrm{SL}(2,\mathbb{Z})$.

To establish our conventions recall that the classical supergravity action has the form

$$S^{\mathrm{EH}} = \frac{1}{(\alpha')^4}\int d^{10}x\, e\, \tau_2^2\, R + \cdots = \frac{1}{\kappa^2}\int d^{10}x\, e\, R + \cdots\,, \tag{2.1}$$

where the ellipsis denotes the presence of many other terms of the same dimension that complete the supersymmetric action of type IIB supergravity.[8] Note that when the dilaton is constant, $(\tau_2^0)^{-1} = g_s$ is the string coupling constant, and therefore $\kappa^2 = (\alpha')^4\, g_s^2$.

The classical type IIB equations of motion were determined in component form in [21] (up to terms quadratic in fermion fields) and in terms of on-shell superfields in [8]. The expression (2.1) is invariant under $\mathrm{SL}(2,\mathbb{Z})$ as is obvious when expressed in the Einstein frame. In order to proceed further we will review properties of higher order terms in the low-energy expansion of the effective action, for which $\mathrm{SL}(2,\mathbb{Z})$ invariance is more subtle since the moduli-dependent coefficients transform as non-holomorphic modular forms.

2.1 Non-holomorphic modular forms

Recall that $\mathrm{SL}(2,\mathbb{Z})$ acts on the scalar field τ as

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{2.2}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. A non-holomorphic modular form $f^{(w,w')}(\tau)$ has holomorphic and anti-holomorphic modular weights (w, w') and its transformation under $\mathrm{SL}(2,\mathbb{Z})$ is given by

$$f^{(w,w')}(\tau) \to (c\tau + d)^{w}\,(c\bar\tau + d)^{w'}\, f^{(w,w')}(\tau)\,. \tag{2.3}$$

Modular covariant derivatives can be defined by

$$\mathcal{D}_w = i\left(\tau_2\frac{\partial}{\partial\tau} - i\,\frac{w}{2}\right), \qquad \bar{\mathcal{D}}_{w'} = -i\left(\tau_2\frac{\partial}{\partial\bar\tau} + i\frac{w'}{2}\right), \tag{2.4}$$

where $\mathcal{D}_w$ transforms $(w, w') \to (w+1, w'-1)$ and $\bar{\mathcal{D}}_{w'}$ transforms $(w, w') \to (w-1, w'+1)$. In other words,

$$\mathcal{D}_w\, f^{(w,w')} := f^{(w+1,w'-1)}\,, \qquad \bar{\mathcal{D}}_{w'}\, f^{(w,w')} := f^{(w-1,w'+1)}\,. \tag{2.5}$$

Non-holomorphic forms for which $w' = -w$ transform by a phase characterised by a $\mathrm{U}(1)$ charge, $q = 2w$, as is evident from (2.3).

[8] The problematic issue of writing an action for the self-dual five-form field strength in the type IIB theory is not relevant here since our focus will be on the on-shell scattering amplitudes.

For future reference we note that since $\tau_2 = (\tau - \bar\tau)/(2i)$, the action of $\mathcal{D}_w$ on a power of $\tau_2^0 = 1/g_s$ is given by

$$\mathcal{D}_w \tau_2^\alpha\Big|_{\tau_2=\tau_2^0} \sim \frac{1}{2}\left(\tau_2 \frac{\partial}{\partial \tau_2} + w\right) \tau_2^\alpha\Big|_{\tau_2=\tau_2^0} = \frac{1}{2}\left(-g_s \frac{\partial}{\partial g_s} + w\right) g_s^{-\alpha}\,. \tag{2.6}$$

In the next sub-section we will describe the coefficients of the first few terms in the low-energy expansion, which are known to be modular forms that are related by first-order differential equations of the form (2.5). These imply that they satisfy various kinds of Laplace equations. The simplest examples of such equations are Laplace eigenvalue equations that have solutions parameterised by $s \in \mathbb{C}$ and have the form

$$\Delta_{(-)w}\, f_s^{(w,-w)}(\tau) = \left(s(s-1) - w(w-1)\right)\, f_s^{(w,-w)}(\tau)\,. \tag{2.7}$$

where $\Delta_{(-)w} := 4\mathcal{D}_{w-1}\bar{\mathcal{D}}_{-w}$ is a laplacian acting on a weight $(w,-w)$ non-holomorphic modular form. This equation has a unique solution, subject to the physically required boundary condition that it has moderate growth (power behaviour) in the large-τ_2 limit (the weak-coupling limit). The solution is given in terms of Eisenstein series as reviewed in appendix B.

2.2 Some coefficients of low-order terms

In order to motivate our subsequent discussion, we will now summarise the known coefficients of terms in the low-energy expansion. In each case the interaction takes its simplest form in the Einstein frame, in which S-duality is manifest. As is seen in (1.1), this is related to the form of the interaction in the string frame by a rescaling of the metric by a dilaton-dependent factor. Since the dilaton is constant in the backgrounds of relevance to this paper, this simply introduces a power of the coupling constant that depends on the dimension of the interaction (the order in α').

2.2.1 Terms at order $(\alpha')^{-1}$

In our conventions the classical supergravity action (2.1) is of order $(\alpha')^{-4}$ and has dimension 2. The first non-leading terms arise from a super-multiplet of 1/2-BPS interactions at order $(\alpha')^{-1}$ (dimension 8), with distinct U(1) charges ranging from 0 to 24.[9] The fully supersymmetric nonlinear effective action has not been determined,[10] but it is relatively straightforward to determine it in linearised approximation. In this description the supergravity fields are the components of a linearised scalar superfield that is a function of a 16-component chiral SO(9,1) spinor, θ, as described in appendix C. The component interactions described by $\mathcal{P}_n(\{\Phi\})$ in (1.1) result from integrating a function of this superfield over θ.

The coefficient of any such interaction with U(1) charge $q = -2w$ is a $(w,-w)$ modular form $F_w^{(0)}(\tau)$, where we recall that $n = 4+w$ indicates the number of factors in the product

[9] As will be discussed later, expanding the coefficient function $F_{w,i}^{(p)}(\tau)$ in (1.1) in fluctuations of τ leads to amplitudes with higher U(1)-charge violation.

[10] However, the fully nonlinear action at $O((\alpha')^{-1})$ has been determined in the special case in which $G = \partial_\mu\tau = 0$, where G is the complex three-form and τ is the complex scalar field [22, 23]

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of n fields, $\mathcal{P}_n(\{\Phi\})$. These component interactions include the R^4 interaction, which has charge $q_{R^4}=0$ and a coefficient that is a weight-$(0,0)$ form (a modular function), as well as many other dimension-8 interactions with non-zero q_U. The $O((\alpha')^{-1})$ interaction with the greatest value of q is the sixteen-dilatino interaction, Λ^{16}, which has $q_{\Lambda^{16}}=-24$ and its coefficient is a modular form $F^{(0)}_{12}(\tau)$.

The coefficient of the effective term R^4 is the solution of (2.7) with $w=0$ and $s=3/2$, which has the form

$$S_4^{(0)}=\frac{1}{\alpha'}\int d^{10}xe\,\tau_2^{\frac{1}{2}}\,E(\tfrac{3}{2},\tau)R^4=\kappa^{-\frac{1}{2}}\int d^{10}xe\,E(\tfrac{3}{2},\tau)R^4\,. \tag{2.8}$$

The weak-coupling $(\tau_2\to\infty)$ expansion of $E(\frac{3}{2},\tau)$ that arises in (2.8) has the form

$$E(\tfrac{3}{2},\tau)=2\zeta(3)\tau_2^{\frac{3}{2}}+4\zeta(2)\,\tau_2^{-\frac{1}{2}}+4\pi\sum_{n\neq0}\sigma_{-2}(|n|)|n|^{\frac{1}{2}}\,e^{2\pi(in\tau_1-|n|\tau_2)}\left(1+O(\tau_2^{-1})\right)\,, \tag{2.9}$$

where $\sigma_{-2}(|n|)$ is the divisor sum defined in (B.5). After including the factor of $\tau_2^{1/2}$ in (2.8), which translates the expression into the string frame, the power-behaved terms in (2.9) correspond to tree-level and one-loop terms in string perturbation theory. Whereas the sum of exponential terms is interpreted as the contribution of D-instantons, each of which has an infinite series of perturbative corrections in powers of $\tau_2=g_s^{-1}$.

More generally, the coefficients of dimension-8 $(p=0)$ interactions with $0\le w\le 12$ are proportional to the modified Eisenstein series', $E_w(s,\tau)$, which are $(w,-w)$ modular forms defined in appendix B, so that

$$F_w^{(0)}(\tau)=c_w^{(0)}\,E_w(\tfrac{3}{2},\tau)\,. \tag{2.10}$$

The normalisation constants $c_w^{(0)}$ may be determined by comparison of the tree-level term in $F_w^{(0)}(\tau)$ (the term of order $\tau_2^{3/2}$) with tree-level superstring perturbation theory (and we have chosen $c_0^{(0)}=1$ for later convenience). The expression $E_w(\frac{3}{2},\tau)$, in (2.10) is obtained by setting $s=3/2$ in (B.8) and (B.9),

$$E_w(\tfrac{3}{2},\tau)=\frac{2^{w-1}\sqrt{\pi}}{\Gamma(\frac{3}{2}+w)}\,\mathcal{D}_{w-1}\ldots\mathcal{D}_0\,E(\tfrac{3}{2},\tau)\;=\sum_{(m,n)\neq(0,0)}\left(\frac{m+n\bar\tau}{m+n\tau}\right)^w\frac{\tau_2^{\frac{3}{2}}}{|m+n\tau|^3}\,. \tag{2.11}$$

This has a weak-coupling expansion (given in (B.14)) that has two terms that are power behaved in $\tau_2=g_s^{-1}$ and an infinite sequence of exponentially suppressed D-instanton and anti D-instanton contributions (where the D-instantons dominate by a factor of τ_2^{2w}). The normalisation in (2.11) has been chosen so that $E_w(\frac{3}{2},\tau)\underset{\tau_2\to\infty}{\to}2\zeta(3)\tau_2^{3/2}+O(\tau_2^{-1/2})$ for all w. The first order differential equations (B.6) and (B.7) satisfied by $E_w(s,\tau)$ imply Laplace eigenvalue equations given in (B.10) and (B.11).

2.2.2 Terms at order $O(\alpha')$

The next order in the low-energy expansion has dimension 12. The subset of the interactions that we are considering are those that are given by integrals of $F_w^{(2)}(\tau)\,d^4\,\mathcal{P}_n(\{\Phi\})$.

The notation does not indicate which of the $n = 4+w$ fields in $\mathcal{P}_n(\{\Phi\})$ the four derivatives act on, or how they are contracted. This is specified precisely by the form of the scattering amplitudes, where derivatives are replaced by momenta and d^4 becomes a quadratic monomial in Mandelstam invariants. The modular forms are given in this case by Eisenstein series with $s = 5/2$,

$$F_w^{(2)}(\tau) = c_w^{(2)}\, E_w(\tfrac{5}{2},\tau)\,, \tag{2.12}$$

where the modified Eisenstein series, $E_w(\frac{5}{2},\tau)$, is defined by setting $s = 5/2$ in (B.8) and (B.9), and $c_w^{(2)}$ are normalisation constants. These may be fixed by comparison of the coefficient of the $(\tau_2)^{5/2}$ term in $F_w^{(2)}$ with the term of order α' in the expansion of n-point tree-level superstring amplitude, where $n = 4 + w$. The coefficient of the $w = 0$ term is $c_0^{(2)} = 1/2$.

2.2.3 Terms at order $O\left((\alpha')^2\right)$

Up to now the only interaction of order $O\left((\alpha')^2\right)$ that has been fully analysed in the literature is the U(1)-conserving interaction $F_0^{(3)}(\tau)\, d^6 R^4$ [10]. This is a $w = 0$ component of the series of maximal U(1) charge-violating interactions defined by the integrals

$$S_{n,i}^{(3)} = \int d^{10}x\, e\, F_{w,i}^{(3)}(\tau)\, d_{(i)}^6 \mathcal{P}_n(\{\Phi\})\,, \tag{2.13}$$

which will be discussed in the following sections. The operator d^6 gives rise to a single kinematic invariant in the $w = 0$ case, namely, the symmetric monomial in Mandelstam invariants, $s^3 + t^3 + u^3$, in the expansion of the four-graviton amplitude. However, as we will discuss, for general $w = n - 4$, in particular for those with $w \geq 2$, there is a two-fold degeneracy in the kinematic invariants that is indicated by the index $i = 1, 2$ in the operator $d_{(i)}^6$ in (2.13).

Whereas the coefficients of the 1/2-BPS and 1/4-BPS interactions satisfy Laplace eigenvalue equations of the form (2.7), the equation satisfied by $F_0^{(3)}(\tau)$ is the inhomogeneous Laplace equation [10],

$$(\Delta - 12)\; F_0^{(3)}(\tau) = -E(\tfrac{3}{2},\tau)^2\,. \tag{2.14}$$

The zero Fourier mode of $F_0^{(3)}(\tau)$ with respect to τ_1 of the solution to this equation contains four power-behaved terms, which correspond to genus-zero to genus-three contributions in string perturbation theory, as well as the contribution of D-instanton/anti D-instanton pairs,

$$F_0^{(3)}(\tau) = \frac{2}{3}\zeta(3)^2\tau_2^3 + \frac{4}{3}\zeta(2)\zeta(3)\,\tau_2 + \frac{8}{5}\zeta(2)^2\tau_2^{-1} + \frac{4}{27}\zeta(6)\tau_2^{-3} + \mathcal{O}(e^{-4\pi\tau_2})\,. \tag{2.15}$$

The first three of the power-behaved terms are easily obtained by equating the coefficients in the expansion of the left-hand and right-hand sides of (2.15) in powers of τ_2 using the expansion (2.9) for $E(\frac{3}{2},\tau)$. However, the τ_2^{-3} term is in the kernel of the operator on the left-hand side and does not arise in the expansion of $E(\frac{3}{2},\tau)^2$ on the right-hand side.

The determination of its coefficient is therefore a little subtle, and originates from the presence of an infinite series of instantonic contributions, as was described in [10] (and amplified in [24]). These coefficients agree with the explicit string perturbation theory calculations. The complete large-τ_2 expansion of $F_0^{(3)}(\tau)$, including its rich assortment of instanton contributions, was determined in [24]. One of the challenges that we address in this paper is the extension of this equation to cases in which $w > 0$.

3 Consistency constraints on 1/8-BPS modular coefficients.

As described in section 2.2 the 1/2-BPS and 1/4-BPS interactions $F_w^{(p)}\,\mathcal{P}_n$ $(p = 0, 2)$ in (1.1) are related by first order differential equations that are implied by supersymmetry. The Laplace eigenvalue equations that follow by iterating these equations determine the τ-dependent coefficients to be modular forms that generalise the standard non-holomorphic Eisenstein series at $s = (3 + p)/2$.

We will here generalise the $p = 3$, $w = 0$ case to coefficients $F_{w,i}^{(3)}$ with $w > 0$ by requiring consistency with the $w = 0$ case. We will consider the cases $w = 1$ and $w = 2$ explicitly although the procedure generalises in an obvious manner to all w. The cases of $w > 2$ will be further studied in section 5 using soft limits. As we will see in section 3.2, for $w = 2$ a two-fold degeneracy arises $(i = 1, 2)$, which is connected with the presence of two possible symmetric polynomials in the Mandelstam variables for massless six-particle scattering. One particular combination arises in the tree-level expansion as we will see later. This implies that there is a second $w = 2$ modular form that is unrelated to the $w = 0$ case (in the manner of (1.5)) that has no tree-level (genus-zero) term in its zero Fourier mode.

A comment on notation. The coefficients of the $O\left((\alpha')^2\right)$ 1/8-BPS interactions are proportional to modular forms denoted $\mathcal{E}_{w,i}^{(3)}(\tau)$

$$F_{w,i}^{(3)}(\tau) = c_{w,i}^{(3)}\,\mathcal{E}_{w,i}^{(3)}(\tau)\,, \tag{3.1}$$

where the index i again allows for a possible degeneracy, which will arise in cases with $w \geq 2$. We will suppress this index for the cases with $w = 0$ and $w = 1$, where there is no degeneracy.

Furthermore, we will choose a normalisation in which $c_0^{(3)} = 1$ so that (2.14) may be rewritten in the form

$$\Delta\mathcal{E}_0^{(3)} = 4\bar{\mathcal{D}}\mathcal{D}\mathcal{E}_0^{(3)} = 12\mathcal{E}_0^{(3)} - \left(E_0(\tfrac{3}{2})\right)^2\,, \tag{3.2}$$

where we are suppressing the arguments τ, and we will often drop the labelling on covariant derivatives since it is implied by the context.[11]

[11] For example, an expression such as $\mathcal{D}f^{(w,-w)}$ is identified with $\mathcal{D}_w f^{(w,-w)}$.

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3.1 The $p = 3$, $w = 1$ case

This is the first example of a U(1)-violating interaction that is related by supersymmetry to the d^6R^4 interaction. It involves fields in $\mathcal{P}_5$ and contributes to amplitudes with $n = 5$ external particles.

We begin by *defining*

$$\mathcal{E}_1^{(3)} := a\mathcal{D}\mathcal{E}_0^{(3)}\,, \tag{3.3}$$

where a is a constant. The coefficient $\mathcal{E}_1^{(3)}$ is proportional to the $q_U = 2$ modular form that is the coefficient of maximal U(1)-violating five-point interactions such as $d^6G^2R^3$.[12]

With the definition (3.3) (and noting (2.6)) the leading term at large τ_2 is given by $\mathcal{E}_1^{(3)} \underset{\tau_2\to\infty}{\to} a\,\zeta(3)^2\tau_2^3$. Applying the covariant derivative $\mathcal{D}$ to the Laplace equation (3.2) leads to the inhomogeneous Laplace equation for $\mathcal{E}_1^{(3)}$,

$$\Delta_{(-)}\mathcal{E}_1^{(3)} = 4\mathcal{D}\bar{\mathcal{D}}\mathcal{E}_1^{(3)} = 12\mathcal{E}_1^{(3)} - \frac{3}{2}\,a\,E_1(\tfrac{3}{2})\,E_0(\tfrac{3}{2})\,, \tag{3.4}$$

where we have used the properties of the Eisenstein series given in (B.6) and (B.7). We will now check the consistency of this equation by applying $\bar{\mathcal{D}}$ to it, which gives

$$\Delta(\bar{\mathcal{D}}\mathcal{E}_1^{(3)}) = 4\bar{\mathcal{D}}\mathcal{D}(\bar{\mathcal{D}}\mathcal{E}_1^{(3)}) = 12\,\bar{\mathcal{D}}\mathcal{E}_1^{(3)} - \frac{3}{2}\,a\,\bar{\mathcal{D}}\left(E_1(\tfrac{3}{2})\,E_0(\tfrac{3}{2})\right)\,. \tag{3.5}$$

Using the relation

$$E_1(\tfrac{3}{2})\,E_0(\tfrac{3}{2}) = \frac{2}{3}\mathcal{D}\,(E_0(\tfrac{3}{2}))^2\,, \tag{3.6}$$

we then obtain

$$\Delta(\bar{\mathcal{D}}\mathcal{E}_1^{(3)}) = 12\,\bar{\mathcal{D}}\mathcal{E}_1^{(3)} - \frac{a}{4}\Delta\left((E_0(\tfrac{3}{2}))^2\right)\,. \tag{3.7}$$

The above equation can be recast as

$$\Delta\left(\bar{\mathcal{D}}\mathcal{E}_1^{(3)} + \frac{a}{4}(E_0(\tfrac{3}{2}))^2\right) = 12\left(\bar{\mathcal{D}}\mathcal{E}_1^{(3)} + \frac{a}{4}(E_0(\tfrac{3}{2}))^2\right) - 3\,a\,(E_0(\tfrac{3}{2}))^2\,. \tag{3.8}$$

This reproduces (3.2) if we make the identification

$$\bar{\mathcal{D}}\mathcal{E}_1^{(3)} = 3a\,\mathcal{E}_0^{(3)} - \frac{a}{4}(E_0(\tfrac{3}{2}))^2\,, \tag{3.9}$$

in which case (3.8) reduces to the $w = 0$ Laplace equation, (3.2).

The value of a is arbitrary, but for later convenience we we will make the choice $a = 2$. With this choice, the first order differential relations for $\mathcal{E}_1^{(3)}$ become

$$\mathcal{E}_1^{(3)} = 2\mathcal{D}\mathcal{E}_0^{(3)}\,, \tag{3.10}$$

$$\bar{\mathcal{D}}\mathcal{E}_1^{(3)} = 6\,\mathcal{E}_0^{(3)} - \frac{1}{2}(E_0(\tfrac{3}{2}))^2\,, \tag{3.11}$$

and the inhomogeneous Laplace equation (3.4) becomes

$$\Delta_{(-)}\mathcal{E}_1^{(3)} = 12\mathcal{E}_1^{(3)} - 3\,E_1(\tfrac{3}{2})\,E_0(\tfrac{3}{2})\,. \tag{3.12}$$

[12] This definition can be modified, for example, by defining $\mathcal{E}_1^{(3)} := a\mathcal{D}\mathcal{E}_0^{(3)} + bE_0E_1$, but such a shift is mathematically trivial and (3.3) coincides with the definition of physical interest as we discuss in the next section.

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3.2 The $p = 3$, $w = 2$ cases

We may anticipate that there is a two-fold degeneracy in $w = 2$, $p = 3$ modular forms, labelled by an index i on $\mathcal{E}^{(3)}_{2,i}$, where $i = 1, 2$. This expectation is based on the following analysis, coupled with known facts about the low-energy expansion of superstring six-particle scattering amplitudes. We will consider the $i = 1$ case in some detail and follow that with a somewhat more conjectural discussion of the $i = 2$ case, which will be justified from the consideration of scattering amplitudes.

The modular form $\mathcal{E}^{(3)}_{2,1}$. The $\mathcal{E}^{(3)}_1$ Laplace equation (3.12) may be rewritten using the identification $\Delta_{(-)} = \Delta_{(+)} + 2$ (see (B.12)), giving

$$\Delta_{(+)}\mathcal{E}^{(3)}_1 = 4\bar{\mathcal{D}}\mathcal{D}\mathcal{E}^{(3)}_1 = 10\mathcal{E}^{(3)}_1 - 3\, E_1(\tfrac{3}{2})\, E_0(\tfrac{3}{2})\,. \tag{3.13}$$

Applying the covariant derivative, $\mathcal{D}$, to this equation gives

$$\Delta_{(-)}(\mathcal{D}\mathcal{E}^{(3)}_1) = 4\mathcal{D}\bar{\mathcal{D}}(\mathcal{D}\mathcal{E}^{(3)}_1) = 10\mathcal{D}\mathcal{E}^{(3)}_1 - 3\,\mathcal{D}\left(E_1(\tfrac{3}{2})\, E_0(\tfrac{3}{2})\right)\,. \tag{3.14}$$

To proceed, we will define

$$\mathcal{E}^{(3)}_{2,1} := b\,\mathcal{D}\mathcal{E}^{(3)}_1\,. \tag{3.15}$$

Substituting (3.15) in (3.14) gives

$$\Delta_{(-)}\mathcal{E}^{(3)}_{2,1} = 4\mathcal{D}\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1} = 10\mathcal{E}^{(3)}_{2,1} - 3b\,\mathcal{D}\left(E_1(\tfrac{3}{2})\, E_0(\tfrac{3}{2})\right)\,. \tag{3.16}$$

Applying $\bar{\mathcal{D}}$ to both sides of the above equation leads to

$$\begin{aligned}\Delta_{(+)}(\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1}) = 4\bar{\mathcal{D}}\mathcal{D}(\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1}) &= 10(\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1}) - 3b\,\bar{\mathcal{D}}\mathcal{D}\left(E_1(\tfrac{3}{2})\, E_0(\tfrac{3}{2})\right)\\ &= 10(\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1}) - \frac{3b}{4}\Delta_{(+)}\left(E_1(\tfrac{3}{2})\, E_0(\tfrac{3}{2})\right)\,.\end{aligned} \tag{3.17}$$

Now we can identify

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1} = \frac{5b}{2}\,\mathcal{E}^{(3)}_1 - \frac{3b}{4}E_1(\tfrac{3}{2})\, E_0(\tfrac{3}{2})\,, \tag{3.18}$$

in which case (3.17) reduces to (3.13). Again, b is arbitrary and is correlated with the normalisation constant $c^{(3)}_{2,1}$. It is again convenient to make the choice $b = 2$ so that the first-order differential equations for $\mathcal{E}^{(3)}_1$ become

$$\mathcal{E}^{(3)}_{2,1} = 2\mathcal{D}\mathcal{E}^{(3)}_1\,, \tag{3.19}$$

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1} = 5\,\mathcal{E}^{(3)}_1 - \frac{3}{2}E_1(\tfrac{3}{2})\, E_0(\tfrac{3}{2})\,. \tag{3.20}$$

The inhomogeneous Laplace equation that follows by combining these equations is

$$\Delta_{(-)}\mathcal{E}^{(3)}_{2,1} = 4\mathcal{D}\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1} = 10\mathcal{E}^{(3)}_{2,1} - \frac{15}{2}\left(E_0(\tfrac{3}{2})E_2(\tfrac{3}{2}) + \frac{3}{5}E_1(\tfrac{3}{2})E_1(\tfrac{3}{2})\right)\,, \tag{3.21}$$

where we have used the relation $\mathcal{D}\left(E_1(\tfrac{3}{2})\, E_0(\tfrac{3}{2})\right) = 5/4E_0(\tfrac{3}{2})E_2(\tfrac{3}{2}) + 3/4E_1(\tfrac{3}{2})E_1(\tfrac{3}{2})$.

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Since $\mathcal{E}^{(3)}_{2,1}(\tau) = 2\mathcal{D}\mathcal{E}^{(3)}_1(\tau) = 4\mathcal{D}\mathcal{D}\,\mathcal{E}^{(3)}_0(\tau)$, it is straightforward to deduce its large-τ_2 expansion from (2.15), which gives

$$\mathcal{E}^{(3)}_{2,1}(\tau) = 8\zeta(3)^2\tau_2^3 + \frac{8}{3}\zeta(2)\zeta(3)\,\tau_2 + \frac{8}{9}\zeta(6)\tau_2^{-3} + \mathcal{O}(e^{-2\pi\tau_2})\,. \tag{3.22}$$

Note that the genus-two term proportional to τ_2^{-1} is absent. This is an example of a general point concerning perturbative terms in the $p=3$ coefficient modular forms with $w \geq 2$, which are positive or negative integer powers of τ_2. The action of successive covariant derivatives on a particular power τ_2^x is

$$\mathcal{D}_{w-1}\ldots\mathcal{D}_u\,\tau_2^x = 2^{u-w}\prod_{j=u}^{w-1}(x+j)\,\tau_2^x\,, \tag{3.23}$$

which is killed by a sufficient number of derivatives if $-x > j$ and $x \in \mathbb{Z}$. The $p=0$ and $p=2$ perturbative interactions have half-integer powers of τ_2 in the Einstein frame and so the above argument does not apply.

3.2.1 Comments concerning six-particle amplitudes

Before discussing the other $p=3$, $w=2$ modular form, $\mathcal{E}^{(3)}_{2,2}$, we will discuss the $w=2$ contribution to six-particle scattering amplitudes with $q_U=-4$. As mentioned earlier, the structure of the six-particle superstring amplitude suggests that there should be two distinct modular forms that contribute to $\mathcal{E}^{(3)}_{2,i}\,d^6_{(i)}\,\mathcal{P}_6$. We have seen that one of these, $\mathcal{E}^{(3)}_{2,1}$, has a large-τ_2 (weak coupling) expansion that contains a component proportional to τ_2^3 that corresponds to a tree-level contribution (in the Einstein frame) to the $d^6_{(1)}\,\mathcal{P}_6$ interaction. This matches the expectation based on the explicit tree-level superstring calculations to be described in section 4, where we will see that the derivative factor $d^6_{(1)}$ translates into a particular cubic polynomial in the Mandelstam invariants of the six-particle amplitude,[13]

$$d^6_{(1)} \to \mathcal{O}^{(3)}_{6,1} := \frac{1}{32}\left(10\sum_{1\leq i<j\leq 6} s_{ij}^3 + 3\sum_{1\leq i<j<k\leq 6} s_{ijk}^3\right)\,. \tag{3.24}$$

This invariant has been chosen to coincide with the combination that arises in the tree-level calculation of the six-particle amplitude as will be shown in section 4. Furthermore, in the soft limit, it reduces to the unique kinematic invariant of the five-particle amplitude,

$$\mathcal{O}^{(3)}_{6,1}\big|_{p_6\to 0} \to \mathcal{O}^{(3)}_5\,, \tag{3.25}$$

where we have defined $\mathcal{O}^{(3)}_5 := 1/2\sum_{1\leq i<j\leq 5} s_{ij}^3$.

The above soft behaviour of the six-particle kinematic invariant $\mathcal{O}^{(3)}_{6,1}$ is consistent with the fact that a six-particle U(1)-violating amplitude with a number of external Z states reduces to a five-particle amplitude with one less Z when one of the Zs becomes soft. As

[13] In the following expressions the Mandelstam invariants are defined by $s_{ij} = -(p_i+p_j)^2$ and $s_{ijk} = -(p_i+p_j+p_k)^2$, where p_i is the momentum for the ith external massless particle, which satisfies $p_i^2=0$ and $\sum_{i=1}^6 p_i = 0$.

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we will see in greater detail in section 5.1 this is related to the fact that the coefficient of the six-particle interaction $F^{(3)}_{2,1}(\tau)$ is a covariant derivative of $F^{(3)}_1(\tau)$.

The other independent kinematic structure translates into

$$d^6_{(2)} \to \mathcal{O}^{(3)}_{6,2} := 2 \sum_{1\le i<j\le 6} s^3_{ij} - \sum_{1\le i<j<k\le 6} s^3_{ijk} = \frac{1}{8} \sum_{\text{permutation}} s_{12}s_{34}s_{56}\,, \tag{3.26}$$

where the sum is over 6! permutations. Up to an overall constant this is the unique symmetric polynomial of degree 3 in the six-particle Mandelstam invariants that vanishes in the single soft limit, $p_i \to 0$ for any i. This soft behaviour implies that in any maximal U(1)-violating amplitude with external axio-dilaton states there is at least one derivative on each Z or $\bar{Z}$. This is important since it shows that the coefficient of $\mathcal{O}^{(3)}_{6,2}$, which is $F^{(3)}_{2,2}(\tau)$, does not come from the expansion of the coefficient of a $n=5$ amplitude. If it did it would contain at least one "naked" Z or $\bar{Z}$ factor (a factor with no derivative acting on it), as we will see in section 4.

The modular form $\boldsymbol{\mathcal{E}^{(3)}_{2,2}}$. The coefficient of $d^6_{(2)}\,\mathcal{P}_6$ is given by $F^{(3)}_{2,2}(\tau) = c^{(3)}_{2,2}\,\mathcal{E}^{(3)}_{2,2}(\tau)$, which is proportional to the second $w=2$, $p=3$ modular form. The following discussion of $\mathcal{E}^{(3)}_{2,2}$ is based on the following inputs.

1 We will assume that $\mathcal{E}^{(3)}_{2,2}(\tau)$ satisfies a SL$(2,\mathbb{Z})$-covariant first-order differential equation analogous to (3.20).[14]

2 Since $\mathcal{O}^{(3)}_{6,2}$ does not contribute to the tree-level $p=3$ and $w=2$ interaction, so the leading term in $\mathcal{E}^{(3)}_{2,2}(\tau)$ in the large-τ_2 limit is the genus-one term of order τ_2.

Item 1 implies that the inhomogeneous term in the first-order differential equation of $\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,2}$ in terms of a linear combination of $\mathcal{E}^{(3)}_1$ and $E_0(\tfrac{3}{2})E_1(\tfrac{3}{2})$, namely

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,2} = c_1\mathcal{E}^{(3)}_1 + c_2 E_0(\tfrac{3}{2})\,E_1(\tfrac{3}{2})\,. \tag{3.27}$$

Item 2 determines the relative coefficients c_1 and c_2 of these two terms that is required for the τ_2^3 contribution to cancel. Making use of the perturbative expansions

$$\mathcal{E}^{(3)}_1 = 2\mathcal{D}\mathcal{E}^{(3)}_0 = 2\zeta(3)^2\tau_2^3 + \frac{4}{3}\zeta(2)\zeta(3)\,\tau_2 - \frac{8}{5}\zeta(2)^2\tau_2^{-1} - \frac{4}{9}\zeta(6)\tau_2^{-3} + \mathcal{O}(e^{-2\pi\tau_2})\,, \tag{3.28}$$

and

$$E_0(\tfrac{3}{2})\,E_1(\tfrac{3}{2}) = 4\zeta(3)^2\tau_2^3 + \frac{16}{3}\zeta(2)\zeta(3)\,\tau_2 - \frac{16}{3}\zeta(2)^2\tau_2^{-1} + \mathcal{O}(e^{-2\pi\tau_2})\,, \tag{3.29}$$

we see that the cancellation of the tree-level term (proportional to τ_2^3) requires $c_2 = -1/2\,c_1$, and therefore

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,2} = c_1\left(\mathcal{E}^{(3)}_1 - \frac{1}{2}E_0(\tfrac{3}{2})\,E_1(\tfrac{3}{2})\right)\,. \tag{3.30}$$

[14] This assumption will be justified later in section 6.

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Since the tree-level term in the large-τ_2 expansion of the right-hand side is designed to be zero, the leading power-behaved term is the genus-one term proportional to τ_2. The value of the constant c_1 may therefore be determined by an explicit evaluation of the six-point maximal U(1)-violating one-loop amplitude.

Comparing (3.20) and (3.30), and using $E_0(\frac{3}{2})E_1(\frac{3}{2}) = 2\bar{\mathcal{D}}(E_1(\frac{3}{2})^2)$, we find

$$\mathcal{E}^{(3)}_{2,2} = \frac{c_1}{5}\left(\mathcal{E}^{(3)}_{2,1} - 2E_1(\tfrac{3}{2})E_1(\tfrac{3}{2})\right) . \tag{3.31}$$

The Laplace equation satisfied by $\mathcal{E}^{(3)}_{2,2}$ follows by applying $\Delta_{(-)}$ to the above equation and using (3.21),

$$\Delta_{(-)}\mathcal{E}^{(3)}_{2,2} = 10\mathcal{E}^{(3)}_{2,2} - \frac{5c_1}{2}\left(E_0(\tfrac{3}{2})E_2(\tfrac{3}{2}) - E_1(\tfrac{3}{2})E_1(\tfrac{3}{2})\right) , \tag{3.32}$$

which has the same eigenvalue as the Laplace equation for $\mathcal{E}^{(3)}_{2,1}$ but with a different inhomogeneous term such that the tree-level contribution vanishes.

By knowing $\mathcal{E}^{(3)}_{2,1} = 4\mathcal{D}\mathcal{D}\mathcal{E}^{(3)}_{0}$, from (3.31) we may obtain $\mathcal{E}^{(3)}_{2,2}$ up to an unknown overall constant c_1. Expanding near the cusp, $\tau_2 \to \infty$, and dropping the constant factor c_1 gives the following weak coupling expansion, (in the Einstein frame, as usual)

$$\mathcal{E}^{(3)}_{2,2}(\tau) = \zeta(2)\zeta(3)\tau_2 - \frac{4}{15}\zeta(2)^2\tau_2^{-1} + \frac{1}{15}\zeta(6)\tau_2^{-3} + \mathcal{O}(e^{-2\pi\tau_2}) , \tag{3.33}$$

which contains contributions corresponding to genus-one, genus-two and genus-three superstring loop amplitudes, but with no tree contribution.

3.3 Comments on coefficients with $w > 2$ and on $p > 3$

As was commented on earlier, the modular form coefficients with weights $w > 2$ are related to the lower-weight ones by applying covariant derivatives. Thus, the modular form $\mathcal{E}^{(3)}_{w,2}(\tau)$ ($w > 2$) accompanying the n-particle $p = 3$ kinematic invariant, $\mathcal{O}^{(3)}_{n-4,2}$ can be obtained by acting with covariant derivates on $\mathcal{E}^{(3)}_{2,2}(\tau)$. This corresponds the expansion of $\mathcal{E}^{(3)}_{2,2}(\tau)$ in powers of Z fields around a fixed background, as will be discuss further in section 4.

Interactions with $p > 3$, which have dimension > 14, are non-BPS terms. Therefore in general they are expected to receive all-loop perturbative contributions, in addition to non-perturbative D-instanton contributions. The first such interaction is d^8R^4, which has a unique kinematic invariant $s^4+t^4+u^4$ and a coefficient $\mathcal{E}^{(4)}_0(\tau)$. Once again, higher-point terms have a degenerate set of kinematic invariants. In fact, already at five points there are two independent kinematic invariants,

$$\mathcal{O}^{(4)}_{5,1} = \sum_{i<j} s_{ij}^4 + \frac{1}{12}\left(\sum_{i<j} s_{ij}^2\right)^2 , \qquad \mathcal{O}^{(4)}_{5,2} = \sum_{i<j} s_{ij}^4 - \frac{1}{4}\left(\sum_{i<j} s_{ij}^2\right)^2 , \tag{3.34}$$

where $\mathcal{O}^{(4)}_{5,1}$ is the invariant arising in the five-point U(1)-violating tree-level string amplitude, which has a single-particle soft limit that results in the unique four-particle kinematic factor. As in the case of $\mathcal{O}^{(3)}_{6,1}$, the coefficient of $\mathcal{O}^{(4)}_{5,1}$ can be obtained by acting with a covariant derivative on the coefficient of d^8R^4, so its coefficient is given by $\mathcal{D}\mathcal{E}^{(4)}_0(\tau)$, as was discussed in [25].

The second $p=4$ five-particle kinematic invariant in (3.34), $\mathcal{O}^{(4)}_{5,2}$, is determined by requiring it to vanish in the soft limit. Therefore, $\mathcal{O}^{(4)}_{5,2}$ does not appear at tree level and first appears at one loop (and its form precisely agrees with the expression obtained from the matrix M'_7 in equation (5.5) of [25]). Clearly, the same analysis applies to the interactions with more general w's and p's.[15] We expect that it is generally true that interactions can be separated into different sets whose coefficients are related by covariant derivatives (which will become more evident in the next section). However, as will be shown in section 6, equations such as (3.20) and (3.30), or the Laplace equations discussed in the previous section, are special properties of F-terms with $p \le 3$.

4 Low-energy expansion of U(1)-violating scattering amplitudes

Before discussing details of the scattering amplitudes we will make some important comments about the special features of maximal U(1)-violating amplitudes.

4.1 Preliminary comments concerning maximal U(1)-violating amplitudes

The simplest class of superstring amplitudes are those n-particle amplitudes that violate U(1) maximally since these do not have any massless poles in any channel. At low orders in the low-energy expansion these amplitudes correspond to the contact interactions in the effective action that was considered in earlier sections. Simply setting the modulus field equal to its background value, $\tau = \tau^0 = \chi + i/g_s$ in the effective action leads immediately to expressions for on-shell maximal-violating n-particle scattering amplitudes, in which each of the fields in $d^{2p}_{(i)}\,\mathcal{P}_n(\{\Phi\})$ is associated with an external on-shell state. The coupling constant dependence is determined by $(\alpha')^{p-1}(\tau_2^0)^{\frac{1-p}{2}}\,F^{(p)}_w(\tau^0) = \kappa^{\frac{p-1}{2}}\,F^{(p)}_w(\tau^0)$ with $w = n-4$. For example, the leading correction to the four-graviton amplitude beyond the classical supergravity amplitude has $p=0$ and is proportional to $\kappa^{-1/2}\,E(\frac{3}{2},\tau^0)\,R^4$, while the sixteen-dilatino amplitude is proportional to $\kappa^{-1/2}\,E_{12}(\frac{3}{2},\tau^0)\,\Lambda^{16}$, and so on.

Among the n-field terms in $\mathcal{P}_n(\{\Phi\})$ there are dimension-8 terms containing at most two powers of $\bar\tau$ and two powers of τ, which are related by supersymmetry to the R^4 interaction. For example there is a term of the form $d^2\tau\, d^2\tau\, d^2\bar\tau\, d^2\bar\tau$. But there can be no "naked" powers of τ or $\bar\tau$ in $\mathcal{P}_n(\{\Phi\})$ — i.e., no factors of τ or $\bar\tau$ that are not acted on by derivatives. The naked τ and $\bar\tau$ fields are moduli that enter into the instanton contributions to the coefficients $F^{(p)}_{w,i}(\tau)$ in (1.1).

There are, however, further maximal U(1)-violating scattering amplitudes that have arbitrary numbers of additional τ fluctuations that are obtained by expanding the modular coefficients in the action in fluctuations of τ. As will be described in the next sub-section,

[15] We have checked that the kinematic invariants obtained from the matrices M'_8 (i.e. $w=1, p=5$) and M'_9, M''_9 (i.e. $w=1, p=6$) in equation (5.5) of [25] also vanish in the soft limit. This is expected from our considerations since M'_8, M'_9, M''_9 do not contribute at tree level (as was also the case with M'_7).

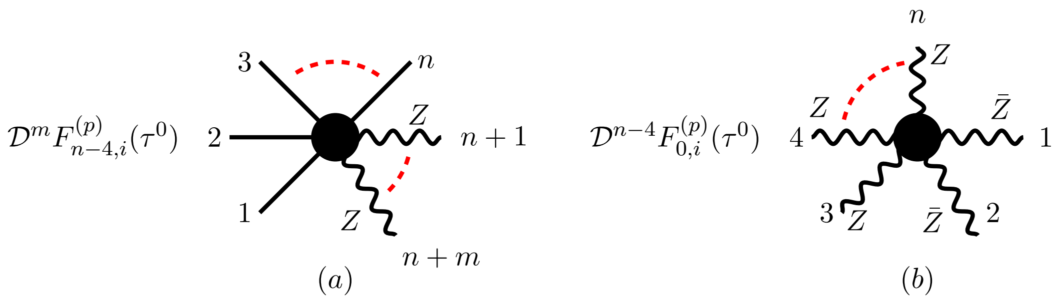

Figure 1. Maximal U(1)-violating amplitude (a) With n particles interacting via $d^{2p}_{(i)}\,\mathcal{P}_n$ and m fluctuations of the complex scalar field Z. (b) With two Z particles and two $\bar{Z}$ particles interacting via $d^{2p}_{(i)}\,\mathcal{P}_4$ and $n-4$ fluctuations of Z.

it is important in performing such an expansion to parameterise the fluctuations in a manner that preserves the induced U(1) symmetry. This is a special case of the general procedure for expanding nonlinear sigma models defined on a $\mathbb{G}/\mathbb{K}$ coset space, in which the fluctuating fields are defined by a normal coordinate expansion that is covariant with respect to the $\mathbb{K}$ symmetry. In our case we need to re-parameterise the fluctuations $\delta\tau$ around the background τ^0 in order that the fluctuations transform with a given U(1) charge. In the next sub-section we will see that the covariant expansion is given in terms of a reparameterisation of the complex scalar τ of the form of a Cayley map from the upper-half plane to the unit disk,

$$\tau \to Z = \frac{\tau - \tau^0}{\tau - \bar{\tau}^0}\,. \tag{4.1}$$

Figure 1(a) illustrates a maximal U(1)-violating amplitude with n external states taken from $\mathcal{P}_n$ and m scalar particle fluctuations (so $q_U = 8 - 2m - 2n$). This amplitude can be expressed in the form[16]

$$\mathcal{D}^m F^{(p)}_{n-4,\,i}(\tau^0)\,\mathcal{O}^{(p)}_{m+n,i}\,\mathcal{P}_n(\{\Phi\})\,Z^m\,, \tag{4.2}$$

$\mathcal{O}^{(p)}_{m+n,i}$ is a monomial in the Mandelstam invariants of the $(m+n)$-particle amplitude of degree p.

Among many such component amplitudes, there are maximal U(1)-violating interactions in which all n external states are complex scalars. These arise by choosing the component of $\mathcal{P}_4(\{\Phi\})$ that has two Z states and two $\bar{Z}$ states and expanding $F^{(4)}_0(\tau)$ to give $n-4$ Z fluctuations which is illustrated in figure 1(b). These maximal U(1)-violating amplitudes of scalars are simply monomials of Mandelstam invariants given by

$$\mathcal{D}^{n-4} F^{(p)}_{0,\,i}(\tau^0)\,\mathcal{O}^{(p)}_{m+n,i}\,\mathcal{O}^{(4)}_{Z^2\bar{Z}^2}\,Z^{m+n-2}\,\bar{Z}^2\,, \tag{4.3}$$

where the Mandelstam invariant $\mathcal{O}^{(4)}_{Z^2\bar{Z}^2}$ represents 8 derivatives acting on two Z states and two $\bar{Z}$ states of $\mathcal{P}_4(\{\Phi\})$. How these derivatives act is determined by maximal supersymmetry.

[16] In this expression, as well as later expressions for amplitudes, each field is to be replaced by its wave function in momentum space (i.e., each field represents an external state of definite momentum).

4.2 Expansion in scalar field fluctuations

In order to discuss the structure of the amplitudes in more detail we need to consider the appropriate definition of the fluctuating scalar fields. To illustrate the issue, consider the expansion of any modular form, $F_w(\tau)$, in powers of normalised small fluctuations around the background, $\delta\tau = \tau - \tau^0$,

$$\hat{\tau} := \frac{i}{2}\frac{\tau - \tau^0}{\tau_2^0} = \frac{i}{2}\frac{\delta\tau}{\tau_2^0}\,, \qquad\qquad \bar{\hat{\tau}} := -\frac{i}{2}\frac{\delta\bar{\tau}}{\tau_2^0}\,. \tag{4.4}$$

The quantity $\hat{\tau}$ does not transform covariantly under $\mathrm{SL}(2,\mathbb{Z})$ acting on τ and τ^0. Consequently, the coefficients in the expansion of a $w=0$ modular form, $F_0(\tau)$, in powers of $\hat{\tau}$,

$$F_0(\tau^0 + \delta\tau) = F_0(\tau^0) + 2i\tau_2^0\,\partial_{\tau^0}F_0(\tau^0)\hat{\tau} - 2(\tau_2^0)^2\,\partial_{\tau^0}^2 F_0(\tau^0)\hat{\tau}^2 + \cdots\,, \tag{4.5}$$

do not transform as modular forms. In such a parameterisation the Feynman rules have contact terms that vanish on shell and the evaluation of covariant amplitudes is very complicated.

The appropriate redefintion of τ is achieved by the $\mathrm{SL}(2,\mathbb{C})$ transformation that defines Z in (4.1) and which has an expansion as an infinite series of powers of $\hat{\tau}$,

$$Z = -(\hat{\tau} + \hat{\tau}^2 + \hat{\tau}^3 + \ldots)\,. \tag{4.6}$$

As required, the transformation of Z under the action of $\mathrm{SL}(2,\mathbb{Z})$ is the linear $\mathrm{U}(1)$ transformation given by

$$Z \to \frac{c\bar{\tau}^0 + d}{c\tau^0 + d}\,Z\,, \tag{4.7}$$

which means that Z is a weight $(-1,1)$ modular form and so carries $\mathrm{U}(1)$ charge $q_Z = -2$,

From the definition of $\delta\tau$ we have

$$\delta\tau = 2i\tau_2^0\frac{Z}{1-Z}\,, \tag{4.8}$$

and it is straightforward to verify that the Taylor expansion of $F_0(\tau)$ around the background $\tau = \tau^0$ given in (4.5) has the required covariant form,

$$F_0(\tau) = \sum_{w=0}^{\infty} 2^w\,\mathcal{D}_{w-1}\ldots\mathcal{D}_0\,F_0(\tau)\Big|_{\tau=\tau^0}\left(\frac{Z^w}{w!}\right) + \cdots\,. \tag{4.9}$$

The reparameterisation has converted the derivatives in (4.5) into covariant derivatives and the coefficients of the powers of Z^w are weight $(w,-w)$ modular forms which compensates for the charge of Z^w. The systematics of this expansion will be reflected in the expressions for n-particle scattering amplitudes in the following.

The parameterisation in terms of Z follows closely the discussion in [21], where the type IIB supergravity equations were formulated in a $\mathrm{SU}(1,1)$-covariant manner. This is briefly reviewed in appendix A.3. However, in considering the type IIB superstring it is

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important to remain in the gauge $\phi = 0$ as in (A.6). In that case after setting $\phi = 0$ in (A.28) we have

$$P_\mu(\tau) := i\frac{\partial_\mu \tau}{2\tau_2} = \frac{\partial_\mu Z}{1-\bar{Z}Z}\left(\frac{1-\bar{Z}}{1-Z}\right). \tag{4.10}$$

The supergravity action expressed in terms of the fluctuations $\hat{\tau}$ has interaction terms that vanish on shell. For example, the expansion of S_τ in (A.19) in powers of $\hat{\tau}$ and $\bar{\hat{\tau}}$ leads to interactions, such as $\partial_\mu\hat{\tau}\,\partial^\mu\bar{\hat{\tau}}\,\hat{\tau}$ that violate U(1) but vanish on shell. It is an important consequence of the parameterisation of the complex scalar field in terms of Z that such on-shell vanishing terms are absent. Thus, the scalar field action S_τ in (A.19) is replaced by

$$S_Z = -\frac{2}{\kappa^2}\int d^{10}x\, e\, \frac{\partial_\mu Z \partial^\mu \bar{Z}}{(1-\bar{Z}Z)^2}\,. \tag{4.11}$$

All terms in the expansion of this expression in powers of $\bar{Z}Z$ transform as U(1) singlets and none of them vanish on shell.

Other fields. For consistency it is important to perform reparameterisations of other massless fields. For example, consider the Dirac lagrangian density for the dilatino of charge $q_\Lambda = -3/2$,

$$\bar{\Lambda}^a\gamma^\mu(\partial_\mu + iq_\Lambda Q_\mu)\,\Lambda^a\,. \tag{4.12}$$

In a fixed background $\tau = \tau^0$ we need to use the expression for Q_μ given in (A.29). The resulting Dirac action contains U(1)-violating contact interactions that vanish on shell, the lowest order being of the form $\bar{\Lambda}\gamma^\mu\partial_\mu Z\,\Lambda$. These again lead to very complicated Feynman rules and are removed by the appropriate field redefinition,

$$\Lambda'_a = \Lambda_a\left(\frac{1-Z}{1-\bar{Z}}\right)^{q_\Lambda/2}. \tag{4.13}$$

It is straightforward to check that the redefined Λ'_a transforms linearly by the induced U(1) transformation,

$$\Lambda'_a \to \left(\frac{c\bar{\tau}^0+d}{c\tau^0+d}\right)^{q_\Lambda/2}\Lambda'_a\,, \tag{4.14}$$

under $SL(2,\mathbb{Z})$. Furthermore, when the interactions are expressed in terms of Λ'_a, all the U(1)-violating (and on-shell vanishing) vertices in the reparameterised (4.12) are removed.

The same considerations apply to the reprameterisation of the gravitino, ψ, which has $q_\psi = -1/2$, as well as the third-rank field strength G with $q_G = -1$. After these reparameterisations the n-particle contact interactions in (1.1) are transformed into

$$F^{(p)}_{n-4,i}(\tau^0 + \delta\tau(Z))\, d^{2p}_{(i)}\mathcal{P}^{(p)}_n(\{\Phi\})\left(\frac{1-Z}{1-\bar{Z}}\right)^{n-4}, \tag{4.15}$$

where we have indicated the Z-dependence in the fluctuations of τ around its background value as well as the explicit Z-dependence coming from the transformation of the fields in $\mathcal{P}^{(p)}_n(\{\Phi\})$. The expression (4.15) is appropriate for performing a covariant expansion in m powers of Z, resulting in an expression for the $(m+n)$-particle amplitude of the form (4.2).

4.3 Supersymmetric scattering amplitudes

Ten-dimensional helicity spinors. In order to describe the super-amplitudes, we introduce the ten dimensional spinor helicity formalism, following [26]. The spinor-helicity formalism expresses the momentum of any massless state in terms of chiral bosonic spinors λ_a^A satisfying the ten-dimensional Dirac equation,

$$(\gamma^\mu)^B{}_A\, p_\mu \lambda_a^A = 0\,, \tag{4.16}$$

where $A = 1, \ldots, 16$ labels the components of a SO(9, 1) chiral spinor under and $a = 1, \ldots, 8$ labels the components of a SO(8) spinor of the little group of massless states. The ten-dimensional gamma matrices, $(\gamma^\mu)^B{}_A$, are projected onto the subspace of 16-dimensional chiral spinors. The momentum is expressed in terms of λ by

$$p^{BA} := (\gamma^\mu)^{BA}\, p_\mu = \lambda^{Ba}\lambda_a^A\,, \tag{4.17}$$

and the supercharges satisfying the on-shell super-algebra

$$\{\bar{q}_i^B,\, q_i^A\} = \lambda_i^{Ba}\lambda_{i,a}^A\,, \tag{4.18}$$

are expressed as [27]

$$q_i^A = \lambda_{i,a}^A\, \eta_i^a\,, \qquad \bar{q}_i^B = \lambda_i^{B,a}\frac{\partial}{\partial \eta_i^a}\,, \tag{4.19}$$

where η^a is a Grassman variable satisfying

$$\left\{\eta^a,\, \frac{\partial}{\partial \eta^b}\right\} = \delta^a{}_b\,. \tag{4.20}$$

Each external single-particle state in a scattering amplitude labelled i is associated with a on-shell super-field that is a function of independent variables (p_i, η_i) and has an expansion in powers of η_i^a given by

$$\phi_0(p_i) + \eta_i^a \phi_a(p_i) + \frac{1}{2!}\eta_i^a\eta_i^b\phi_{ab}(p_i) + \cdots + \frac{1}{8!}(\eta_i)^8\bar{\phi}_0(p_i)\,. \tag{4.21}$$

The 256 component fields in this expansion correspond to the massless fields of type IIB supergravity that arise in the linearised on-shell superfield in appendix C.[17] Thus,

$$\phi_0 \sim Z\,, \qquad \phi_a \sim \Lambda'_a\,,\ \ldots, \qquad \bar{\phi}_0 \sim \bar{Z}\,, \tag{4.22}$$

where we have explicitly used the redefined component fields. As with the superfield defined in (C.5) this field has U(1) charge $q_\phi = -2$. If we assign a U(1) charge $q_\eta = -1/2$ to η, a component field with m SO(8) spinor indices has a charge $q_m = -2 + m/2$.

[17] This is also very similar to the expansion of the superfield in the light-cone gauge formulation of type IIB supergravity.

There are independent supersymmetry generators, $(q_i^A, \bar{q}_i^B)$, of the form (4.19) on each leg of the diagram with variables (η_i, λ_i). The total supercharge for a n-particle amplitude are

$$Q_n^A = \sum_{i=1}^n q_i^A = \sum_{i=1}^n \lambda_{i,a}^A \, \eta_i^a \,, \qquad \bar{Q}_n^B = \sum_{i=1}^n \bar{q}_i^B = \sum_{i=1}^n \lambda_i^{B,a} \frac{\partial}{\partial \eta_i^a} \,, \tag{4.23}$$

and the amplitude satisfies the overall supersymmetry conditions,

$$Q_n^A \, A_n = 0 = \bar{Q}_n^A \, A_n \,, \tag{4.24}$$

in addition to overall momentum conservation. This means that an amplitude with n massless external states has the form[18]

$$A_n = \delta^{10}\left(\sum_{i=1}^n p_i\right) \delta^{16}(Q_n) \, \hat{A}_n \,, \tag{4.25}$$

where

$$\delta^{16}(Q) = \frac{1}{16!} \epsilon_{A_1 \dots A_{16}} Q^{A_1} \dots Q^{A_{16}} \,, \tag{4.26}$$

and

$$\bar{Q}_n^A \hat{A}_n = 0 \,. \tag{4.27}$$

The relation (4.26) and the condition (4.27) ensure that the amplitude A_n is annihilated by the thirty-two supersymmetries.

Apart from the three-particle on-shell amplitude, which has degenerate kinematics, these conditions imply that scattering amplitudes vanish unless the total number of η's from external states is at least 16. Amplitudes in which there are exactly sixteen η variables are those for which $q_U = -2(n-4)$, (these are called "maximal R-symmetry violating" amplitudes in [7]). In this case the quantity $\hat{A}$ contains no factors of η but it is a function of the Mandelstam invariants that encodes the α'-dependence characteristic of string theory, as well as the dependence on the complex coupling constant, τ^0.

In considering the low-energy expansion of amplitudes it is important to take into account non-analytic features that come from the effects of higher genus contributions and non-perturbative effects. Although this is very complicated in general, the first three terms in the low-energy expansion of the ten-dimensional amplitude, which are protected by supersymmetry, are analytic in the Mandelstam invariants. For these terms $\hat{A}$ has an expansion in a series of symmetric polynomials of degree $p = 0$, $p = 2$ and $p = 3$ in the Mandelstam invariants, since maximal U(1)-violating amplitudes cannot have poles in momenta.

This leads to BPS terms in the low-energy limits of n-particle superstring amplitudes in the form,

$$A_n^{(p)} = \kappa^{\frac{p-1}{2}} F_{n-4}^{(p)}(\tau^0) \, \delta^{16}(Q_n) \, \hat{A}_n^{(p)}(s_{ij}) \,, \tag{4.28}$$

[18]We will suppress the momentum conservation delta function from hereon.

where the subscript $(n-4)$ indicates the weight, w. In this expression, which includes amplitudes of the form (4.2), the factor $\hat{A}_n^{(p)}(s_{ij})$ is simply a symmetric homogeneous degree-p polynomial of Mandelstam variables. Note that in our normalisation the overall power of κ for a n-particle amplitude is independent of n.

Since these amplitudes have no poles they may be viewed as on-shell supervertices. For $p \leq 3$ they are BPS F-terms, whose coefficients $F_{n-4}^{(p)}(\tau^0)$ are constrained by supersymmetry, as will be shown in later sections. For such amplitudes $\hat{A}_n^{(p)}(s_{ij})$ may contain powers of Mandelstam invariants but these cannot be re-expressed in terms of the other sixteen supercharges $\bar{Q}^A$ [11].

Terms of higher order in the low-energy expansion — i.e. of dimension ≥ 16 (or $p \geq 4$) — are D-terms and they can be written in terms of 32 supercharges. For example if $\hat{A}_n^{(4)}(s_{ij})$ is a symmetric polynomial in Mandelstam invariants of degree 4 it can be expressed in the schematic form

$$\hat{A}_n^{(4)}(s_{ij}) \sim \sum_{\text{permutations}} (\bar{Q})^{16} \eta_i^8 \eta_j^8 \,. \tag{4.29}$$

This is simply a consequence of power counting since $(\bar{Q})^{16}$ is of order s_{ij}^4. This is the on-shell amplitude description of D-terms. Indeed as we will see later such terms are, unsurprisingly, unconstrained and do not appear to be protected by supersymmetry.

The coefficient function $F_{n-4}^{(p)}(\tau^0)$ contains the full non-perturbative dependence on the complex type IIB coupling constant in the Einstein frame. The leading term in the weak coupling limit is the tree-level contribution, which is given by $(\tau_2^0)^{\frac{3+p}{2}}$ multiplied by a rational multiple of a weight-$(3+p)$ odd zeta value. We will now discuss examples of these BPS terms that emerge explicitly from the expansion of tree-level maximal U(1)-violating superstring amplitudes.

4.4 Low-energy expansion of tree-level maximal U(1)-violating amplitudes

In recent years various methods have been devised for calculating n-particle superstring theory tree amplitudes [28–30]. Closed-string amplitudes are efficiently expressed in the KLT manner by doubling open-string amplitudes, which are stringy extensions of Yang-Mills theory. This results in expressions for the super closed-string tree amplitudes that are conveniently expressed in the following manner

$$A^n_{closed} = A^n_{\text{YM tree}}\, S_{\text{KLT}}(s_{ij})\, G(\alpha' s_{ij})\, \tilde{A}^n_{\text{YM tree}} \tag{4.30}$$

where $A^n_{\text{YM tree}}$ and $\tilde{A}^n_{\text{YM tree}}$ are n-particle colour-stripped super Yang-Mills tree amplitudes with different permutations of the cyclic order, $S_{\text{KLT}}(s_{ij})$ is the KLT kernel, and $G(\alpha' s_{ij})$ contains the stringy corrections to the field theory expression so $\lim_{\alpha'\to 0} G(\alpha' s_{ij}) = 1$. The low-energy expansion involves expanding $G(\alpha' s_{ij})$ in a power series in $\alpha' s_{ij}$.

Such expressions may be efficiently evaluated in the four-dimensional spinor-helicity formalism in which MHV amplitudes play a distinguished rôle. For instance, the MHV supergravity amplitudes are obtained if the states in both Yang-Mills factors are chosen to be MHV. However, U(1)-violating amplitudes arise when the helicity assignments in

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the Yang-Mills factors are distinct. The maximal U(1)-violating closed-string amplitudes result from the choices in which one Yang-Mills factor is MHV and the other is $\overline{\text{MHV}}$. This four-dimensional formalism is convenient for describing the compactification to four-dimensional $\mathcal{N}=8$ supergravity but it obscures its origins in ten dimensions. In particular, it obscures the rôle of the ten-dimensional complex axio-dilaton.

A more direct procedure is to consider the KLT construction of (4.30) in ten dimensions, where the maximally supersymmetric amplitudes have been constructed by use of the pure spinor formalism. The form of the maximal U(1)-violating amplitudes is tightly constrained by supersymmetry. One may compute component amplitudes with particular external states, which have the form (4.28). The results of this analysis give the following low-order terms in the low-energy expansion of $\hat{A}_n(s_{ij})$,[19]

$$\begin{aligned}
\hat{A}_4(s_{ij}) &= 2\,\kappa^{-\frac{1}{2}}\,\tau_2^{\frac{3}{2}}\,\zeta(3) + \kappa^{\frac{1}{2}}\,\tau_2^{\frac{5}{2}}\,\zeta(5)\,\mathcal{O}_4^{(2)} + \frac{2}{3}\,\kappa\,\tau_2^3\,\zeta(3)^2\,\mathcal{O}_4^{(3)} + \cdots \\
\hat{A}_5(s_{ij}) &= 3\,\kappa^{-\frac{1}{2}}\,\tau_2^{\frac{3}{2}}\zeta(3) + \frac{5}{2}\kappa^{\frac{1}{2}}\,\tau_2^{\frac{5}{2}}\zeta(5)\,\mathcal{O}_5^{(2)} + 2\,\kappa\,\tau_2^3\zeta(3)^2\,\mathcal{O}_5^{(3)} + \cdots \\
\hat{A}_6(s_{ij}) &= \frac{15}{2}\,\kappa^{-\frac{1}{2}}\,\tau_2^{\frac{3}{2}}\zeta(3) + \frac{35}{4}\kappa^{\frac{1}{2}}\,\tau_2^{\frac{5}{2}}\zeta(5)\,\mathcal{O}_6^{(2)} + 8\,\kappa\,\tau_2^3\zeta(3)^2\,\mathcal{O}_{6,1}^{(3)} + \cdots\,,
\end{aligned} \tag{4.31}$$

where we have expressed the amplitudes in the Einstein frame, and

$$\mathcal{O}_n^{(2)} := \frac{1}{2}\sum_{1\le i<j\le n} s_{ij}^2\,, \qquad \mathcal{O}_n^{(3)} := \frac{1}{2}\sum_{1\le i<j\le n} s_{ij}^3\,, \tag{4.32}$$

and $\mathcal{O}_{6,1}^{(3)}$ is the kinematics structure defined in (3.24). Each amplitude has been normalised to be consistent with the convention that will be defined in (5.2).

Although the overall normalisations of the tree-level n-particle amplitudes depend on conventions, the above equations determine the relative coefficients of the $p=2$ and $p=3$ terms in the low-energy expansion (the dimension-12 and dimension-14 terms) in terms of the $p=0$ coefficients. This relates the values of constants $c_{w,i}^{(p)}$ that arose in sections 2 and 3 as follows

$$\begin{aligned}
c_0^{(2)} &= \frac{1}{2}c_0^{(0)}\,, & c_0^{(3)} &= c_0^{(0)} \\
c_1^{(2)} &= \frac{5}{6}c_1^{(0)}\,, & c_1^{(3)} &= \frac{2}{3}c_1^{(0)} \\
c_2^{(2)} &= \frac{7}{6}c_2^{(0)}\,, & c_{2,1}^{(3)} &= \frac{4}{15}c_2^{(0)}\,.
\end{aligned} \tag{4.33}$$

Note that the choice of overall normalisations of the amplitudes given in (4.31) translates into the choices $c_0^{(0)}=1$, $c_1^{(0)}=3/2$, $c_2^{(0)}=15/4$. This gives the values for the $p=3$ coefficients, $c_0^{(3)}=c_1^{(3)}=1$ and $c_{2,1}^{(3)}=1$, which is consistent with the choice of normalisation to be made in (5.3) (based on consideration of the soft Z limit).

These tree-level amplitudes are the lowest order terms in the expansion of $\mathrm{SL}(2,\mathbb{Z})$-covariant amplitudes so the coefficients in the expansions of amplitudes with different values

[19] We are very grateful to Oliver Schlotterer for providing us with the coefficients for the six-point amplitude $\hat{A}_6(s_{ij})$ in the following equations [31].

of n (and hence of w) must be related to each other by $\mathrm{SL}(2,\mathbb{Z})$. Since the amplitudes with $(n+1)$ external particles and with n external particles have different kinematics we cannot simply compare the coefficients. However, a $(n+1)$-particle amplitude is expected to reduce to a n-particle amplitude in the soft axio-dilaton limit, as we will now discuss.

5 The soft Z limit and covariant derivatives

As discussed previously, the general $(n+m)$-particle maximal U(1)-violating amplitude for n particles in $\mathcal{P}_n(\{\Phi\})$ together with m axio-dilaton particles, Z^m, is obtained by expanding the expression in (4.15), giving interactions $\mathcal{P}_n(\{\Phi\})Z^m/m!$ with a coefficient modular form

$$F^{(p)}_{m+n-4}(\tau^0) = 2^m\,\mathcal{D}_{m+n-4}\ldots\mathcal{D}_{n-4}\,F^{(p)}_{n-4}(\tau)\Big|_{\tau=\tau^0}\,. \tag{5.1}$$

Importantly the Z fields are trivially attached to the lower-point vertex $\mathcal{P}_n(\{\Phi\})$, therefore we see that not only are the coefficients related by covariant derivatives as in (5.1), but also the kinematic factors in the amplitudes are related by the soft limit on the momentum of a Z field.

Indeed, this is the general property of scalars of a coset space. It is well-known that the amplitudes vanish in the soft scalar limit for the classical theory where the duality symmetry is unbroken [32]. The soft behaviour reflects the fact that the scalars parameterising the coset space are Goldstone bosons. However, for the case of interest in this paper, the continuous symmetries are in general broken, and correspondingly the amplitudes are non-vanishing in the soft scalar limit. In fact, as we indicated above, the soft Z limit relates a n-point amplitude to a $(n-1)$-point amplitude with the soft particle Z_n removed [14],[20]

$$A_n(X,Z_n)\big|_{p_n\to 0} = 2\,\mathcal{D}A_{n-1}(X)\,, \tag{5.2}$$

where X represents the hard particles. More precisely, both $A_{n-1}(X)$ and $A_n(X,Z_n)$ are products of modular forms and kinematic factors, where the modular forms of $A_n(X,Z_n)$ are related to those of $A_{n-1}(X)$ by a covariant derivative $\mathcal{D}$, whereas the kinematic parts of $A_n(X,Z_n)$ reduce to those of $A_{n-1}(X)$, so that (5.2) takes the form

$$F^{(p)}_n(\tau^0)\,\mathcal{O}^{(p)}_{n,i}\big|_{p_n\to 0} = 2\,\mathcal{D}F^{(p)}_{n-1}(\tau)\big|_{\tau=\tau^0}\mathcal{O}^{(p)}_{n-1,i}\,. \tag{5.3}$$

The soft limit (5.2) has also been explicitly checked against the known results such as the tree-level amplitudes given in (4.31) (as well as higher-order terms up to order τ_2^4 which we did not exhibit). On the other hand, the soft limits (5.2) impose highly non-trivial constraints on the amplitudes, and may be utilised to determine higher-point interactions from lower-point ones as will be analysed in the following section.

[20] Analogous soft scalar limits for U(1)-violating amplitudes in the four-dimensional $\mathcal{N}=4$ supergravity were studied in [33, 34].

Note on connection with the standard soft dilaton limit. There is a well-studied soft limit that involves only the real part of Z field, namely the dilaton, which states that[21]

$$A_n(X,\varphi_n)\big|_{p_n\to 0} = \left(\alpha'\frac{\partial}{\partial\alpha'} - 2g_s\frac{\partial}{\partial g_s}\right) A_{n-1}(X)\,, \tag{5.4}$$

where φ_n is the dilaton fluctuation corresponding to the particle with momentum p_n. This soft-dilaton limit has been known since the 70's [35, 36], and has been revisited recently (see for example, [37–39]). In order to compare with the soft-Z limit (5.2), we will transform (5.4) to the Einstein frame. To do so, we express amplitudes in terms of $\kappa = (\alpha')^2 g_s$ and $g_s = \tau_2^{-1}$, such as those in (4.31). We further use the fact that the differential operator in (5.4) annihilates κ, then (5.4) translates into

$$A_n(X,\varphi_n)\big|_{p_n\to 0} = 2\tau_2\frac{\partial}{\partial\tau_2} A_{n-1}(X)\,. \tag{5.5}$$

Since $\tau_2 = e^{-\varphi}$, to lowest order the dilaton is related to Z by

$$\varphi \sim \frac{\tau_2-\tau_2^0}{\tau_2^0} = \left(\frac{Z}{1-Z} + \frac{\bar Z}{1-\bar Z}\right) \sim Z+\bar Z\,. \tag{5.6}$$

It follows that (5.5) is a consequence of the sum of (5.2) and its conjugate equation, $A_n(X,\bar Z_n)\big|_{p_n\to 0} = 2\,\bar{\mathcal{D}} A_{n-1}(X)$,

$$A_n(X,Z_n{+}\bar Z_n)\big|_{p_n\to 0} = 2\left(\mathcal{D}_w + \bar{\mathcal{D}}_{-w}\right) A_{n-1}(X)\,. \tag{5.7}$$

Upon using (2.6), (5.7) indeed reduces to (5.5). Of course, taking the soft limit on φ is unnatural in the content of $\mathrm{SL}(2,\mathbb{Z})$ symmetry. In particular the right-hand side of (5.7) is a sum of modular functions of different $\mathrm{SL}(2,\mathbb{Z})$ weights.

5.1 Applications of the soft Z limit

Here we consider the consequences of soft Z limits for the maximal U(1)-violating amplitudes. As discussed in the previous sections, the amplitudes take the form

$$A_n^{(p)} = \kappa^{\frac{p-1}{2}}\, F_{n-4}^{(p)}(\tau^0)\,\delta^{16}(Q_n)\,\hat A_n^{(p)}(s_{ij})\,, \tag{5.8}$$

where $\hat A_n^{(p)}(s_{ij})$ has a unique kinematic structure when $p=0$ or $p=2$, given by

$$\hat A_n^{(0)}(s_{ij}) = 1\,, \qquad \hat A_n^{(2)}(s_{ij}) = \mathcal{O}_n^{(2)}\,, \tag{5.9}$$

where $\mathcal{O}_n^{(2)}$ is defined in (4.32). The soft limits relate $\hat A_n^{(p)}(s_{ij})$ to $\hat A_{n-1}^{(p)}(s_{ij})$ trivially for these cases, and the coefficients are again related by a covariant derivative

$$F_{n-4}^{(p)}(\tau^0) = 2\,\mathcal{D}F_{n-5}^{(p)}(\tau)\big|_{\tau=\tau^0}\,. \tag{5.10}$$

The cases with $p=3$ are more interesting. For all $n\geq 6$, there are two independent kinematic invariants which are denoted $\mathcal{O}_{n,1}^{(3)}$ and $\mathcal{O}_{n,2}^{(3)}$. They satisfy the soft relations,

$$\mathcal{O}_{n,1}^{(3)}\big|_{p_n\to 0} = \mathcal{O}_{n-1,1}^{(3)}\,, \qquad \mathcal{O}_{n,2}^{(3)}\big|_{p_n\to 0} = \mathcal{O}_{n-1,2}^{(3)}\,. \tag{5.11}$$

[21] In our normalisation of the amplitudes, the following soft factors have no overall factor of κ.

The kinematic invariants for $n > 6$ are uniquely determined using the expressions of $\mathcal{O}^{(3)}_{6,1}$ and $\mathcal{O}^{(3)}_{6,2}$ given in (3.24) and (3.26) and the above soft relations,

$$\begin{aligned}\mathcal{O}^{(3)}_{n,1} &= \frac{1}{32}\left((28-3n)\sum_{i<j} s^3_{ij} + 3\sum_{i<j<k} s^3_{ijk}\right),\\ \mathcal{O}^{(3)}_{n,2} &= (n-4)\sum_{i<j} s^3_{ij} - \sum_{i<j<k} s^3_{ijk}\,,\end{aligned} \tag{5.12}$$

where $\mathcal{O}^{(3)}_{4,1} = s^3 + t^3 + u^3$ corresponds to d^6R^4. The maximal U(1)-violating amplitudes are then given by

$$A^{(3)}_{n,1} = \kappa\, F^{(3)}_{n-4,1}(\tau^0)\,\delta^{16}(Q_n)\,\mathcal{O}^{(3)}_{n,1}\,, \qquad A^{(3)}_{n,2} = \kappa\, F^{(3)}_{n-4,2}(\tau^0)\,\delta^{16}(Q_n)\,\mathcal{O}^{(3)}_{n,2}\,, \tag{5.13}$$

where the coefficients are related by covariant derivatives, i.e. $F^{(3)}_{m,i}(\tau^0) = 2\mathcal{D}F^{(3)}_{m-1,i}(\tau^0)$ for $i = 1, 2$.

In these expressions the coefficient $F^{(3)}_{n-4,1}(\tau^0)$ is determined by nested covariant derivatives acting on $F^{(3)}_0(\tau^0)$, i.e., on the coefficient of d^6R^4. Since these coefficients are associated with kinematic factors $\mathcal{O}^{(3)}_{n,1}$, their tree-level contributions are related and non-zero. However, $F^{(3)}_{n-4,2}(\tau^0)$ with $n > 6$ is determined in terms of nested derivatives acting on the coefficient $F^{(3)}_{2,2}(\tau^0)$. These terms have no tree-level contributions. As discussed in the previous section, $F^{(3)}_{2,2}(\tau^0)$ is constrained by supersymmetry and satisfies (3.30), which will also be seen to emerge from the structure of super-amplitudes in the next section.

6 Super-amplitude constraints on first-order differential equations

We have seen how type IIB amplitudes with different numbers of particles are related by consideration of the soft Z limit. This relates the $(n+1)$-particle amplitude with one soft Z state to the n-particle amplitude with the soft Z removed. In the case of maximal U(1)-violating amplitudes this involves the relation between the coefficient modular forms of the form $F^{(p)}_{w+1}(\tau) \sim \mathcal{D}\,F^{(p)}_w(\tau)$, that was encountered in sections 2 and 3 and applies to the coefficients for any value of p. This relationship applies to terms for which the kinematic factors are related in the soft limit in the manner of (5.3).

In order to show how the conjugate first order differential equations involving $\bar{\mathcal{D}}$ are determined by supersymmetry constraints, we will extend the procedure devised in [11] for determining the constraints based on the structure of super-amplitudes. The key ingredients in this procedure are encapsulated in the following statements:

- *Supersymmetric F-terms are contact interactions corresponding to $p \leq 3$ terms in the low-energy expansion of maximal* U(1)*-violating amplitudes.*
- *Supersymmetric contact terms of dimension ≤ 14 are not allowed for non-maximal* U(1)*-violating processes. The absence of a supersymmetric contact term provides powerful constraints on the F-term effective interactions.*

- *Interactions with dimension more than 14 are D-terms, whose couplings in general are not constrained by supersymmetry.*

As an example, let us consider low-order terms in the low-energy expansion of a six-particle amplitude (terms with $p = 0, 2, 3$), such as the amplitude with four gravitons, one Z field and one $\bar{Z}$ field. It is straightforward to see that a supersymmetric contact term with $p \leq 3$ (i.e. with a number of derivatives not greater than 14) does not exist for such an amplitude. Indeed for this particular case, the corresponding super-amplitude contains 24 η's which enter into a supersymmetric invariant that can be expressed in the following form,

$$\delta^{16}(Q_n)(\bar{Q}_n)^{16}(\eta_i)^8(\eta_j)^8(\eta_k)^8\,, \tag{6.1}$$

since $(\bar{Q}_n)^{16}$ annihilates 16 η's (recalling that $\bar{Q}_n$ is defined in (4.23)). This has 16 powers of momentum whereas BPS terms ($p \leq 3$) have at most 14 powers. Therefore, in order to describe a supersymmetric term there must be intermediate poles (inverse momentum factors) so that supersymmetric contact terms are not allowed.

This argument has an important and subtle loophole for $n = 5$. For instance $R^4\bar{Z}$ also requires 24 η's. But since $R^4\bar{Z}$ is just the complex conjugate of $R^4 Z$ it obviously does have a supersymmetric completion. It turns out that the following expression for the $R^4\bar{Z}$ amplitude, which appears to have a higher-order pole is actually a contact term

$$\delta^{16}(Q_5)\frac{(\bar{Q}_5)^{16}(\eta_1)^8(\eta_2)^8(\eta_3)^8}{(s_{45})^4}\,. \tag{6.2}$$

To see that this is non-singular as $s_{45} \to 0$ it is sufficient to note that this expression is in fact invariant under permutations of the external states, although this is not manifest.

The fact that it is not possible to write a supersymmetric contact term for a non-maximal U(1)-violating process of dimension ≤ 14 implies that the low-energy expansion up to dimension 14 of the super-amplitude is uniquely determined by lower-point amplitudes via factorisation on intermediate poles as determined by tree-level unitarity.

This strongly constrains the components of the effective action. In particular, the contact terms in the component action are related to the non-local factorisation diagrams. In other words, the contact terms that enter the component action are not independent vertices since they cannot be supersymmetrised in isolation from the rest of the amplitude. This implies that there must be a linear relation between the coefficients of component contact terms and those of the factorisation terms. This approach using only on-shell data and tree-level unitarity is an efficient way of imposing supersymmetric constraints on the coefficient modular functions of F-terms.

Before applying the above idea to derive first-order differential equations satisfied by the modular forms that are the coefficients of the F-terms, we should emphasise again that the D-terms are in general not constrained. The existence of supersymmetric contact terms such as (6.1) imply that tree-level unitarity is not enough to determine the D-term contributions to super-amplitudes. In other words, one can always add contact terms with arbitrary coefficients to a given expression without modifying the factorisation conditions. Therefore, as is well-known, D-terms are not constrained by supersymmetry.

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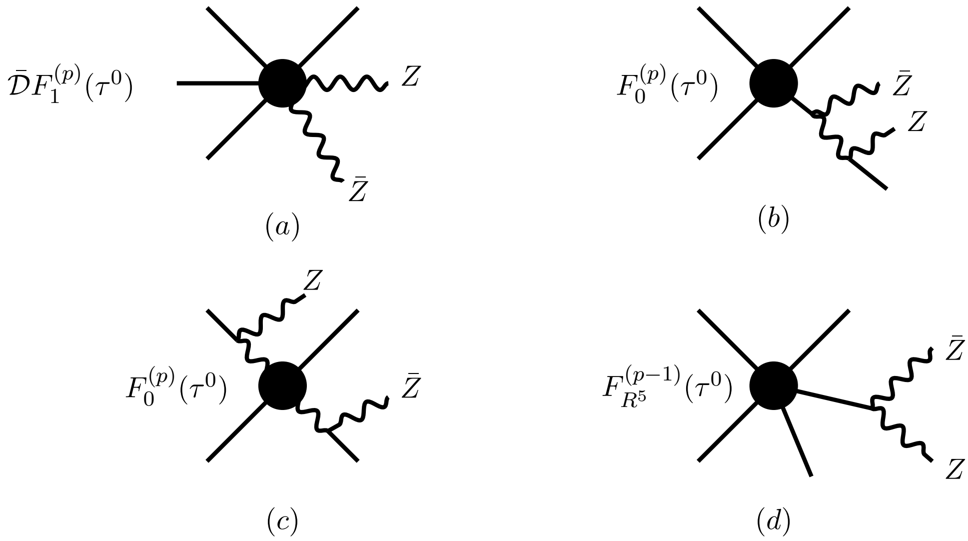

Figure 2. A diagrammatic interpretation of the pieces of the first-order differential equation relating $F_1^{(p)}(\tau^0)$ to $F_0^{(p)}(\tau^0)$ for a dimension-8 ($p = 0$) or dimension-12 ($p = 2$) contribution to the amplitude for 4 gravitons together with a Z and a $\bar{Z}$. (a) A contact term describing the emission of a $\bar{Z}$ from a $d^{2p}R^4Z$ five-particle contact term. (b) and (c) Two examples of factorisation contributions formed by attaching Z and $\bar{Z}$ to the external legs of the $d^{2p}R^4$-type contact terms with supergravity vertices. (d) The factorisation contribution by attaching a Z-$\bar{Z}$-graviton vertex to $d^{2p-2}R^5$ (it does not contribute when $p = 0$).

Terms with $p = 0$ and $p = 2$. These supersymmetry constraints lead to particularly simple first-order equations in the $p = 0$ and $p = 2$ cases that are illustrated by the processes depicted in figure 2. This shows the contributions to the four-graviton-Z-$\bar{Z}$ amplitude. Such an ampliude is not maximal U(1) violating and has massless intermediate poles. Figure 2(a) illustrates a contact interaction in which the coefficient is expressed as $\bar{\mathcal{D}}F_1^{(p)}(\tau^0)$ where $F_1^{(p)}(\tau^0)$ is the coefficient of a five-particle maximal U(1)-violating amplitude for four gravitons and one Z.

The absence of a supersymmetric contact term implies that there is a linear relation between the coefficients of each term that contributes to the component amplitude as shown in figure 2. There are two classes of diagrams. One is the contact interaction of figure 2(a), while the others are factorisation contributions that contain intermediate poles, such as the processes shown in figure 2(b), (c) and (d). In these factorisation diagrams Z and $\bar{Z}$ states are attached via supergravity interactions to external legs of the four-graviton $d^{2p}R^4$ interaction, or (in the case of figure 2(d)) the five-graviton $d^{2p-2}R^5$ interaction. There are several other analogous diagrams to take into account that we have not drawn. It would be complicated to calculate all of these pole contributions precisely. However the contributions from figures 2(b) and (c) are proportional to $F_0^{(p)}(\tau^0)$. In order to complete this discussion we will now demonstrate that $F_{R^5}^{(p-1)}$ (the coefficient of $d^{2p-2}R^5$) in figure 2(d), is also proportional to $F_0^{(p)}(\tau^0)$.

In order to determine properties of $F_{R^5}^{(p-1)}$ we need to consider properties of the five-graviton amplitude, which is not a maximal U(1)-violating process. The diagrams that

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contribute to this interaction are the local vertex, $d^{2p-2}R^5$ (with coefficient $F^{(p-1)}_{R^5}(\tau^0)$) and pole terms arising from attaching a three-graviton vertex to $d^{2p}R^4$. Since the interaction $d^{2p-2}R^5$ has dimension ≤ 14 (when $p \leq 3$), our previous argument implies that the super-amplitude containing this process cannot have a supersymmetric contact term. The absence of such a contact term implies that $d^{2p-2}R^5$ is related to $d^{2p}R^4$, which leads to a linear relation between their coefficients

$$F^{(p-1)}_{R^5}(\tau^0) + a\, F^{(p)}_0(\tau^0) = 0\,, \tag{6.3}$$

that is in agreement with [40]. Therefore, $F^{(p-1)}_{R^5}(\tau^0)$ is proportional to $F^{(p)}_0(\tau^0)$ (when $p \leq 3$). We note, in particular that the absence of a four-graviton interaction with $p = 1$, of the form d^2R^4 implies the absence of a R^5 contact interaction.

Returning to our consideration of the contributions in figure 2 we now see that the uniqueness of the super-amplitude implies that there must be a linear relation between the coefficients of the various contributions, of the form

$$\bar{\mathcal{D}} F^{(p)}_1(\tau^0) = c\, F^{(p)}_0(\tau^0)\,, \qquad\qquad p = 0, 2\,. \tag{6.4}$$

This is the structure of the relationship between the coefficients that was discussed in section 2 where the modular form coefficients were identified with Eisenstein modular forms that satisfy (B.7). In principle the value of c should be determined by explicitly constructing the super-amplitude and evaluating the various supergravity insertions, but we have not done this. An indirect way of fixing the value of c is to note that (6.4), together with the relation $F^{(p)}_1(\tau^0) = 2\mathcal{D}\, F^{(p)}_0(\tau^0)$, imply the well-known Laplace eigenvalue equation $(4\bar{\mathcal{D}}\mathcal{D} - 2c)F^{(p)}_0(\tau^0) = 0$. From our earlier discussion of such equations we know that when $p = 0$ we must have $c = 3/8$ (so the eigenvalue is $3/4$) and when $p = 2$ we must have $c = 15/8$ (so the eigenvalue is $15/4$). Equivalently, the value of c is also be fixed by inputting the string theory tree-level contribution to $F^{(p)}_0(\tau^0)$ in (4.31).

Terms with $p = 3$ and $w = 1$. In considering the first-order differential equations for the $p = 3$ terms (dimension-14 terms) in the low-energy expansion we expect to meet the novel features of the coefficients that were described in section 3. This may again be seen by considering the absence of a contact term of the dimension-14 super-amplitude that contains the component amplitude of four gravitons, one Z field and one $\bar{Z}$ field. The contribution with $p = 3$ to the low-energy expansion of this amplitude receives contributions that are schematically shown in figure 3. In this case, the terms such as figures 3(b)-3(e) are of the same form as those in the $p = 0, 2$ cases. However, dimensional counting shows that the amplitude also has a contribution in which it factorises on an intermediate pole that separates two four-particle higher-dimensional interactions, as shown in figure 3(f).

Again, the absence of a supersymmetric contact term implies that there must be a relation among all three types of terms shown in the figure 3

$$\bar{\mathcal{D}}\, \mathcal{E}^{(3)}_1(\tau^0) + a\, \mathcal{E}^{(3)}_0(\tau^0) + b\, E_0(\tfrac{3}{2}, \tau^0) E_0(\tfrac{3}{2}, \tau^0) = 0\,, \tag{6.5}$$

which is the same form as (3.9). Using $\mathcal{E}^{(3)}_1 = 2\,\mathcal{D}\mathcal{E}^{(3)}_0$ (as follows from the soft limit (5.3)), the above equation leads to an inhomogeneous Laplace equation for $\mathcal{E}^{(3)}_0$ of the form (3.2).

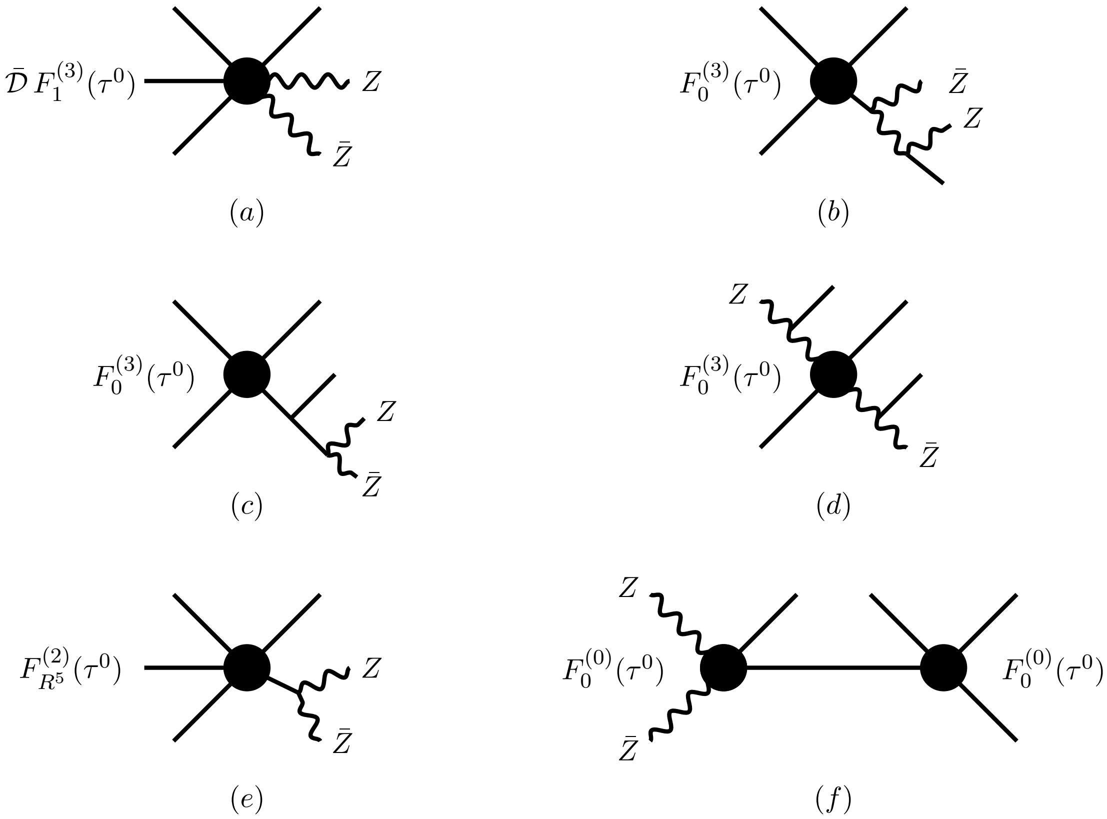

Figure 3. A subset of the many contributions to the first-order differential equation (3.9) in terms of a $\bar{Z}$ insertion in a maximal U(1)-violating five-point function. (a) A contact contribution obtained by applying $\bar{\mathcal{D}}$ to a five-point contact interaction, which comes from expanding $F_1^{(3)}(\tau^0)\, d^6R^4 Z$. (b) A contribution in which a pair of supergravity $gZ\bar{Z}$ vertices is attached to an external line on $F_0^{(3)}(\tau^0)\, d^6R^4$. (c) Another contribution with a supergravity $ggZ\bar{Z}$ tree attached to a d^6R^4 contact interaction. (d) A $F_0^{(3)}(\tau^0)d^6R^2Z\bar{Z}$ contact term with two $gZ\bar{Z}$ vertices attached. (e) A contribution with a $gZ\bar{Z}$ vertex attached to an external graviton line on $F_0^{(3)}(\tau^0)\, d^4R^5$. (f) A contribution to the inhomogeneous term from the product of two R^4-type vertices with coefficients $F_0^{(0)}(\tau^0)$.

The constants a and b would be determined if we were to evaluate all contributions to this six-point amplitude, including those shown in figure 3 and others, which we have not done. However, a shortcut is to input the known tree-level and one-loop terms of $\mathcal{E}_0^{(3)}$, which leads to $a = -6$, $b = 1/2$, as given in (3.11).

Terms with $p = 3$ and $w = 2$. Let us now extend the argument to impose constraints on the six-particle $p = 3$ terms with coefficients $F_{2,1}^{(3)}(\tau)$ and $F_{2,2}^{(3)}(\tau)$. We will proceed by considering the example of the seven-particle amplitude with external states consisting of four gravitons, two Z's and one $\bar{Z}$. Again, a supersymmetric contact term cannot exist for such an amplitude, and therefore the super-amplitude is fully determined by the lower-point amplitudes via tree-level factorisitions. The terms which contribute to this amplitude are schematically shown in figure 4. In this case, there are two independent contact vertices shown in figure 4(a) and figure 4(b), and the examples of factorising contributions are shown in figure 4(c), (d) and (e).

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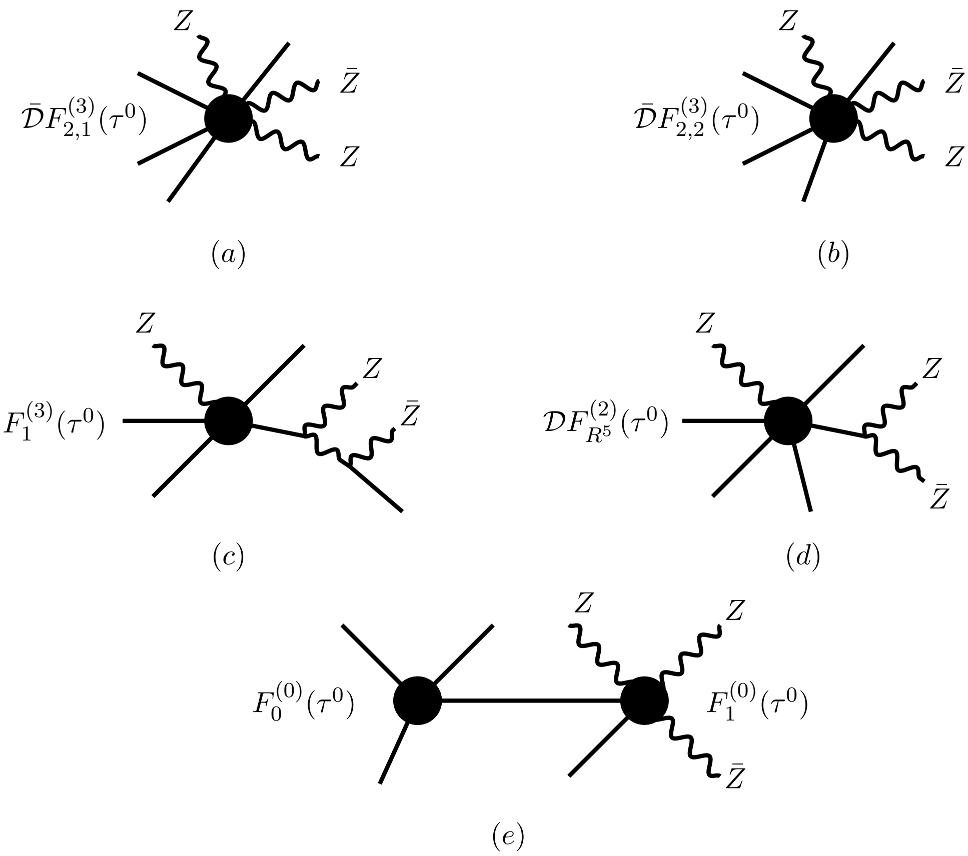

Figure 4. Terms that contribute to the $p = 3$ contribution to the seven-particle amplitude of four gravitons, two Z's and one $\bar{Z}$. (a) The contact interaction obtained by expanding $F^{(3)}_0(\tau)d^6R^4$ to give $\bar{\mathcal{D}}F^{(3)}_{2,1}(\tau)d^6_{(1)}R^4Z^2\bar{Z}$. (b) The other contact interaction obtained by expanding $F^{(3)}_{2,2}(\tau)\, d^6_{(2)}R^4Z^2$, to give the seven-point interaction $\bar{\mathcal{D}}F^{(3)}_{2,2}(\tau)\, d^6_{(2)}R^4Z^2\bar{Z}$. (c) A contribution arising from the Z and $\bar{Z}$ joining to a graviton attached to a leg of a $p = 3$, $n = 5$ interaction. (d) A contribution arising from a $gZ\bar{Z}$ vertex attached to a graviton line of d^4R^5Z (which has coefficient proportional to $\mathcal{D}F^{(2)}_{R^5}(\tau^0)$). (e) A factorising contribution with a pole linking a $p = 0$, $n = 4$ interaction with a $p = 0$, $n = 5$ interaction.

The contribution of 4(d) represents the vertex d^4R^5Z, which is proportional to $\mathcal{D}F^{(2)}_{R^5}(\tau^0)$. As we have argued previously in (6.3) that $F^{(2)}_{R^5}(\tau^0)$ is proportional $F^{(3)}_0(\tau^0)$ therefore we have $\mathcal{D}F^{(2)}_{R^5}(\tau^0) \sim \mathcal{D}F^{(3)}_0(\tau^0) \sim F^{(3)}_1(\tau^0)$. Again, the amplitude also has a factorisation contribution that involves a four-particle and five-point higher-dimensional interactions, as shown in figure 4(e). The absence of supersymmetric contact terms implies that the coefficient of each contact vertex is linearly related to the coefficients of the factorising terms, therefore we have following relations,

$$\bar{\mathcal{D}}\,\mathcal{E}^{(3)}_{2,1}(\tau^0) + b_1\,\mathcal{E}^{(3)}_1(\tau^0) + b_2\,E_0(\tfrac{3}{2},\tau^0)E_1(\tfrac{3}{2},\tau^0) = 0\,, \tag{6.6}$$

and

$$\bar{\mathcal{D}}\,\mathcal{E}^{(3)}_{2,2}(\tau^0) + c_1\,\mathcal{E}^{(3)}_1(\tau^0) + c_2\,E_0(\tfrac{3}{2},\tau^0)E_1(\tfrac{3}{2},\tau^0) = 0\,. \tag{6.7}$$

As discussed earlier, $\mathcal{E}^{(3)}_{2,1}$ (the coefficient of $\mathcal{O}^{(3)}_{6,1}$) is related to $\mathcal{E}^{(3)}_0$ by the action of covariant derivatives. With the normalisation of section 3.2, we have $\mathcal{E}^{(3)}_{2,1} = 4\,\mathcal{D}\mathcal{D}\mathcal{E}^{(3)}_0$. From this, we find

$$b_1 = -5\,, \qquad b_2 = \frac{3}{2}\,, \tag{6.8}$$

as shown in (3.20) .

The modular function $\mathcal{E}^{(3)}_{2,2}$ is genuinely new and more interesting. The fact that $\mathcal{E}^{(3)}_{2,2}$ does not contain a tree-level term (which requires that $c_1 = -2\,c_2$), results in the expression (3.30) for $\mathcal{E}^{(3)}_{2,2}$,

$$\bar{\mathcal{D}}\,\mathcal{E}^{(3)}_{2,2}(\tau^0) = c_1\left(\mathcal{E}^{(3)}_1(\tau^0) - \frac{1}{2}E_0(\tfrac{3}{2},\tau^0)\,E_1(\tfrac{3}{2},\tau^0)\right)\,. \tag{6.9}$$

We see that the modular forms $\mathcal{E}^{(3)}_{2,1}(\tau)$ and $\mathcal{E}^{(3)}_{2,2}(\tau)$ satisfy two distinct first-order differential relations that involve different linear combinations of $\mathcal{E}^{(3)}_1(\tau)$ and $E_0(\frac{3}{2},\tau)E_1(\frac{3}{2},\tau)$. Various features of $\mathcal{E}^{(3)}_{2,1}$ and $\mathcal{E}^{(3)}_{2,2}$, such as their perturbative expansions, were discussed in section 3.

The constant c_1 is in principle determined by supersymmetry by considering the seven-particle super-amplitude. This constant could also be fixed by an explicit evaluation of the dimension-14 contribution to the low-energy expansion of the six-particle one-loop string amplitude although we have not done this. Finally, once the six-particle terms are obtained, all the BPS maximal U(1)-violating interactions are completely fixed by soft limits as discussed in section 5.1. In particular, the coefficient of the n-particle kinematic structure $\mathcal{O}^{(3)}_{n,2}$ with $n > 6$ is determined by acting with covariant derivatives on $\mathcal{E}^{(3)}_{2,2}(\tau)$. So we conclude that all the BPS maximal U(1)-violating amplitudes are determined up to the constant c_1 that we have not evaluated.

7 Summary and discussion

The aim of this paper has been to determine the first-order differential equations that determine the moduli-dependent coefficients of BPS-protected terms in the low-energy expansion of type IIB superstring theory in a flat ten-dimensional Minkowski space background. The terms in the action (1.1) are SL(2, $\mathbb{Z}$)-invariant higher-derivative interactions that have the form of moduli-independent U(1)-violating interactions multiplied by τ-dependent coefficients. These coefficients are modular forms, which transform by a phase under the action of SL(2, $\mathbb{Z}$), which compensates for the U(1)-violation. The BPS-protected interactions are the ones with $p = 0$, $p = 2$ and $p = 3$, which have dimension ≤ 14 (where classical Einstein gravity has dimension 2).

The considerations of this paper followed two interrelated paths, investigating the higher-derivative effective action (1.1) in sections 2 and 3 and properties of U(1)-violating scattering amplitudes in sections 4, 5 and 6.[22]

[22] There are a number of other conjectured generalisations to the coefficients of higher-dimension interactions, such as those of [41, 42], which are distinct from the consideration of this paper.

Summary of the effective interactions. The lowest-order terms in the low-energy expansion beyond classical supergravity are those with $p = 0$ (of order R^4) and $p = 2$ (of order d^4R^4), which were the subject of earlier work. These have coefficients proportional to Eisenstein modular forms with properties summarised in appendix B,

$$F_w^{(p)}(\tau) = c_w^{(p)}\, E_w(s,\tau)\,, \qquad s = \frac{3+p}{2}\,, \qquad p = 0, 2\,, \tag{7.1}$$

where $c_w^{(p)}$ are numerical constants that according to our convention, are determined by (5.3). These functions are related to each other by covariant derivatives that raise and lower the modular weights as in (B.6) and (B.7),

$$E_{w+1}(s,\tau) = \frac{2}{s+w}\,\mathcal{D}_w E_w(s,\tau)\,, \qquad E_{w-1}(s,\tau) = \frac{2}{s-w}\,\bar{\mathcal{D}}_{-w} E_w(s,\tau)\,, \tag{7.2}$$

which imply the Laplace eigenvalue equations

$$\big(\Delta_{(-)} - s(s-1) + w(w-1)\big)\, E_w(s,\tau) = 0\,. \tag{7.3}$$

The structure of the 1/8-BPS terms, for which $p = 3$, were determined in section (3) based on consistency with the coefficient of the $w = 0$ case (the d^6R^4 interaction). In these cases the coefficients are modular forms given by

$$F_{w,i}^{(3)}(\tau) = c_{w,i}^{(3)}\, \mathcal{E}_{w,i}^{(3)}(\tau)\,, \tag{7.4}$$

where $\mathcal{E}_{w,i}^{(3)}(\tau)$ satisfy the following first-order differential equations[23]

$$\begin{aligned}
\mathcal{D}\mathcal{E}_0^{(3)}(\tau) &= \frac{1}{2}\,\mathcal{E}_1^{(3)}(\tau)\,, & \bar{\mathcal{D}}\mathcal{E}_1^{(3)}(\tau) &= 6\,\mathcal{E}_0^{(3)}(\tau) - \frac{1}{2}\,(E_0(\tfrac{3}{2},\tau))^2\,,\\
\mathcal{D}\mathcal{E}_1^{(3)}(\tau) &= \frac{1}{2}\,\mathcal{E}_{2,1}^{(3)}(\tau)\,, & \bar{\mathcal{D}}\mathcal{E}_{2,1}^{(3)}(\tau) &= 5\,\mathcal{E}_1^{(3)}(\tau) - \frac{3}{2}E_0(\tfrac{3}{2},\tau)E_1(\tfrac{3}{2},\tau)\,,\\
\bar{\mathcal{D}}\mathcal{E}_{2,2}^{(3)}(\tau) &= c_1\left(\mathcal{E}_1^{(3)}(\tau) - \frac{1}{2}E_0(\tfrac{3}{2},\tau)E_1(\tfrac{3}{2},\tau)\right)\,,
\end{aligned} \tag{7.5}$$

which imply the following inhomogeneous Laplace eigenvalue equations

$$\begin{aligned}
\big(\Delta_{(-)} - 12\big)\, \mathcal{E}_0^{(3)}(\tau) &= -\,(E_0(\tfrac{3}{2},\tau))^2\,,\\
\big(\Delta_{(-)} - 12\big)\, \mathcal{E}_1^{(3)}(\tau) &= -3\,E_0(\tfrac{3}{2},\tau)E_1(\tfrac{3}{2},\tau)\,,\\
\big(\Delta_{(-)} - 10\big)\, \mathcal{E}_{2,1}^{(3)}(\tau) &= -\frac{15}{2}\left(E_0(\tfrac{3}{2},\tau)E_2(\tfrac{3}{2},\tau) + \frac{3}{5}\,(E_1(\tfrac{3}{2},\tau))^2\right)\,,\\
\big(\Delta_{(-)} - 10\big)\, \mathcal{E}_{2,2}^{(3)}(\tau) &= -\frac{5c_1}{2}\left(E_0(\tfrac{3}{2},\tau)E_2(\tfrac{3}{2},\tau) - (E_1(\tfrac{3}{2},\tau))^2\right)\,.
\end{aligned} \tag{7.6}$$

We note, in particular, that the six-particle coefficient, $\mathcal{E}_{2,1}^{(3)}(\tau)$, has perturbative tree-level, one-loop and three-loop contributions. It multiplies the kinematic invariant $\mathcal{O}_{6,1}^{(3)}$,

[23] We have chosen the normalisation such that $c_{w,i}^{(3)} = 1$ for all w.

which reduces to the unique five-particle invariant $\mathcal{O}^{(3)}_{5,1}$ when any of the six external momenta vanishes. Combined with the first-order differential relation (7.5) this relates the non-perturbative $p = 3$ term in the low-energy expansion of the six-particle amplitude to the $p = 3$ term in the expansion of the five-particle amplitude.

By contrast, the coefficient $\mathcal{E}^{(3)}_{2,2}(\tau)$ has no perturbative tree-level coefficient, which is consistent with the fact that it multiplies the six-particle kinematic invariant $\mathcal{O}^{(3)}_{6,2}$. This invariant vanishes when any of the six particles has zero momentum, so it is not related to a five-particle amplitude in the soft limit.

Summary of constraints from maximal U(1)-violating amplitudes. Scattering amplitudes are evaluated in backgrounds with constant background, $\tau = \tau^0$ leading to maximal U(1)-violating amplitudes depend on the complex coupling in a manner that is described by appropriate modular forms. In section 4 we described the relation between the higher-derivative protected terms in the action and such amplitudes.

Although "naked" factors of the modulus τ cannot arise in $\mathcal{P}_n(\{\Phi\})$, general maximal U(1)-violating scattering amplitudes have external complex scalar states, in addition to the fields in $\mathcal{P}_n(\{\Phi\})$. These are obtained by expanding the modular form coefficients, $F^{(p)}_{w,i}(\tau)$, in fluctuations of τ. It is essential to choose an appropriate parameterisation of these moduli fluctuations in order to preserve manifest invariance under $\mathrm{SL}(2,\mathbb{Z})$ acting on τ^0 as well as on the fluctuations. This procedure, which is the normal coordinate expansion for the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ non-linear sigma model, leads to an expansion in powers of the field Z defined in (1.3). The coefficients of the terms with the same power counting but with different numbers of Z fields are related by covariant derivatives. This procedure is consistent with the soft limits that relate higher-point amplitudes with lower-point ones, which were confirmed explicitly in (4.31) using coefficients of the low-energy expansion of the $n = 4$, 5 and 6 superstring tree amplitudes (which were kindly provided by Oliver Schloterer [31]). The explicit tree low-energy expansion of the tree amplitudes determine ratios of the coefficients $c^{(p)}_{w,i}$, as given in (4.33).

The constraints imposed by supersymmetry were analysed in section 6 by extending the procedure in [11]. This considers n-particle BPS-protected vertices together with some extra external Z and $\bar{Z}$ states. Supersymmetry forbids contact interactions for such augmented amplitudes, which leads to conditions that constrain the modular coefficients of the BPS-protected vertices. In this manner we recover the conditions on the coefficient modular forms of sections 2 and 3, which demonstrates directly that the first-order equations are indeed a direct consequence of supersymmetry.

7.1 Discussion

- We saw from (3.23) that a term in the large-τ_2 expansion that contributes a negative integer power of τ_2 is annihilated by a sufficient number of covariant derivatives. As a result, we saw that the modular form $F^{(3)}_{2,1}(\tau) \sim \mathcal{D}_1\mathcal{D}_0 F^{(3)}_0(\tau)$, which is coefficient of $d^6_{(1)}\, R^4 Z^2$, has a vanishing two-loop term. Similarly, it is easy to see that the coefficients $F^{(3)}_{m,1}(\tau)$ with $m > 3$ have vanishing two-loop and three-loop contributions. Similarly, $\mathcal{D}_{v+1} \dots \mathcal{D}_2 F^{(3)}_{2,2}(\tau)$ not only has no tree-level term, but the three-loop term also vanishes for all $v > 2$.

- There has been a significant literature on the generalisation of the equations for the modular-invariant coefficients of the $w = 0$ BPS-protected interactions (such as $d^{2p}R^4$ with $p \leq 3$) to type II superstring theory compactified on a d-torus to $D = 10 - d$ dimensions. The solutions are specific automorphic functions associated with the higher-rank duality groups in the E_{d+1} series (see, for example, [43, 44] and references therein). It would be of interest to generalise these considerations to include processes in which the R-symmetry is broken, perhaps along the lines of [45] and [14].

- Finally, we note that one of the motivations for studying the constraints imposed by maximal supersymmetry and $\mathrm{SL}(2,\mathbb{Z})$ duality in the ten-dimensional type IIB theory is to better understand the holographic connection with $\mathrm{SL}(2,\mathbb{Z})$ Montonen-Olive duality in four-dimensional $\mathcal{N} = 4$, $\mathrm{SU}(N)$ supersymmetric Yang-Mills theory. In particular, the pattern of U(1)-violation in type IIB superstring amplitudes is the holographic image of the violation of the "bonus" U(1) of [46] in the gauge theory. In order to exhibit the $\mathrm{SL}(2,\mathbb{Z})$ duality it is necessary to choose g_{YM} and N (rather than the 't Hooft coupling $\lambda = g_{YM}^2 N$ and N) as independent parameters in the large-N limit of the Yang-Mills theory. This was discussed in [47] making use of properties of the operator product expansion of the composite gauge invariant Yang-Mills operator that is the holographic dual of τ. Plausibility arguments were given that in the large-N fixed limit the dependence on g_{YM} of certain BPS-protected correlation functions of gauge invariant operators in the 1/2-BPS Yang-Mills current supermultiplet is determined by the same $\mathrm{SL}(2,\mathbb{Z})$-covariant differential equations as those satisfied by the 1/2-BPS terms in the low-energy expansion of type IIB superstring amplitudes. The same limit also entered in [48] in the context of the holographic connection between four-dimensional maximally supersymmetric large-N Yang-Mills and the flat-space limit of the $AdS_5 \times S^5$ type IIB superstring theory.

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A Review of classical type IIB supergravity

We here review some features of type IIB supergravity that are needed in the main body of the paper. This is to some extent based on [21] and the appendices of [6].[24]

[24] Whereas in [21] the scalar fields were taken to parameterise the coset space $\mathrm{SU}(1,1,\mathbb{R})/\mathrm{U}(1)$, in [6] they were taken to paramterise $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$.

A.1 The field content

The fields of type IIB supergravity transform in representations of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2)$, where $\mathrm{SL}(2,\mathbb{R})$ is a global symmetry and $\mathrm{SO}(2)\sim\mathrm{U}(1)$ R-symmetry is a local symmetry. The fermions are charged under SO(2) but are $\mathrm{SL}(2,\mathbb{R})$ singlets. With the exception of the scalar fields, the bosons are neutral with respect to the SO(2) but transform in non-trivial representations of $\mathrm{SL}(2,\mathbb{R})$.

The scalar fields parameterise a $\mathrm{SL}(2,\mathbb{R})$ matrix, which has three independent real components. But this description is redundant since the local SO(2) symmetry can be used to eliminate one scalar field, which restricts the scalar fields to the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. In order to understand the parameterisation of the fields it is useful to review properties of these scalar fields and their restriction to the coset.

The scalar fields and the $\mathbf{SL(2,\mathbb{R})/U(1)}$ coset. A general $\mathrm{SL}(2,\mathbb{R})$ matrix can be written in the $N\times A\times K$ Iwazawa form,

$$\hat{V}(\tau_1,\tau_2,\phi) = \begin{pmatrix}1 & \tau_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\tau_2^{1/2} & 0\\ 0 & \tau_2^{-1/2}\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix} \tag{A.1}$$

$$= \frac{1}{\sqrt{\tau_2}}\begin{pmatrix}\tau_2\cos\phi+\tau_1\sin\phi & -\tau_2\sin\phi+\tau_1\cos\phi\\ \sin\phi & \cos\phi\end{pmatrix}. \tag{A.2}$$

The indices on the matrix $\hat{V}^{\alpha}{}_{i}$ indicate that it transforms on the left by the global $\mathrm{SL}(2,\mathbb{R})$ and on the right by the local SO(2), i.e.,

$$\hat{V}^{\alpha}_{j} \to (U^{-1})^{\alpha}{}_{\beta}\,\hat{V}^{\beta}_{i}\,R_{ij}(\Sigma)\,, \tag{A.3}$$

where

$$U^{\alpha}{}_{\beta} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}, \qquad a,b,c,d\in\mathbb{R}, \qquad \det U = 1\,, \tag{A.4}$$

is a $\mathrm{SL}(2,\mathbb{R})$ matrix and $R_{ij}(\Sigma)$ is a rotation through an angle Σ. Note that

$$\hat{V}^{\alpha}{}_{i}\hat{V}_{i}{}^{\beta} := M^{\alpha\beta} = \frac{1}{\tau_2}\begin{pmatrix}\tau_1^2+\tau_2^2 & \tau_1\\ \tau_1 & 1\end{pmatrix}, \tag{A.5}$$

is a $\mathrm{SL}(2,\mathbb{R})$ matrix that is independent of ϕ.

The local SO(2) gauge symmetry can be used to set $\phi=0$, which restricts the scalar fields to the two-dimensional coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. In that case we have

$$\hat{V}(\tau_1,\tau_2,0) = \frac{1}{\sqrt{\tau_2}}\begin{pmatrix}\tau_2 & \tau_1\\ 0 & 1\end{pmatrix}. \tag{A.6}$$

We often use a complex U(1) basis by adding and subtracting the columns in (A.2) and setting $\phi=0$, which takes $\hat{V}(\tau_1,\tau_2,0)\to V(\tau)$ defined by

$$V(\tau) = \begin{pmatrix}V_-^1 & V_+^1\\ V_-^2 & V_+^2\end{pmatrix} = \frac{1}{\sqrt{-2i\tau_2}}\begin{pmatrix}\bar{\tau} & \tau\\ 1 & 1\end{pmatrix}, \tag{A.7}$$

where $\tau=\tau_1+i\tau_2$ and $V_\pm = V_1\pm iV_2$. In this basis the coset space is $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$.

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After making the gauge choice $\phi = 0$, the action of the global $\mathrm{SL}(2,\mathbb{R})$ must be accompanied by a compensating gauge transformation together with a nonlinear redefinition of τ in order to ensure that $V(\tau)$ remains of the form (A.7). A general transformation (A.3) that preserves the gauge combines a $\mathrm{SL}(2,\mathbb{R})$ transformation with a compensating $\mathrm{U}(1)$ transformation that leaves the form of $\hat{V}$ in (A.7) unchanged has the form

$$(V_+^\alpha(\tau), V_-^\alpha(\tau)) \to (U^{-1})^\alpha{}_\beta \left(V_+^\beta(\tau')e^{-i\Sigma}, V_-^\beta(\tau')e^{i\Sigma}\right) \tag{A.8}$$

where

$$\tau' = \frac{a\tau + b}{c\tau + d}\,, \tag{A.9}$$

and the compensating $\mathrm{U}(1)$ transformation is given by

$$e^{i\Sigma} = \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)^{\frac{1}{2}}\,. \tag{A.10}$$

So we see that after restricting the scalar fields to the coset, a $\mathrm{SL}(2,\mathbb{R})$ transformation induces a $\mathrm{U}(1)$ transformation that acts on the fermions, even though they were originally $\mathrm{SL}(2,\mathbb{R})$ singlets.

The supergravity fields. There are 128 physical bosonic states, of which 64 come from the Neveu-Schwarz/Neveu-Schwarz ($NSNS$) and 64 from the Ramond-Ramond (RR) sector. The fields of the $NSNS$ sector consist of the graviton, which is a $\mathrm{U}(1)$ and $\mathrm{SL}(2,\mathbb{Z})$ singlet; the second-rank antisymmetric potential B_2 with field strength $H = dB_2$, which is a $\mathrm{U}(1)$ singlet and forms part of a $\mathrm{SL}(2,\mathbb{Z})$ doublet; the dilaton φ, which enters into the imaginary part of the complex modulus field, $\tau_2 = e^{-\varphi}$.

The fields of the RR sector consist of the even-rank potentials, $C^{(p)}$ ($p = 0, 2, 4$), with field strengths $F^{(p+1)} = dC^{(p)}$. The pseudoscalar defines the real part of the complex modulus, $\tau_1 = C^{(0)}$. The RR field strength $F^{(3)}$ is a $\mathrm{U}(1)$ singlet that forms the other part of the $\mathrm{SL}(2,\mathbb{Z})$ doublet. The five-form field strength $F^{(5)}$ is a $\mathrm{U}(1)$ and $\mathrm{SL}(2,\mathbb{Z})$ singlet which satisfies a self-duality condition, $F^{(5)} = - * F^{(5)}$.

The 128 physical fermionic states are described by fermions in the $NSNS \times RR$ sector together with those of the $RR \times NSNS$ sector, which can be combined to form a complex chiral gravitino, $\psi_\mu = \psi_{1\,\mu} + i\psi_{2\,\mu}$, and a complex spin-half dilatino, $\Lambda = \Lambda_1 + i\Lambda_2$ of the opposite chirality (where the subscripts 1 and 2 indicate the $NS \times R$ sector and the $R \times NS$ sector, respectively). These fermion fields are invariant under $\mathrm{SL}(2,\mathbb{R})$, while ψ carries a $\mathrm{U}(1)$ charge $q_\psi = -1/2$ and Λ carries charge $q_\Lambda = -3/2$.

The $q = \pm 2$ scalar fields. The two-derivative supergravity action can be conveniently expressed in a covariant form by appropriate parameterisation of the scalar fields. The $\mathrm{SL}(2,\mathbb{R})$ singlet expressions

$$P_\mu = -\epsilon_{\alpha\beta} V_+^\alpha \partial_\mu V_+^\beta = i\frac{\partial_\mu \tau}{2\tau_2}\, e^{-2\pi i\phi}, \qquad \bar{P}_\mu = -\epsilon_{\alpha\beta} V_-^\alpha \partial_\mu V_-^\beta = -i\frac{\partial_\mu \bar{\tau}}{2\tau_2}\, e^{2\pi i\phi}, \tag{A.11}$$

manifestly transform with U(1) charges $q_P = -2$ and $q_{\bar{P}} = 2$, respectively. Upon fixing the gauge $\phi = 0$ they transform int the following manner under the U(1) transformations induced from the SL$(2,\mathbb{R})$ transformations

$$P_\mu \to \left(\frac{c\bar{\tau}+d}{c\tau+d}\right) P_\mu \,, \qquad \bar{P}_\mu \to \left(\frac{c\tau+d}{c\bar{\tau}+d}\right) \bar{P}_\mu \,. \tag{A.12}$$

The U(1) connection and covariant derivatives. Space-time derivatives need to be augmented with a U(1) gauge connection in order to express the action in a SL$(2,\mathbb{R})$-invariant manner. The SL$(2,\mathbb{R})$ singlet expression,

$$Q_\mu = -\frac{i}{2}\epsilon_{\alpha\beta}\left(V_+^\alpha \partial_\mu V_-^\beta - V_-^\beta \partial_\mu V_+^\alpha\right) = \frac{\partial_\mu \tau_1}{2\tau_2} - \partial_\mu \phi \,, \tag{A.13}$$

is the composite U(1) gauge connection that transforms as $Q \to Q - \partial_\mu \Sigma$ under the local transformation (A.8). Thus, we define the covariant space-time derivative acting on charge-$q = -2w$ fields

$$D_w := \partial_\mu + i\, q\, Q_\mu \,. \tag{A.14}$$

It is easy to verify that under the induced U(1) transformation that accompanies a SL$(2,\mathbb{R})$ transformation in the gauge $\phi = 0$, the transformation of Q_μ is given by

$$\begin{aligned} Q_\mu = \frac{\partial_\mu \tau_1}{2\tau_2} &\to \frac{\partial_\mu \tau}{4\tau_2}\left(\frac{c\bar{\tau}+d}{c\tau+d}\right) + \frac{\partial_\mu \bar{\tau}}{4\tau_2}\left(\frac{c\tau+d}{c\bar{\tau}+d}\right) \\ &= Q_\mu + \partial_\mu \Sigma \,, \end{aligned} \tag{A.15}$$

where Σ was defined in (A.10).

To verify that D_w is indeed a U(1)-covariant derivative note that under a SL$(2,\mathbb{Z})$ transformation (2.2) a charge-$q = -2w$ field transforms as

$$\Phi_q \to \frac{(c\bar{\tau}+d)^w}{(c\tau+d)^w}\,\Phi_q = \Phi_q\, e^{iq\Sigma} \,, \tag{A.16}$$

so we have

$$\begin{aligned} \partial_\mu \Phi_q \to & \left(\frac{c\, w\, \partial_\mu \bar{\tau}}{c\bar{\tau}+d} - \frac{c\, w\, \partial_\mu \tau}{c\tau+d}\right) \frac{(c\bar{\tau}+d)^w}{(c\tau+d)^w}\,\Phi_q + \frac{(c\bar{\tau}+d)^w}{(c\tau+d)^w}\,\partial_\mu \Phi_q \\ = & \frac{(c\bar{\tau}+d)^w}{(c\tau+d)^w}\left(\partial_\mu - iq\partial_\mu \Sigma\right)\Phi_q \,. \end{aligned} \tag{A.17}$$

Therefore

$$\left(\partial_\mu + iqQ_\mu\right)\Phi_q \to \frac{(c\bar{\tau}+d)^w}{(c\tau+d)^w}\left(\partial_\mu + iqQ_\mu\right)\Phi_q \,. \tag{A.18}$$

A.2 Terms in the type IIB supergravity action

The scalar field action: the scalar field kinetic term has the form (in Einstein frame)

$$S_\tau = -\frac{1}{\kappa^2}\int d^{10}x\, e\, \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{2\tau_2^2} = -\frac{2}{\kappa^2}\int d^{10}x\, e\, P_\mu\, \bar{P}^\mu \,. \tag{A.19}$$

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The other bosonic fields: the two antisymmetric second-rank potentials, $B_{\mu\nu}$ and $C^{(2)}_{\mu\nu}$, have field strengths $H = dB_2$ and $F^{(3)} = dC^{(2)}$. that form an $\mathrm{SL}(2,\mathbb{R})$ doublet, F^α. In discussing the $\mathrm{SL}(2,\mathbb{Z})$ properties of the theory is very natural to package them into the $\mathrm{SL}(2,\mathbb{R})$ singlet fields,

$$G = -\epsilon_{\alpha\beta} V_+^\alpha F^\beta, \qquad \bar{G} = -\epsilon_{\alpha\beta} V_-^\alpha F^\beta\,, \tag{A.20}$$

which carry U(1) charges $q_G = -1$ and $q_{\bar{G}} = +1$, respectively. The kinetic term involving G in the action is given by

$$S_G = -\frac{1}{\kappa^2}\int d^{10}x\, e\, \frac{1}{2} G\,\bar{G}\,. \tag{A.21}$$

The antisymmetric fourth-rank potential, $C^{(4)}$, with self-dual field strength $F_5 = dC^{(4)}$, has an equation of motion that is expressed by the self-duality condition $F_5 = *F_5$, which cannot be obtained from a globally well-defined Lagrangian.

The fermion field action: the covariant Dirac action for the dillatino has the form

$$S_\Lambda = \frac{i}{\kappa^2}\int d^{10}x\, e\, \bar{\Lambda}\gamma^\mu \left(\partial_\mu + \frac{3}{2} i\, Q_\mu\right)\Lambda\,. \tag{A.22}$$

Similarly, in a fixed gauge $\gamma_\mu \psi^\mu = 0$ the Rarita-Schwinger equation for ψ_μ reduces to $\partial_\mu \psi^\mu = 0$ and $\gamma \cdot D\psi_\mu = 0$ and the action for the Rarita-Schwinger field can be written as

$$S_\psi = \frac{i}{\kappa^2}\int d^{10}x\, e\, \bar{\psi}^\nu\, \gamma^\mu \left(\partial_\mu - \frac{1}{2} i\, Q_\mu\right)\psi_\nu\,. \tag{A.23}$$

Interaction terms: although we do not need the explicit supergravity interaction terms in this paper we note that they are invariant under $\mathrm{SL}(2,\mathbb{Z})$ and they conserve the local U(1), which means that the phase ϕ cancels out of the action. For example, the complex scalar field interacting with the fermions has the form

$$S^P_{\Lambda\psi*} = \frac{i}{\kappa^2}\int d^{10}x\, e\, \bar{\Lambda}\gamma^\mu\gamma^\omega \bar{\psi}_\mu P_\omega + \text{c.c.}\,. \tag{A.24}$$

A.3 The SU(1, 1) parameterisation of the complex scalar field fluctuations

For much of this paper we use moduli fields that parameterise the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$, which is the upper half τ plane. This is well suited to making the discrete identifications that are implied by invariance under the T transformation, $\tau \to \tau + 1$, and the S transformation, $\tau \to -1/\tau$, which restrict τ to a fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$.

However, as is common in coset space nonlinear sigma models, in discussing amplitudes with external scalar fields it is important that we define the fluctuating fields in a parameterisation that transforms covariantly under the symmetry. Therefore we want to consider fluctuations of the bosonic fields around a constant background $\tau = \tau^0$, that transform covariantly under the U(1) induced by $\mathrm{SL}(2,\mathbb{Z})$ transformations. This is realised by the field redefinition of (1.3)

$$Z = \frac{\tau - \tau^0}{\tau - \bar{\tau}^0}\,, \tag{A.25}$$

which is a $\mathrm{SL}(2,\mathbb{C})$ transformation that maps the upper-half τ plane to the unit disk in the Z plane. The origin of the disk is the mapping of the point $\tau = \tau^0$ and its boundary is the real axis of the τ plane. It is easy to see that transforming τ and τ^0 by $\mathrm{SL}(2,\mathbb{Z})$ gives the linear transformation

$$Z \to \frac{c\tau^0 + d}{c\bar{\tau}^0 + d} Z \,. \tag{A.26}$$

The advantage of describing the background in the $\mathrm{SL}(2,\mathbb{R})$ parameterisation is that the duality transformations lie in the arithmetic subgroup $\mathrm{SL}(2,\mathbb{Z}) \in \mathrm{SL}(2,\mathbb{R})$ which is obtained by making discrete identifications of τ that restrict it to a single fundamental domain. This restriction is very unnatural in the $\mathrm{SU}(1,1)$ parameterisation.

The definition of Z given in (A.25) leads to the expression

$$\tau_2 = \tau_2^0 \frac{1 - \bar{Z}Z}{(1-Z)(1-\bar{Z})} \,, \tag{A.27}$$

and the field P_μ in (A.11) becomes

$$P_\mu = i \frac{\partial_\mu \tau}{2\tau_2} e^{-2\pi i \phi} = \frac{\partial_\mu Z}{1 - \bar{Z}Z} \left(\frac{1-\bar{Z}}{1-Z} \right) e^{-2\pi i \phi} \,. \tag{A.28}$$

In our analysis we are setting $\phi = 0$ in order to describe the coset in terms of τ.

Likewise the expression for the connection becomes

$$Q_\mu = \frac{\partial_\mu \tau_1}{2\tau_2} = \frac{i}{2} \frac{Z\partial_\mu \bar{Z} - \bar{Z}\partial_\mu Z}{1 - \bar{Z}Z} + \frac{i}{2} \partial_\mu \log \left(\frac{1-\bar{Z}}{1-Z} \right) . \tag{A.29}$$

It is very simple to transform terms in the action, such as the scalar kinetic term S_τ in (A.19), or the interaction term $S^P_{\Lambda\psi*}$ (A.24), from functions of τ to functions of Z.

Although we want to stay in the gauge $\phi = 0$, we note that in order for the transformation to reproduce the form of the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset with $W^\alpha_\pm$ it would be necessary to change the $\mathrm{U}(1)$ gauge so that

$$e^{2\pi i \phi} = \left(\frac{1-\bar{Z}}{1-Z} \right) , \tag{A.30}$$

in which case (A.28) and (A.29) are the same as those in [21].

B Properties of modular forms and Eisenstein series

We will here discuss some properties of the modular functions and modular forms that arise in the text. The simplest examples are the Laplace equations (2.7) for $w = 0$ and $s \in \mathbb{C}$,

$$(\Delta - s(s-1))\, f^{(0,0)}(\tau) = 0 \,, \tag{B.1}$$

where $\Delta = 4\tau_2^2(\partial_\tau\, \partial_{\bar{\tau}})$ and $f^{(0,0)}(\tau)$ is a $\mathrm{SL}(2,\mathbb{Z})$ modular function (so $w = 0$) that satisfies the boundary condition $\lim_{\tau_2 \to \infty} f^{(0,-0)}(\tau) < \tau_2^a$, where a is a real number. This condition of power boundedness follows from string perturbation theory, where the most singular term has the tree-level behaviour. The unique solution to this Laplace eigenvalue equation with these boundary conditions is the non-holomorphic Eisenstein series, which has the form

$$E(s,\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}} = \sum_{N\in\mathbb{Z}} \mathcal{F}_N(s,\tau_2)\, e^{2\pi i N\tau_1}\,, \tag{B.2}$$

where the zero mode consists of two power behaved terms,

$$\mathcal{F}_0(s,\tau_2) = 2\zeta(2s)\,\tau_2^s \;+\; \frac{2\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)}\,\tau_2^{1-s}\,, \tag{B.3}$$

and the non-zero modes are proportional to K-Bessel functions,

$$\mathcal{F}_N(s,\tau_2) = \frac{4\,\pi^s}{\Gamma(s)}\,|N|^{s-\frac{1}{2}}\,\sigma_{1-2s}(|N|)\sqrt{\tau_2}\,K\Big(s-\frac{1}{2},2\pi|N|\tau_2\Big)\,, \quad N\neq 0\,, \tag{B.4}$$

where the divisor sum is defined by

$$\sigma_p(N) = \sum_{d>0,d|N} d^{2p}\,, \quad \text{for} \quad N>0\,, \tag{B.5}$$

and $\sigma_{-p}(N) = N^{-p}\,\sigma_p(N)$.

The lowest order example of such a modular invariant coefficient is $F_0^{(0)}(\tau) = E(\frac{3}{2},\tau)$, the coefficient of the R^4 interaction, which is the $p=0$ (i.e dimension-8) term in the low-energy expansion of the four-graviton amplitude. This has a zero mode that contains two power-behaved terms given by (B.3) with $s=3/2$. Taking into account the power of $\tau_2^{1/2}$ in transforming to the string frame in (1.1), these powers are τ_2^2 and τ_2^0, which correspond to tree-level and one-loop perturbative superstring contributions. The $p=2$ term of order d^4R^4 has a coefficient $E(\frac{5}{2},\tau)$ that has tree-level and two-loop perturbative contributions.

We are generally interested in modular forms with weights $(w,-w)$, or U(1) charge $q=2w$. Using the definitions of covariant derivatives in (2.4) we have,

$$\mathcal{D}_w\,E_w(s,\tau) = \frac{s+w}{2}\,E_{w+1}(s,\tau)\,, \tag{B.6}$$

and

$$\bar{\mathcal{D}}_{-w}\,E_w(s,\tau) = \frac{s-w}{2}\,E_{w-1}(s,\tau)\,. \tag{B.7}$$

Note, in particular, that with this normalisation

$$E_w(s,\tau) = \frac{2^w\Gamma(s)}{\Gamma(s+w)}\,\mathcal{D}_{w-1}\cdots\mathcal{D}_0\,E_0(s,\tau)\,, \tag{B.8}$$

where $E_0(s,\tau) := E(s,\tau)$. It is straightforward to show that

$$E_w(s,\tau) = \sum_{(m,n)\neq(0,0)} \left(\frac{m+n\bar{\tau}}{m+n\tau}\right)^w \frac{\tau_2^s}{|m+n\tau|^{2s}}\,. \tag{B.9}$$

Iterating these equations gives the Laplace equations

$$\Delta^{(w)}_{(+)} E_w(s,\tau) := 4\bar{\mathcal{D}}_{-w-1}\mathcal{D}_w E_w(s,\tau) = (s+w)(s-w-1)\, E_w(s,\tau)\,, \tag{B.10}$$

$$\Delta^{(w)}_{(-)} E_w(s,\tau) := 4\mathcal{D}_{w-1}\bar{\mathcal{D}}_{-w} E_w(s,\tau) = (s-w)(s+w-1)\, E_w(s,\tau)\,. \tag{B.11}$$

Note that the two laplacians acting on weight-$(w,-w)$ modular forms satisfy

$$\Delta^{(w)}_{(+)} - \Delta^{(w)}_{(-)} = -2w\,, \qquad \Delta^{(0)}_{(+)} = \Delta^{(0)}_{(-)} = \Delta\,. \tag{B.12}$$

Hence we see that the non-holomorphic modular form $f^{(w,-w)}$ satisfying the $\mathrm{SL}(2,\mathbb{Z})$-covariant Laplace eigenvalue equation (2.7) has the solution

$$f^{(w,-w)}_s(\tau) := E_w(s,\tau)\,. \tag{B.13}$$

In the case $s = 3/2$ that is relevant for the coefficients of the $O\big((\alpha')^{-1}\big)$ terms, this has a Fourier expansion of the form

$$\begin{aligned} E_w(\tfrac{3}{2},\tau) &= 2\zeta(3)\,\tau_2^{\frac{3}{2}} + \frac{4\zeta(2)}{1-4w^2}\,\tau_2^{-\frac{1}{2}} \\ &\quad + \sum_{N=1}^{\infty} \left(\mathcal{F}_{N,4-w}(\tfrac{3}{2},\tau_2) e^{2\pi i N\tau_1} + \mathcal{F}_{N,4+w}(\tfrac{3}{2},\tau_2) e^{-2\pi i N\tau_1}\right)\,. \end{aligned} \tag{B.14}$$

The first two terms in (B.14) have the interpretation of contributions that should arise in string perturbation theory at tree-level and one loop, while the instanton and anti-instanton terms are contained in

$$\mathcal{F}_{N,4+w}(\tfrac{3}{2},\tau_2) = (8\pi)^{\frac{1}{2}}\,\sigma_{-2}(N)\,(2\pi N)^{\frac{1}{2}} \sum_{k=w}^{\infty} \frac{a_{4+w,k}}{(2\pi N\tau_2)^k}\, e^{-2\pi N\tau_2} \tag{B.15}$$

where

$$a_{n,k} = \frac{(-1)^n}{2^k(k-n+4)!}\,\frac{\Gamma(\frac{3}{2})}{\Gamma(n-\frac{5}{2})}\,\frac{\Gamma(k-\frac{1}{2})}{\Gamma(-k-\frac{1}{2})}\,. \tag{B.16}$$

The instanton sum in (B.15) begins with the power τ_2^w for D-instantons (which have phases $e^{2\pi i N\tau_1}$) while the series of corrections to the anti D-instanton (with phases $e^{-2\pi i N\tau_1}$) starts with the power τ_2^{-w}. These powers are consistent with the requirement of saturating the fermionic zero modes that are present in the D-instanton background.

C Linearised supersymmetry and higher derivative terms

The supersymmetries of the ten-dimensional type IIB theory are associated with two sixteen-component chiral fermionic $\mathrm{SO}(9,1)$ spinors, θ_1 and θ_2, which have the same chirality. It is convenient to combine these into a complex supercharge $\theta = \theta_1 + i\theta_2$, and its complex conjugate, $\bar{\theta}$. The linearised expressions for the effective interactions that preserve half of the 32 supersymmetries can be simply obtained by packaging the physical fields or their field strengths into a constrained superfield $\Phi(x^\mu - i\bar{\theta}\gamma^\mu\theta, \theta)$ where θ^A $(A = 1,\ldots,16)$

is a complex Grassmann coordinate that transforms as a Weyl spinor of SO(9, 1). This superfield satisfies the holomorphic condition [8],

$$\bar{D}_\theta \Phi = 0, \tag{C.1}$$

and is further constrained by imposing the condition,

$$D_\theta{}^4 \Phi = \bar{D}_\theta{}^4 \bar{\Phi}, \tag{C.2}$$

where

$$D_{\theta A} = \frac{\partial}{\partial \theta^A} + 2i(\gamma^\mu \bar{\theta})_A \partial_\mu, \qquad \bar{D}_{\theta A} = -\frac{\partial}{\partial \bar{\theta}^A} \tag{C.3}$$

are the holomorphic and anti-holomorphic covariant derivatives that anticommute with the rigid supersymmetries

$$Q_A = \frac{\partial}{\partial \theta^A}, \qquad \bar{Q}_A = -\frac{\partial}{\partial \bar{\theta}^A} + 2i(\bar{\theta}\gamma^\mu)_A \partial_\mu. \tag{C.4}$$

The constraints (C.1) and (C.2) ensure that the field Φ has an expansion in powers of θ (but not $\bar{\theta}$), that terminates after the θ^8 term and the 256 component fields satisfy the linearised field equations and Bianchi identities.

$$\begin{aligned} \Phi &= \tau_2^0 + \tau_2^0\, \Delta \\ &= \tau_2^0 + \tau_2^0(\hat{\tau} + \theta\Lambda + \theta^2 G + \theta^3 \partial\psi + \theta^4(\mathcal{R}_{\mu\sigma\nu\tau} + \partial F_5) + \cdots + \theta^8 \partial^4 \hat{\bar{\tau}}) \\ &:= \tau_2^0 + \tau_2^0 \sum_{r=0}^{8} \theta^r \Phi^{(r)}, \end{aligned} \tag{C.5}$$

where we have suppressed all details of the spinor and tensor indices. The quantity $\tau_2^0\,\Delta$ is the linearised fluctuation around a constant purely imaginary flat background, $\tau_2^0 = g_s^{-1}$. The fields G and $\bar{G}$ are complex combinations of the RR and $NSNS$ field strengths. The θ^4 terms are the Weyl curvature, R, and the RR five-form field strength, F_5. The fermionic field Λ is the complex dilatino and ψ is the complex gravitino. The terms indicated by $\cdots$ in (C.5) fill in the remaining members of the ten-dimensional $N = 2$ chiral supermultiplet, comprising (in symbolic notation) $\partial\psi$, $\partial^2\bar{G}$ and $\partial^3\bar{\Lambda}$. The complex conjugate superfield $\bar{\Phi}$ is a function of $\bar{\theta}$ and has a similar expansion with the component fields interchanged with their complex conjugates.

The U(1) R-symmetry charge q_r of any component $\Phi^{(r)}$ is correlated with the powers of θ. Assigning a charge $-1/2$ to θ and an overall charge -2 to the superfield leads to the charge for the field with r powers of θ,

$$q_r = -2 + \frac{r}{2}\,. \tag{C.6}$$

Thus, $q_{\hat{\tau}} = -2$; $q_\Lambda = -3/2$; $q_G = -1$; $q_\psi = -1/2$; $q_R = q_{F_5} = 0$.

Although the linearised theory cannot capture the full structure of the terms in the effective action it can be used to relate various terms in the limit of weak coupling, $\tau_2^0 = g_s^{-1} \to \infty$ (where $g = e^{\phi_0}$ is the string coupling constant). The linearised approximations

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to the complete interactions are those that arise by integrating a function of ϕ over the sixteen components of θ,

$$S_{\rm linear} = \int d^{10}x d^{16}\theta \, e \, H[\Phi] + \text{ c.c.}, \tag{C.7}$$

which is manifestly invariant under the rigid supersymmetry transformations, (C.4). The various component interactions contained in (C.7) are obtained from the θ^{16} term in the expansion,

$$H[\Phi] = H(\tau_2^0) + \Delta \frac{\partial}{\partial \tau_2^0} H(\tau_2^0) + \frac{1}{2}\Delta^2 \left(\frac{\partial}{\partial \tau_2^0}\right)^2 H(\tau_2^0) + \cdots . \tag{C.8}$$

Using the expression for Δ in (C.5) and substituting into (C.7) leads to all the possible interactions at order $1/\alpha'$,

$$\begin{aligned} S_{\rm linear} = \int d^{10}x \, e \, \Big(& h^{(12,-12)}\Lambda^{16} + h^{(11,-11)} G\Lambda^{14} + \dots \\ & + h^{(8,-8)} G^8 + \dots + h^{(0,0)} \, R^4 + \dots + h^{(-12,12)} \Lambda^{*\,16} \Big) , \end{aligned} \tag{C.9}$$

where $h^{(w,-w)}$ are functions of τ_2^0.

The superscripts that label the coefficients $h^{(w,-w)}$ are related to the violation of the U(1) charge. Thus, the linearised form of the general term in (C.9) contains a product of p fields,

$$\int d^{10}x \, e \, h^{(w,-w)} \prod_{k=1}^{p} \Phi^{(r_k)}, \tag{C.10}$$

which violates the U(1) charge by the units of

$$\sum_{k=1}^{p} q_{r_k} = -2w = 8 - 2p \, , \tag{C.11}$$

where we have used (C.6) and the fact that the total power of θ must be $\sum_k r_k = 16$. For example the R^4 term ($w = 0$) conserves the U(1) charge while the Λ^{16} term ($w = 12$) violates the U(1) charge by -24 and there are many other terms that violate the charge by any even number.

In the linearised approximation, $g_s \to 0$ ($\tau_2^0 \to \infty$), the coefficients $h^{(w,-w)}$ are constants that are related to each other by use of the Taylor expansion, (C.8). For example, the R^4 term has coefficient $\partial^4_{\tau_2^0} H$ while the Λ^{16} term has coefficient $\partial^{16}_{\tau_2^0} H$ so that, at the linearised level,

$$h^{(12,-12)} \sim \left(\tau_2^0 \frac{\partial}{\partial \tau_2^0}\right)^{12} h^{(0,0)}, \tag{C.12}$$

where for the moment we are not concerned about the overall constant. In writing this we have used the fact that the linearised approximation is valid only if the inhomogeneous term in the modular covariant derivative, $\mathcal{D}$ is negligible, which requires that

$$2\tau_2^0 \partial_{\tau_2^0} h^{(w,-w)} \gg w h^{(w,-w)} \tag{C.13}$$

since only in this case does the modular covariant derivative reduce to the ordinary derivative. This inequality is obviously not satisfied by terms in the expansion of $h^{(w,-w)}$ that are powers of τ_2^0. However, when acting on a factor such as $(\tau_2^0)^n e^{-2\pi|N|\tau_2^0}$ (where n is any constant) which is characteristic of a charge-N D-instanton, the inhomogeneous term may be neglected in the limit $\tau_2^0 \to \infty$ and the covariant derivative linearises. Therefore, a linearised superspace expression such as (C.7) should contain the exact leading multi-instanton contributions to the R^4 and related terms. These leading instanton terms arise by substituting the expression

$$F_N(\phi) = c_N e^{2\pi i|N|\phi} \tag{C.14}$$

into (C.7).

In the nonlinear theory the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the type IIB theory requires that the $h^{(w,-w)}(\tau)$ are modular forms with holomorphic and anti-holomorphic weights as indicated in the superscripts. The relative coefficients of the interactions of different U(1) charge could, in principle, be determined by supersymmetry, but we have not determined them in that manner.

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